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Hypersingular Integro-Differential Equation Containing Polynomials and Their Derivatives in Coefficients

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Abstract: A new linear integro-differential equation is considered on a closed curve located on the complex plane. The integrals in the equation are understood in the sense of a finite part, according to Hadamard. The order of the equation can be higher than one. The coefficients of the equation have a special structure. A characteristic of the coefficients is that they contain two arbitrary polynomials and their derivatives. Solvability conditions are explicitly stated. Whenever they are satisfied, an exact analytical solution is given. Generalized Sokhotsky formulas, the theory of Riemann boundary value problems, methods for solving linear differential equations, and the properties of analytic functions of a complex variable are used. An example is given.

Keywords: integro-differential equation; hypersingular integral; Riemann boundary problem; linear differential equation; analytic function

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1. Introduction

Hypersingular integrals in the sense of a finite part, according to Hadamard [1], have a rather wide range of applications in physical and engineering science (e.g., [2–4] and others). Equations with such integrals are interesting from a purely mathematical point of view and could be significant for further applications. These equations are solved mainly numerically on the basis of approximate approaches. A good overview of the relevant results is provided in this long article [5], which specifically states that "there are currently a few disparate results only which are dedicated to analytical methods for solving hypersingular integral equations" [5] (p. 248). It is mostly related to hypersingular integro-differential equations. Approximate methods for solving hypersingular integro-differential equations can be found in [6,7], and exact analytical methods are at the initial stage of investigation.

Let L be a simple smooth closed curve on the complex plane, D_+ and D_- are the interior and exterior domains of the complex plane with the boundary L, respectively, $0 \in D_+$, $\infty \in D_-$. We choose the orientation on the curve L, which leaves the domain D_+ left. Henceforth, the function $\varphi(t)$, $t \in L$, will be desired. It will require the presence of H-continuous (i.e., satisfying Hölder condition) derivatives up to some finite order n from it. Thereafter, the hypersingular integral $\int\limits_L \frac{\varphi(\tau)d\tau}{(\tau-t)^{n+1}}$ will be understood in the sense of Hadamard finite part. In respect of such integrals, the following formula is valid

$$\int\limits_L \frac{\varphi(\tau)d\tau}{(\tau-t)^{n+1}} = \frac{\pi i}{n!}\varphi^{(n)}(t) + \int\limits_L \frac{\varphi(\tau) - \sum\limits_{k=0}^{n} \frac{\varphi^{(k)}(t)}{k!}(\tau-t)^k}{(\tau-t)^{n+1}}\,d\tau, \tag{1}$$

justified in [8]. The integral on the right-hand side of (1) exists as improper in the usual sense. It should be noted that the singular integral is a special case of hypersingular one because (1) is also valid when $n = 0$.

For analytic functions

$$\Phi_{\pm}(z) = \frac{1}{2\pi i}\int\limits_L \frac{\varphi(\tau)d\tau}{\tau - z},\ \ z \in D_{\pm}, \tag{2}$$

in [8], there are proven generalized Sokhotski formulae. These formulas have the following form:

$$\Phi_{\pm}^{(k)}(t) = \pm\frac{1}{2}\varphi^{(k)}(t) + \frac{k!}{2\pi i}\int\limits_L \frac{\varphi(\tau)d\tau}{(\tau - t)^{k+1}},\ \ k = \overline{0,n}\ ,\ \ t \in L, \tag{3}$$

and evaluate the L curve values of the corresponding derivatives $\Phi_{\pm}^{(k)}(z)$. Formula (3) applied to hypersingular integrals allows us to investigate the hypersingular integro-differential equation

$$\sum_{k=0}^{n}\left(a_k\varphi^{(k)}(t) + \frac{b_k k!}{\pi i}\int\limits_L \frac{\varphi(\tau)d\tau}{(\tau - t)^{k+1}}\right) = f(t)\,,\ \ t \in L, \tag{4}$$

along with more general equations. In [9], for the case of constant coefficients $a_k,\ b_k$, the analytical solution of Equation (4) is provided. The author is tasked with finding and investigating cases of variable coefficients in Equation (4) with arbitrary finite order n when it remains the possibility of an exact analytical solution. This study is devoted to one of such cases.

