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SENSOR LOCATION PROBLEM FOR THE BIDIRECTIONAL GRAPH: OPTIMAL SOLUTIONS

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Combining the recent achievements of sparse matrix and network analysis, graph theory, and theoretical computer science, we propose a new approach to the construction of numerical methods for finding optimal solutions to the sensor location problem for a bidirectional graph.

Keywords: NP-complete problem; transport network analysis; traffic sensor location; sparse underdetermined system.

The problem of minimizing the set M of monitored nodes being the locations of sensors that collect the necessary information about the network flow [1, 2] is NP-complete [3]. A brute-force search for the optimal solution (the minimum set of monitored nodes), is associated with huge computational costs [3, 4].

Consider a finite connected oriented bidirectional graph (network) $G=(I,U)$, where the set of arcs U is defined on the direct product $I\times I$, $|I|<\infty$, $|U|<\infty$. Let x_{ij} be the unknown flow on the arc $(i,j)\in U$. A bidirectional graph G has the following property: if there exists an arc $(i,j)\in U$ with a flow x_{ij} then the corresponding arc $(j,i)\in U$ with some flow x_{ji} also exists. The flow function $x:U\to\mathbb{R}$ satisfies the following system:

$$\sum_{j\in I_i^+(U)} x_{ij} - \sum_{j\in I_i^-(U)} x_{ji} = \begin{cases} 0, & i\in I\setminus I^*, \\ x_i, & i\in I^*, \end{cases} \tag{1}$$

where x_i is the external flow into the node $i\in I^*\subseteq I$, x_{ij} is the flow on the arc $(i,j)\in U$, $I_i^+(U)=\{j\in I:(i,j)\in U\}$, $I_i^-(U)=\{j\in I:(j,i)\in U\}$. For external flows, the following condition holds: $\sum_{i\in I^*} x_i = 0$. For each arc $(i,j)\in U$ one knows the fraction $p_{ij}\in(0,1]$ of x_{ij} in the total outgoing flow $\sum_{\tilde{j}\in I_i^+(U)} x_{i\tilde{j}}$ from the node i. Suppose for each node $i\in I$ there exists a *canonical* arc $(i,k)\in U$, where $k\in I_i^+(U)$ and $x_{ik}\neq 0$. Using special programmable devices (sensors) we monitor the nodes $M\subseteq I$ and get the following data:

$$x_{ij}=f_{ij},\ j\in I_i^+(U);\ x_{ji}=f_{ji},\ j\in I_i^-(U);$$
$$x_i=f_i,\ i\in M\cap I^*; \tag{2}$$

$$x_{ij}=\beta_{ij}f_{ik},\ \beta_{ij}=\frac{p_{ij}}{p_{ik}},\ j\in I_i^+(U)\setminus\{k\},\ i\in I_k^-(U),\ k\in M. \tag{3}$$

Note that in the equation (3) we have $\left|I_i^+(U)\setminus\{k\}\right|\geq 1$.

Given the set $M \subseteq I$ of monitored nodes of the graph G we build the graph $\overline{G} = \left(\overline{I}, \overline{U}\right)$ being the unobserved part of G. Let us remove the arcs and nodes with the known flows (2)–(3) from the graph G. Then the system of equations to get the unknown flows x_{ij}, $(i, j) \in \overline{U}$, x_i, $i \in \overline{I}^* = \overline{I} \cap I^*$, has the form

$$\sum_{j \in I_i^+(\overline{U})} x_{ij} - \sum_{j \in I_i^-(\overline{U})} x_{ji} = \begin{cases} a_i, & i \in \overline{I} \setminus \overline{I}^*, \\ x_i + a_i, & i \in \overline{I}^*, \end{cases} \qquad (4)$$

$$x_{ij} = \beta_{ij} f_{ik}, \; \beta_{ij} = \frac{p_{ij}}{p_{ik}}, \; j \in I_i^+(U) \setminus \{k\}, \; \left|I_i^+(U)\right| > 1, \; i \in \overline{I}. \qquad (5)$$

where a_i, $i \in \overline{I}$, are constants obtained from the system (1) using the a priori information (2)–(3) and $(i, k) \in \overline{U}$ are the canonical arcs of the nodes $i \in \overline{I}$.

