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О C^*-АЛГЕБРАХ, ПОРОЖДЕННЫХ ИДЕМПОТЕНТАМИ

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Банаховы алгебры, порожденные идемпотентами, начали интересовать специалистов давно. В 1968–1969 гг. П. Р. Халмош и Г. К. Педерсен изучали структуру C^*-алгебр, порожденных двумя самосопряженными проекторами. Банаховы алгебры, порожденные двумя идемпотентами, были описаны С. Рохом и Б. Зильберманом в 1988 г. Такие алгебры могут иметь неприводимые представления первого или второго порядка. Теория банаховых алгебр, порожденных тремя идемпотентами, полностью не разработана. Такие алгебры могут иметь неприводимые представления любого порядка. В 1974 г. Ф. Краусс и Т. Лоусон описали структуру n-однородных C^*-алгебр над сферами S^2, S^3, S^4. С использованием этих результатов в настоящей работе доказывается, что n-однородная ($n > 2$) C^*-алгебра с пространством примитивных идеалов $\mathrm{Prim}\,A \cong S^4$ может быть порождена конечным набором идемпотентов.

Ключевые слова: C^*-алгебра; идемпотент; конечно-порожденная алгебра; число порождающих элементов; примитивный идеал; база расслоения; алгебраическое расслоение; операторная алгебра; неприводимые представления.

ON C^*-ALGEBRAS GENERATED BY IDEMPOTENTS

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Banach algebras generated by two idempotents appear in many places. In 1968–1969 P. R. Halmos and G. K. Pedersen studied C^*-algebras generated by two self-adjoint projections. The Banach algebras generated by two idempotents were described by S. Roch and B. Silbermann in 1988. Such algebras can have irreducible representations of first or second order. The theory of Banach algebras generated by three idempotents has not yet been constructed. Such algebras can have irreducible representations of any order. In 1974 F. Krauss and T. Lawson described the n-homogeneous C^*-algebras over spheres S^2, S^3, S^4. By using these results we prove that n-homogeneous ($n > 2$) C^*-algebra such that $\mathrm{Prim}\,A \cong S^4$ can be generated by finite number of idempotents.

Keywords: C^*-algebra; idempotent; finitely generated algebra; number of generators; primitive ideals; base space; algebraic bundle; operator algebra; irreducible representation.

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Introduction

An element a from an algebra A is idempotent if $a^2 = a$. Moreover, if the Banach algebra A has the operation of involution * and the idempotent a is self-adjoint ($a^* = a$) then the element a is called a projection.

Algebras generated by two idempotents appear in many places. In 1960s P. R. Halmos [1] and G. K. Pedersen [2] studied an C^*-algebra A generated by two projections (p and q) such that the spectrum $\sigma(prp)$ equals to the segment $[0, 1]$. Furthermore, P. R. Halmos supposed that any two sets from $\mathrm{Ker}(p)$, $\mathrm{Im}(p)$, $\mathrm{Ker}(q)$, $\mathrm{Im}(q)$ have the intersection equals to zero.

The Banach algebras generated by two idempotents were studied by S. Roch and B. Silbermann [3]. The approach of the paper uses the PI-algebras that were studied by N. Krupnik. As a result, the next theorem appeared.

Theorem 1 (two projections theorem) [3]. *Let A be a Banach algebra with identity e, and let p and r be idempotents in A. The smallest closed subalgebra of A, which contains p, r and e, will be denoted by B. Then:*

(i) for each $x \in \sigma_B(e - p - r + pr + rp) \setminus \{0, 1\}$*, the mapping* $F_x : e, p, r \to C^{2\times 2}$ *given by*

$$F_x(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ F_x(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ F_x(r) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix},$$

where $\sqrt{x(1-x)}$ *denotes any number with* $\left(\sqrt{x(1-x)}\right)^2 = x(1-x)$*, extends to a continuous algebra homomorphism from B onto* $C^{2\times 2}$*, which we denote also by* F_x*;*

(ii) for each $m \in \sigma_B(p + 2r) \cap \{0, 1, 2, 3\}$*, the mapping* $G_m : \{e, p, r\} \to C$ *given by* $G_0(e) = 1$, $G_0(p) = G_0(r) = 0$, $G_1(e) = G_1(p) = 1$, $G_1(r) = 0$, $G_2(e) = G_2(r) = 1$, $G_2(p) = 0$, $G_3(e) = G_3(p) = G_3(r) = 1$ *extends to a continuous algebra homomorphism from B onto C;*

(iii) an element $a \in B$ *is invertible in B if and only if the matrices* $F_x(a)$ *are invertible for all* $x \in \sigma_B(e - p - r + pr + rp) \setminus \{0, 1\}$ *and if the numbers* $G_m(a)$ *are non-zero for all* $m \in \sigma_B(p + 2r) \cap \{0, 1, 2, 3\}$.