2. Problem Defenition and the General Solution Scheme

Assume that $a(t) \neq 0$, $b(t) \neq 0$, and $f(t)$ are H-continuous functions, $t \in L$, and $P(z) = b_m z^m + b_{m-1}z^{m-1} + \ldots + b_0$, $Q(z) = d_l z^l + d_{l-1}z^{l-1} + \ldots + d_0$ are polynomials with complex coefficients, where $m \geq 1,\ \ l \geq 1,\ \ b_m \neq 0,\ \ d_l \neq 0$. Also, let us have the complex numbers $\alpha,\ \ \beta,\ \ A_k,\ \ B_k,\ \ k = \overline{0, n+1},\ \ n \geq 2$, such that $A_0 = B_0 = A_{n+1} = B_{n+1} = 0$, $A_n = B_n = 1$. We look for a solution of the equation

$$\sum_{k=0}^{n}\Bigg[(a(t)((A_k - \alpha A_{k+1})P(t) - A_{k+1}P'(t)) + b(t)((B_k - \beta B_{k+1})Q(t) - B_{k+1}Q'(t)))\varphi^{(k)}(t) + \\ + \frac{(a(t)((A_k - \alpha A_{k+1})P(t) - A_{k+1}P'(t)) - b(t)((B_k - \beta B_{k+1})Q(t) - B_{k+1}Q'(t)))k!}{\pi i}\int\limits_L \frac{\varphi(\tau)d\tau}{(\tau - t)^{k+1}}\Bigg] = \\ = f(t),\ t \in L. \tag{5}$$

In case $P(z) = Q(z) = z$ Equation (5) has been solved in [10].

We denote $G(\lambda) = \sum\limits_{k=0}^{n-1} A_{k+1}\lambda^k$, $\quad H(\mu) = \sum\limits_{k=0}^{n-1} B_{k+1}\mu^k$. Assume that the roots $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ of equation $G(\lambda) = 0$ and the roots $\mu_1, \mu_2, \ldots, \mu_{n-1}$ of equation $H(\mu) = 0$ are simple, such that $G(\alpha) \neq 0,\ \ H(\beta) \neq 0$. Let us take the integral of Cauchy type (2) and substitute into Equation (5) the values

$$\varphi^{(k)}(t) = \Phi_{+}^{(k)}(t) - \Phi_{-}^{(k)}(t), \tag{6}$$

$$\frac{k!}{\pi i}\int\limits_L \frac{\varphi(\tau)d\tau}{(\tau - t)^{k+1}} = \Phi_{+}^{(k)}(t) + \Phi_{-}^{(k)}(t),\ \ k = \overline{0,n}, \tag{7}$$

that follow from Formula (3). As a result, for analytic functions

$$F_{+}(z) = \sum_{k=0}^{n}\left((A_k - \alpha A_{k+1})P(z) - A_{k+1}P'(z)\right)\Phi_{+}^{(k)}(z),\ z \in D_{+}, \tag{8}$$

$$F_-(z) = \sum_{k=0}^{n} \big((B_k - \beta B_{k+1})Q(z) - B_{k+1}Q'(z)\big)\Phi_-^{(k)}(z),\ z \in D_-, \quad (9)$$

with H-continuous limits $F_\pm(t),\ t \in L$, we obtain the Riemann boundary value problem

$$F_+(t) = \frac{b(t)}{a(t)}F_-(t) + \frac{f(t)}{2a(t)},\ t \in L. \quad (10)$$

Problem (10) should be solved in the class of functions that admit at infinity a pole of order not greater than $l-1$ (if $l=1$ the "pole" of zero means boundness). The behavior of $F_-(z)$ at infinity is seen from Formula (9) based on the fact that the Cauchy-type integral at infinity has a zero, generally speaking, of the first order.

If Riemann's problem (10) has no solution, then Equation (5) has no solution, too. If the Riemann problem turns out to be solvable, and its solution is found, then relations (8) and (9) should be considered as linear differential equations with respect to analytic functions $\Phi_\pm(z)$, with $\Phi_-(\infty) = 0$. If functions $\Phi_\pm(z)$ are found, then the solution to Equation (5) is obtained using Formula (6) with $k = 0$.