The sparse system of equations (4)–(5), uniquely determined by the set $M \subseteq I$ of the graph G, connects the unknown arc flows x_{ij}, $(i, j) \in \overline{U}$ and external flows x_i, $i \in \overline{I}^*$, of the unmonitored part $\overline{G}$ of the network and can be: 1) *underdetermined*; 2) *overdetermined*; 3) *exactly determined*. In [5], a constructive theory of decomposition of basis graphs for solving the sparse underdetermined systems (4)–(5) is developed. In cases 1) and 2) one should rebuild the set of monitored nodes M, the unmonitored part $\overline{G} = \left(\overline{I}, \overline{U}\right)$ and the system (4)–(5). The rebuild is finished when the matrix of the system (4)–(5) has rank equal to the number of unknowns. In case 3 the set of monitored nodes M of the graph G is suitable for finding the unknown flows of the unmonitored part $\overline{G} = \left(\overline{I}, \overline{U}\right)$ from the system (4)–(5). The graph $\overline{G} = \left(\overline{I}, \overline{U}\right)$, i.e. the unmonitored part of $G = (I, U)$, may be disconnected. Some connectivity components of graph G may not contain nodes from the set $\overline{I}^*$ with non-zero external flow. The basis graph of such a component is a spanning tree. For those connectivity components that have some nodes from the set $\overline{I}^*$ the basis graph is a forest with the properties described in [4].

Example. Optimal solution. Fig. 1 shows the finite connected biderectional graph $G = (I, U)$, $I = \{1, 2, 3, 4, 5, 6\}$, $U = \{(1,2), (1,3), (2,1), (2,4), (2,6), (3,1), (3,5), (4,2), (4,5), (4,6), (5,3), (5,4), (5,6), (6,2), (6,4), (6,5)\}$, and the set of nodes with non-zero external flow $I^* = \{2, 4, 5, 6\}$.

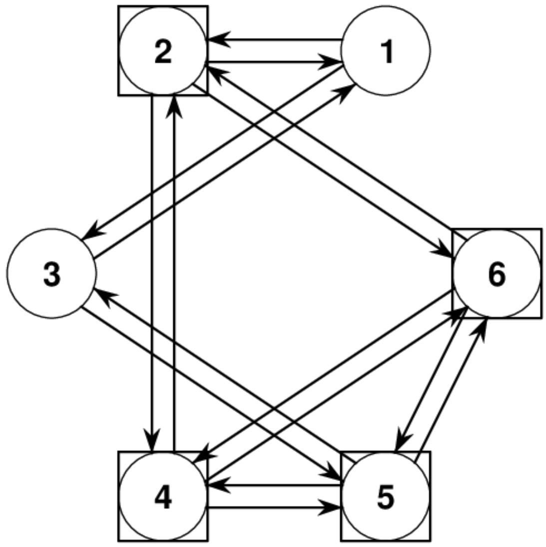

Fig. 1. Initial graph G

For the given graph the system of equations (1) has the form

$$
\begin{aligned}
x_{1,2}+x_{1,3}-x_{2,1}-x_{3,1}&=0,\\
x_{2,1}+x_{2,4}+x_{2,6}-x_{1,2}-x_{4,2}-x_{6,2}&=x_2,\\
x_{3,1}+x_{3,5}-x_{1,3}-x_{5,3}&=0,\\
x_{4,2}+x_{4,5}+x_{4,6}-x_{2,4}-x_{5,4}-x_{6,4}&=x_4,\\
x_{5,3}+x_{5,4}+x_{5,6}-x_{3,5}-x_{4,5}-x_{6,5}&=x_5,\\
x_{6,2}+x_{6,4}+x_{6,5}-x_{2,6}-x_{4,6}-x_{5,6}&=x_6.
\end{aligned}
\tag{6}
$$

Suppose the set of monitored nodes for the graph G shown in Fig. 1 is $M=\{2\}$. Consider the cut of the network G with the source set M (and the sink set $I\setminus M$). Let $CS(M)$ be the cut-set (of arcs) and $I(CS(M))$ be the set of nodes incident to the edges of $CS(M)$. We form the sets $M^{+}=I(CS(M))\setminus M=\{1,4,6\}$, $M^{*}=M\cup M^{+}=\{1,2,4,6\}$, $I\setminus M^{*}=\{3,5\}$.