The theory of Banach algebras generated by three idempotents is still not constructed. There are some results showing that Banach algebras generated by three idempotents can have a complicated structure. The next theorem appeared in 1955.

Theorem 2 [4]. *The ring B of all bounded operators on separable Hilbert space can be generated by three idempotents in weak topology.*

In this theorem the therm «generate» means that the smallest weakly-closed self-adjoint algebra A that contains the idempotents p, q, r and constants coincides with B. On the other hand, the next question is interesting: which Banach algebras can be generated by three and more idempotents in uniform topology?

Theorem 3 [5]. *The algebra* $M_n(c)$ *(*$n \geq 3$*) can be generated by three idempotents. The algebra cannot be generated by two idempotents.*

In fact, the Banach algebra A generated by three idempotents can have irreducible representations of any dimension. On the other hand, several algebras that have the same space of irreducible representations can be non-isomorphic. Suppose A is a n-homogeneous C^*-algebra. It means that all irreducible representations for the algebra A are of the order n. If the algebra A is isomorphic to the algebra $M_n(B)$ then the algebra A is called trivial. Here $M_n(B)$ denotes the algebra of all continuous matrix-functions from the set B to the algebra of matrices $C^{n\times n}$. There are also non-trivial n-homogeneous C^*-algebras. It was shown by J. M. G. Fell [6], J. Tomiyama and M. Takesaki [7] in 1961 that every n-homogeneous C^*-algebra is isomorphic to the algebra $\Gamma(E)$ of all continuous sections for the appropriate algebraic bundle (E, B, p). Here the base space B for the bundle is homeomorphic to the space $\mathrm{Prim}A$ of primitive ideals for the n-homogeneous C^*-algebra A.

Let us remind that a triple (E, B, p) is called bundle if the following conditions hold.

(I) E and B are topological spaces.

(II) $p : E \to B$ is a continuous surjection.

The space E is called a bundle space; the space B is said to be the base space. The surjection p is called a projection. The set $F = p^{-1}(x)$ is the fiber over a point $x \in B$. For example, consider the product-bundle $E = B \times F$, where B and F are topological spaces. By p denote the projection $B \times F \to B$ to the first multiplier.

The bundle ξ is said to be the trivial bundle if it is isomorphic to a product-bundle. On the other hand, consider the Mobius tape M. Note that the Mobius tape M is a non-trivial bundle. The circle S^1 is the bundle space. The interval I is the fiber. However M is not isomorphic to the product-bundle $S^1 \times I$. At the same time M is locally trivial. Such bundles cab be called twisted bundles.

A G-bundle $\xi = (E,\ B,\ p)$ is called as the algebraic bundle if the following conditions hold.

(I) The fiber F_x is the algebra $\mathrm{Mat}(n) = C^{n \times n}$ of square matrices of the order n.

(II) The group G is the group $\mathrm{Aut}(n)$ of all automorphisms for the algebra $\mathrm{Mat}(n)$.

Bundles $\xi_1 = (E_1,\ B_1,\ p_1)$ and $\xi_2 = (E_2,\ B_2,\ p_2)$ are said to be isomorphic if there exists a homeomorphism $\gamma : E_1 \to E_2$ such that $\gamma(F_x) = F_{\alpha(x)}$. Here $\alpha : B_1 \to B_2$ is a homeomorphism of the bases, the set $F_{\alpha(x)} = p_2^{-1}(\alpha(x))$ is the fiber over the point $\alpha(x) \in B_2$. The author considered the n-homogeneous C^*-algebras over the sphere S^2.

Theorem 4 [8]. *Let A be a n-homogeneous ($n \geq 2$) C^*-algebra. Suppose the space* Prim*A is homeomorphic to the sphere S^2. In this case, there are three idempotents $p,\ q,\ r \in A$ such that the minimal Banach algebra containing these elements coincides with the algebra A.*

The next theorem extends the result to a n-homogeneous C^*-algebra A such that $\mathrm{Prim}A \cong P_k$. Here P_k denotes an two-dimensional oriented manifold. The manifold P_k can be realised as the sphere S^2 with k handles attached.