The theory of the Riemann problem is presented in [11]. Let $X_\pm(z)$, $z \in D_\pm \cup L$, be canonical functions of this problem. $X_\pm(z)$ have been written explicitly in [11], expressions for which are not given here. Let us denote $\gamma = \mathrm{Ind}_L \frac{b(t)}{a(t)}$. If $\gamma \geq -l$, then the problem (10) is unconditionally solvable, but if $\gamma < -l$, for its solvability, it is necessary and sufficient to satisfy the following conditions:

$$\int_L \frac{f(\tau)\tau^k d\tau}{a(\tau)X_+(\tau)} = 0,\ \ k = \overline{0, -\gamma - l - 1}. \quad (11)$$

If the Riemann problem is solvable, then its solution is given by formula

$$F_\pm(z) = X_\pm(z)(\Psi_\pm(z) + R(z)),\ z \in D_\pm,$$

where $\Psi_\pm(z) = \frac{1}{4\pi i}\int\limits_L \frac{f(\tau)d\tau}{a(\tau)X_+(\tau)(\tau - z)}$, $R(z)$ is a polynomial of degree $\gamma + l - 1$ with arbitrary complex coefficients if $\gamma + l - 1 \geq 0$, $R(z) \equiv 0$ if $\gamma + l - 1 < 0$.

Assuming that problem (10) is solvable, and its solution has been found, we proceed to solving Equations (8) and (9).

3. Solution of the Differential Equation for the Interior Domain

Let us write the homogeneous Equation (8) $(F_+(z) \equiv 0)$ in the form

$$P(z)\sum_{k=0}^{n-1} A_{k+1}\Phi_+^{(k+1)}(z) = \big(\alpha P(z) + P'(z)\big)\sum_{k=0}^{n-1} A_{k+1}\Phi_+^{(k)}(z). \quad (12)$$

Then, its solution reduces to the sequential solving of equations

$$P(z)Y'(z) = \big(\alpha P(z) + P'(z)\big)Y(z), \quad (13)$$

$$\sum_{k=0}^{n-1} A_{k+1}\Phi_+^{(k)}(z) = Y(z). \quad (14)$$

The fundamental system of solutions to Equation (13) consists of one function $e^{\alpha z}P(z)$. The fundamental system of solutions to Equation (12) consists of a fundamental system of solutions $e^{\lambda_1 z}, e^{\lambda_2 z}, \ldots, e^{\lambda_{n-1} z}$ to the homogeneous Equation (14) and a partical solution to the equation

$$\sum_{k=0}^{n-1} A_{k+1}\Phi_+^{(k)}(z) = e^{\alpha z}P(z).$$

We find this partical solution using the classical method of undetermined coefficients, according to which it has the form $e^{\alpha z}\widetilde{P}(z)$, where $\widetilde{P}(z) = \widetilde{b}_m z^m + \widetilde{b}_{m-1}z^{m-1} + \ldots + \widetilde{b}_0$. The coefficients $\widetilde{b}_k,\ \ k = \overline{0,m}$, which are initially undetermined, are found as a (unique) solution to a system of linear algebraic equations

$$\begin{cases} G(\alpha)C_m^m\widetilde{b}_m = b_m, \\ G'(\alpha)C_m^{m-1}\widetilde{b}_m + G(\alpha)C_{m-1}^{m-1}\widetilde{b}_{m-1} = b_{m-1}, \\ G''(\alpha)C_m^{m-2}\widetilde{b}_m + G'(\alpha)C_{m-1}^{m-2}\widetilde{b}_{m-1} + G(\alpha)C_{m-2}^{m-2}\widetilde{b}_m = b_{m-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ G^{(m)}(\alpha)C_m^0\widetilde{b}_m + G^{(m-1)}(\alpha)C_{m-1}^0\widetilde{b}_{m-1} + G^{(m-2)}(\alpha)C_{m-2}^0\widetilde{b}_{m-2} + \ldots + G(\alpha)C_0^0\widetilde{b}_0 = b_0. \end{cases} \tag{15}$$

Here and below, notation C_p^q is used for binomial coefficients $C_p^q = \frac{p!}{q!(p-q)!}$.

We denote this using V, Vandermonde's determinant of numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \alpha$. Under our assumptions, this is a constant. Using derivatives of this determinant, we will further understand its derivatives with respect to variable α. Then, for $k = 0, 1, 2, \ldots$ we obtain

$$V^{(k)} = \begin{vmatrix} 1 & \ldots & 1 & \left(\alpha^0\right)^{(k)} \\ \lambda_1 & \ldots & \lambda_{n-1} & (\alpha)^{(k)} \\ \lambda_1^2 & \ldots & \lambda_{n-1}^2 & \left(\alpha^2\right)^{(k)} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1^{n-1} & \ldots & \lambda_{n-1}^{n-1} & \left(\alpha^{n-1}\right)^{(k)} \end{vmatrix}. \tag{16}$$