In sensor location problem (SLP) the flows on every incoming and outgoing arc for each node $i\in M$ (M is the set of monitored nodes) are known as well as the external flows $x_i=f_i$, $i\in M\cap I^{*}$:

$$
\begin{aligned}
&x_{1,2}=f_{1,2},\ x_{2,1}=f_{2,1},\ x_{2,4}=f_{2,4},\\
&x_{4,2}=f_{4,2},\ x_{2,6}=f_{2,6},\ x_{6,2}=f_{6,2},\ x_2=f_2.
\end{aligned}
\tag{7}
$$

We substitute the known values of the variables (7) into the system of equations (6) and remove the arcs $CS(M)$ and the only monitored node 2 from the graph G. The resulting graph G' is shown in Fig. 2. The rest of the flows for the outgoing arcs of the nodes $M^{+}=I(CS(M))\setminus M=\{1,4,6\}$ can be expressed by the following equations:

$$x_{1,3}=\frac{p_{1,3}}{p_{1,2}}f_{1,2},$$

$$
\begin{aligned}
x_{4,5} &= \frac{p_{4,5}}{p_{4,2}} f_{4,2}, \\
x_{4,6} &= \frac{p_{4,6}}{p_{4,2}} f_{4,2}, \\
x_{6,4} &= \frac{p_{6,4}}{p_{6,2}} f_{6,2}, \\
x_{6,5} &= \frac{p_{6,5}}{p_{6,2}} f_{6,2}.
\end{aligned}
\tag{8}
$$

Let us substitute (7) and (8) into the system of equations (6) and remove the arc with known flows (8) from the graph G'. The resulting graph $\overline{G} = \left(\overline{I}, \overline{U}\right)$ is shown in Fig. 3. And the system of equations (6) takes the following form:

$$
\begin{gathered}
f_{1,2} + \frac{p_{1,3}}{p_{1,2}} f_{1,2} - f_{2,1} - x_{3,1} = 0, \\
f_{2,1} + f_{2,4} + f_{2,6} - f_{1,2} - f_{4,2} - f_{6,2} = f_2, \\
x_{3,1} + x_{3,5} - \frac{p_{1,3}}{p_{1,2}} f_{1,2} - x_{5,3} = 0, \\
f_{4,2} + \frac{p_{4,5}}{p_{4,2}} f_{4,2} + \frac{p_{4,6}}{p_{4,2}} f_{4,6} - x_{2,4} - x_{5,4} - \frac{p_{6,4}}{p_{6,2}} f_{6,2} = x_4, \\
x_{5,3} + x_{5,4} + x_{5,6} - x_{3,5} - \frac{p_{4,5}}{p_{4,2}} f_{4,2} - \frac{p_{6,5}}{p_{6,2}} f_{6,5} = x_5, \\
f_{6,2} + \frac{p_{6,4}}{p_{6,2}} f_{6,2} + \frac{p_{6,5}}{p_{6,2}} f_{6,2} - f_{2,6} - \frac{p_{5,6}}{p_{4,2}} f_{4,2} - x_{5,6} = x_6.
\end{gathered}
\tag{9}
$$