Theorem 5 [9]. *Let A be a n-homogeneous ($n \geq 2$) C^*-algebra. We suppose that the space* Prim*A is homeomorphic to the two-dimensional oriented connected manifold P_k. In this case, the algebra A can be generated by three idempotents. The algebra A can not be generated by two idempotents.*

In fact, these results allow to find the minimal number of idempotent generators for several algebras. The next result describes the class of n-homogeneous C^*-algebras that can be generated by finite number of idempotents.

Proposition 1 [10]. *Let A be a finitely generated n-homogeneous ($n \geq 2$) C^*-algebra. We suppose that the algebra A contains at least one idempotent a ($a \neq 0,\ a \neq 1$). In this case, there exist idempotents $p_1,\ \ldots,\ p_k \in A$ such that the algebra A can be generated by these elements $p_1,\ \ldots,\ p_k$.*

Moreover, the class of algebras that can be generated by three idempotents is extremely large. Let us remind that a topological space is separable if it possesses a countable dense subset.

Theorem 6 [11]. *Every separable Banach algebra is isomorphic to a subalgebra of a Banach algebra generated by three idempotents.*

On the other hand, if the Banach algebra A can be generated by N idempotents with some concrete relations between generators then the structure of the algebra A can be described. Let A be a Banach algebra with the identity element e, and let $p_1, p_2, \ldots, p_{2N}$ be a partition of the identity into non-zero idempotents: $p_i p_j = \delta_{ij} p_i$ for all $i,\ j = 1,\ \ldots,\ 2N$. Here δ_{ij} is the Kronecker delta, and $p_1 + p_2 + \ldots + p_{2N} = e$. Further, let P be an idempotent element of A, set $Q = e - P$ and $p_{2N+1} = p_1$. We suppose that the conditions

$$P(p_{2i-1})P = (p_{2i-1} + p_{2i})P \tag{1}$$

and

$$Q(p_{2i} + p_{2i+1})Q = (p_{2i} + p_{2i+1})Q \tag{2}$$

hold for all $i,\ j = 1,\ \ldots,\ N$.

Let us define $p_k = p_l$, where $l \in \overline{1,\ 2N}$ whenever $k - l$ is divisible by $2N$. We suppose that the algebra B is the smallest Banach algebra that contains $p_1,\ \ldots,\ p_{2N}$ as well as P. Further, set

$$X = \sum_{i=1}^{N} \left(p_{2i-1} P p_{2i-1} + p_{2i} P p_{2i} \right) \tag{3}$$

and

$$Y = \sum_{i=1}^{N} \left(p_{2i-1} P + p_{2i} Q \right) + \sum_{i=1}^{2N} (2i-1) p_i. \tag{4}$$

Theorem 7 (N projections theorem) [11]. *Let A be a Banach algebra with the identity e, and suppose $p_1,\ \ldots,\ p_{2N}$ and P are non-zero elements of A satisfying (1)–(4). Further, let B stand for the smallest closed subalgebra of A containing the elements P and $p_1,\ \ldots,\ p_{2N}$. Then the following assertions hold.*

(I) If $x \in \sigma_B(X) \backslash \{0,\ 1\}$, then the mapping $F_x : P,\ p_1,\ \ldots,\ p_{2N} \to C^{2N \times 2N}$ given by $F_x(p_i) = \mathrm{diag}(0,\ \ldots,\ 0,\ 1,\ 0,\ \ldots,\ 0)$ with one standing at the i^{th} place and

$$F_x(p_i) = \operatorname{diag}(1, -1, 1, -1, \dots, 1, -1)\begin{pmatrix} x & x-1 & x-1 & x-1 & \cdots & x-1 & x-1 \\ x & x-1 & x-1 & x-1 & \cdots & x-1 & x-1 \\ x & x & x & x-1 & \cdots & x-1 & x-1 \\ x & x & x & x-1 & \cdots & x-1 & x-1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x & x & x & x & \cdots & x & x-1 \\ x & x & x & x & \cdots & x & x-1 \end{pmatrix}$$

extends to a continuous algebra homomorphism from B onto $C^{2N\times 2N}$.