Since $V = \prod\limits_{1\le j<s\le n-1}\left(\lambda_s - \lambda_j\right)\prod\limits_{j=1}^{n-1}\left(\alpha - \lambda_j\right)$, and $\prod\limits_{j=1}^{n-1}\left(\alpha - \lambda_j\right) = G(\alpha)$, then $V^{(k)} = \prod\limits_{1\le j<s\le n-1}\left(\lambda_s - \lambda_j\right)\ G^{(k)}(\alpha)$. Taking $\prod\limits_{1\le j<s\le n-1}\left(\lambda_s - \lambda_j\right) = \frac{V}{G(\alpha)}$, we obtain together with (16)

$$V^{(k)} = \frac{VG^{(k)}(\alpha)}{G(\alpha)}. \tag{17}$$

Let $W(z)$ be the Wronskian of functions $e^{\lambda_1 z}, e^{\lambda_2 z}, \ldots, e^{\lambda_{n-1} z}, e^{\alpha z}\widetilde{P}(z)$, forming a fundamental system of solutions to the homogeneous Equation (8). To understand the properties of this determinant, which are necessary for further reasoning, let us express it through the initial data.

Lemma 1. *The following formula for $W(z)$ is valid*

$$W(z) = \frac{e^{(\lambda_1+\lambda_2+\ldots+\lambda_{n-1}+\alpha)z}\ V\,P(z)}{G(\alpha)}. \tag{18}$$

Proof. At first, we can easily obtain

$$W(z) = e^{(\lambda_1+\lambda_2+\ldots+\lambda_{n-1}+\alpha)z}W^*(z),$$

where

$$W^*(z) = \begin{vmatrix} 1 & \dots & 1 & \sum\limits_{j=0}^{0} \alpha^{0-j} C_0^j \widetilde{P}^{(j)}(z) \\ \lambda_1 & \dots & \lambda_{n-1} & \sum\limits_{j=0}^{1} \alpha^{1-j} C_1^j \widetilde{P}^{(j)}(z) \\ \lambda_1^2 & \dots & \lambda_{n-1}^2 & \sum\limits_{j=0}^{2} \alpha^{2-j} C_2^j \widetilde{P}^{(j)}(z) \\ \dots & \dots & \dots & \dots \\ \lambda_1^{n-1} & \dots & \lambda_{n-1}^{n-1} & \sum\limits_{j=0}^{n-1} \alpha^{n-1-j} C_{n-1}^j \widetilde{P}^{(j)}(z) \end{vmatrix}.$$

If we try to express the polynomial $\widetilde{P}(z)$ in terms of a polynomial $P(z)$, then it will be difficult to obtain this due to the very cumbersome formulas of solutions to the system in (15). Let us proceed differently. In the sums of the last column of the determinant $W^*(z)$, we regroup the similar terms and re-order the terms in descending order of powers of z. Then, we obtain

$$\begin{pmatrix} \sum\limits_{j=0}^{0} \alpha^{0-j} C_0^j \widetilde{P}^{(j)}(z) \\ \sum\limits_{j=0}^{1} \alpha^{1-j} C_1^j \widetilde{P}^{(j)}(z) \\ \sum\limits_{j=0}^{2} \alpha^{2-j} C_2^j \widetilde{P}^{(j)}(z) \\ \dots\dots\dots\dots\dots\dots \\ \sum\limits_{j=0}^{n-1} \alpha^{n-1-j} C_{n-1}^j \widetilde{P}^{(j)}(z) \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \\ \dots \\ \alpha^{n-1} \end{pmatrix} \widetilde{b}_m z^m + \left(\begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \\ \dots \\ \alpha^{n-1} \end{pmatrix} \widetilde{b}_{m-1} + \begin{pmatrix} 0 \\ 1 \\ 2\alpha \\ \dots \\ (n-1)\alpha^{n-2} \end{pmatrix} m\widetilde{b}_m \right) z^{m-1} +$$

$$+ \left(\begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \\ \dots \\ \alpha^{n-1} \end{pmatrix} \widetilde{b}_{m-2} + \begin{pmatrix} 0 \\ 1 \\ 2\alpha \\ \dots \\ (n-1)\alpha^{n-2} \end{pmatrix} (m-1)\widetilde{b}_{m-1} + \begin{pmatrix} 0 \\ 0 \\ 2\cdot 1 \\ \dots \\ (n-1)(n-2)\alpha^{n-3} \end{pmatrix} \frac{m(m-1)}{2} \widetilde{b}_m \right) z^{m-2} + \dots +$$

$$+ \left(\begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \\ \dots \\ \alpha^{n-1} \end{pmatrix} \widetilde{b}_0 + \begin{pmatrix} 0 \\ 1 \\ 2\alpha \\ \dots \\ (n-1)\alpha^{n-2} \end{pmatrix} \widetilde{b}_1 + \begin{pmatrix} 0 \\ 0 \\ 2\cdot 1 \\ \dots \\ (n-1)(n-2)\alpha^{n-3} \end{pmatrix} \widetilde{b}_2 + \dots + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ (n-1)(n-2)\cdot\ldots\cdot(n-m)\alpha^{n-m-1} \end{pmatrix} \widetilde{b}_m \right) z^0$$