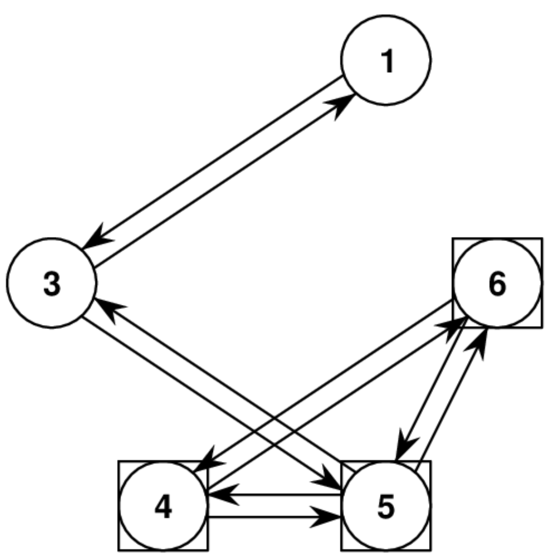

Fig. 2. Graph G'

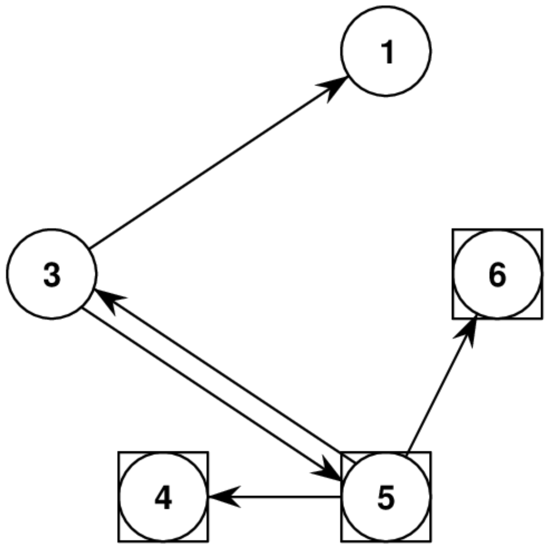

Fig. 3. Graph $\overline{G}=\left(\overline{I},\overline{U}\right)$

The flows $x_{i,j}$, $\left(i,j\right)\in\overline{U}$, on the arcs outgoing from the nodes $I\setminus M^{*}=\left\{3,5\right\}$ are unknown. So we form the additional equations of type (5):

Thus, the additional equations are

$$x_{3,1}=\frac{p_{3,1}}{p_{3,5}}x_{3,5},$$
$$x_{5,3}=\frac{p_{5,3}}{p_{5,6}}x_{5,6}, \qquad (10)$$
$$x_{5,4}=\frac{p_{5,4}}{p_{5,6}}x_{5,6}.$$

Suppose the sensors are ideal, so that the second equation of the system (9) is consistent though degenerate, and rewrite the system (9)–(10) omitting the degenerate equation:

$$x_{3,1}=f_{1,2}+\frac{p_{1,3}}{p_{1,2}}f_{1,2}-f_{2,1},$$
$$x_{3,1}+x_{3,5}-x_{5,3}=\frac{p_{1,3}}{p_{1,2}}f_{1,2},$$
$$x_{2,4}+x_{5,4}+x_{4}=f_{4,2}+\frac{p_{4,5}}{p_{4,2}}f_{4,2}+\frac{p_{4,6}}{p_{4,2}}f_{4,6}-\frac{p_{6,4}}{p_{6,2}}f_{6,2},$$
$$x_{5,3}+x_{5,4}+x_{5,6}-x_{3,5}-x_{5}=\frac{p_{4,5}}{p_{4,2}}f_{4,2}+\frac{p_{6,5}}{p_{6,2}}f_{6,5},$$
$$x_{5,6}+x_{6}=f_{6,2}+\frac{p_{6,4}}{p_{6,2}}f_{6,2}+\frac{p_{6,5}}{p_{6,2}}f_{6,2}-f_{2,6}-\frac{p_{5,6}}{p_{4,2}}f_{4,2}, \qquad (11)$$
$$x_{3,1}-\frac{p_{3,1}}{p_{3,5}}x_{3,5}=0,$$

$$x_{5,3} - \frac{p_{5,3}}{p_{5,6}} x_{5,6} = 0,$$

$$x_{5,4} - \frac{p_{5,4}}{p_{5,6}} x_{5,6} = 0.$$

The number of both equations and unknowns in the system (11) is 8.

We compute the rank of the matrix of the system (11). If the matrix is of full rank, then the system (11) has a unique solution.

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