(II) If $m \in \sigma_B(Y) \cap \{1, \dots, 4N\}$, *then the mapping* $G_m : P,\ p_1, \dots, p_{2N} \to C$ *defined by*

$$G_{4k}(p_i) = \begin{cases} 1, & i = 2k, \\ 0, & i \neq 2k, \end{cases} \quad G_{4k}(P) = 0,$$

$$G_{4k-1}(p_i) = \begin{cases} 1, & i = 2k, \\ 0, & i \neq 2k, \end{cases} \quad G_{4k-1}(P) = 0,$$

$$G_{4k-2}(p_i) = \begin{cases} 1, & i = 2k-1, \\ 0, & i \neq 2k-1, \end{cases} \quad G_{4k-2}(P) = 0,$$

$$G_{4k-3}(p_i) = \begin{cases} 1, & i = 2k-1, \\ 0, & i \neq 2k-1, \end{cases} \quad G_{4k-3}(P) = 0,$$

where $k = 1, \dots, N$, *extends to a continuous algebra homomorphism from B onto C.*

(III) An element $b \in B$ *is invertible in B if and only if the matrices* $F_x(b)$ *are invertible for all* $x \in \sigma_B(X)\setminus\{0, 1\}$ *and if the numbers* $G_m(b)$ *are non-zero for all* $m \in \sigma_B(Y) \cap \{1, \dots, 4N\}$.

(IV) An element $b \in B$ *is invertible in A if and only if the matrices* $F_x(b)$ *are invertible for all* $x \in \sigma_A(X)\setminus\{0, 1\}$ *and if the numbers* $G_m(b)$ *are non-zero for all* $m \in \sigma_A(Y) \cap \{1, \dots, 4N\}$.

The main results

Let $\coprod$ denote the coproduct of two spaces. Syppose e_+^4 and e_-^4 be the upper and lower half-sphere for the sphere S^4.

Proposition 2 [12]. *Let* $n \geq 2$ *denote the degree of homogeneity for the algebra A, let p be any integer, and let* $\operatorname{Prim} A \cong S^4$. *The algebra A is* C^**-isomorphic to one of the* C^**-algebras* $A_{n,p} = g_p\left(e_+^4 \coprod e_-^4,\ M_n(C)\right)$ *given as follows: if* $f \in A_{n,p}$ *and* $(z, w) \in S^3$, *then*

$$f_+(z, w) = g_p(z, w)\cdot f_-(z, w) = \begin{pmatrix} z & -\overline{w} & \dots & 0 \\ w & \overline{z} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}^p \cdot f_-(z, w)\begin{pmatrix} \overline{z} & \overline{w} & \dots & 0 \\ -w & z & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}^p.$$

Theorem 8. *Suppose A is the n-homogeneous (n > 2)* C^**-algebra such that* $\operatorname{Prim} A \cong S^4$. *In this case, the algebra A can be generated by finite number of idempotents.*

P r o o f. We will construct an idempotent $p \in A$ using proposition 2. Let $E_{n,m}$ be a $n \times n$ matrix of the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

The matrix $E_{n,m}$ contains one one and $n^2 - 1$ zeros. The number 1 is on the intersection of n^{th} line and n^{th} row. Let the element p equal $E_{n,m}$ on the lower half of S^4. One can see that the element $E_{n,m}$ does not change under the transformation described in proposition 2. Therefore, p can be easily extended to the upper half-sphere. We can define the element p easily to be $E_{n,m}$ for all points x on the upper half-sphere. Therefore, the algebra A contains at least one non-trivial idempotent p ($p \neq 0, p \neq 1$). Using proposition 1, we can construct the idempotents $p_1, \ldots, p_l$ such that the minimal Banach algebra containing $p_1, \ldots, p_l$ coincides with A. The proof is finished.

Conclusions

Finally, the theory of Banach algebras generated by two idempotents is well known now. The theory of Banach algebras generated by three idempotents is still not formulated. One can see that complicated Banach algebras can be generated by three and more idempotents.

It is still an open question: can 2-homogeneous C^*-algebra ($\mathrm{Prim} A \cong S^4$) be generated by finite number of idempotents?

Several authors considered other questions regarding operators and idempotents. For example, the next proposition exists.

Proposition 3 [13]. *Let H be a separable Hilbert space (finite- or countable-dimensional), let $L(H)$ be the space of bounded operators on H, and let I be the identity operator on H. In this case, for any $\lambda \in C$ and for any operator $B \in L(H)$ there are idempotents $P_1, P_2, P_3, P_4, P_5 \in L(H)$ such that $P_1 + P_2 + P_3 + P_4 + P_5 = \lambda I$.*

Also, in work [14] the numbers $\lambda \in C$ such that $P_1 + P_2 = \lambda I$, $P_1 + P_2 + P_3 = \lambda I$, $P_1 + P_2 + P_3 + P_4 = \lambda I$ were described.

Works [15; 16] contain results for the two projections theory.

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