(if $m \geq n$, then $(n-1)(n-2)\cdot\ldots\cdot(n-m)\alpha^{n-m-1}$ and similar products are equal to zero). Now, let us represent the determinant $W^*(z)$ as a sum of determinants corresponding to the resulting sum for its last column. Then, we will make further calculations using representations (16) and (17) and also replace the arising numbers 1, m, $m-1$, $\frac{m(m-1)}{2}$ etc., with the corresponding binomial coefficients. As a result, we obtain

$$W^*(z)=\begin{vmatrix}1&\dots&1&1\\ \lambda_1&\dots&\lambda_{n-1}&\alpha\\ \lambda_1^2&\dots&\lambda_{n-1}^2&\alpha^2\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&\alpha^{n-1}\end{vmatrix}C_m^m\widetilde{b}_m z^m+$$

$$+\left(\begin{vmatrix}1&\dots&1&1\\ \lambda_1&\dots&\lambda_{n-1}&\alpha\\ \lambda_1^2&\dots&\lambda_{n-1}^2&\alpha^2\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&\alpha^{n-1}\end{vmatrix}C_{m-1}^{m-1}\widetilde{b}_{m-1}+\begin{vmatrix}1&\dots&1&0\\ \lambda_1&\dots&\lambda_{n-1}&1\\ \lambda_1^2&\dots&\lambda_{n-1}^2&2\alpha\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&(n-1)\alpha^{n-2}\end{vmatrix}C_m^{m-1}\widetilde{b}_m\right)z^{m-1}+$$

$$+\left(\begin{vmatrix}1&\dots&1&1\\ \lambda_1&\dots&\lambda_{n-1}&\alpha\\ \lambda_1^2&\dots&\lambda_{n-1}^2&\alpha^2\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&\alpha^{n-1}\end{vmatrix}C_{m-2}^{m-2}\widetilde{b}_{m-2}+\begin{vmatrix}1&\dots&1&0\\ \lambda_1&\dots&\lambda_{n-1}&1\\ \lambda_1^2&\dots&\lambda_{n-1}^2&2\alpha\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&(n-1)\alpha^{n-2}\end{vmatrix}C_{m-1}^{m-2}\widetilde{b}_{m-1}+\right.$$

$$\left.+\begin{vmatrix}1&\dots&1&0\\ \lambda_1&\dots&\lambda_{n-1}&0\\ \lambda_1^2&\dots&\lambda_{n-1}^2&2\cdot 1\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&(n-1)(n-2)\alpha^{n-3}\end{vmatrix}C_m^{m-2}\widetilde{b}_m\right)z^{m-2}+\dots+\left(\begin{vmatrix}1&\dots&1&1\\ \lambda_1&\dots&\lambda_{n-1}&\alpha\\ \lambda_1^2&\dots&\lambda_{n-1}^2&\alpha^2\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&\alpha^{n-1}\end{vmatrix}C_0^0\widetilde{b}_0+\right.$$

$$+\begin{vmatrix}1&\dots&1&0\\ \lambda_1&\dots&\lambda_{n-1}&1\\ \lambda_1^2&\dots&\lambda_{n-1}^2&2\alpha\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&(n-1)\alpha^{n-2}\end{vmatrix}C_1^0\widetilde{b}_1+\begin{vmatrix}1&\dots&1&0\\ \lambda_1&\dots&\lambda_{n-1}&0\\ \lambda_1^2&\dots&\lambda_{n-1}^2&2\cdot 1\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&(n-1)(n-2)\alpha^{n-3}\end{vmatrix}C_2^0\widetilde{b}_2+\dots+$$

$$\left.+\begin{vmatrix}1&\dots&1&0\\ \lambda_1&\dots&\lambda_{n-1}&0\\ \lambda_1^2&\dots&\lambda_{n-1}^2&0\\ \dots&\dots&\dots&\dots\\ \lambda_1^{n-1}&\dots&\lambda_{n-1}^{n-1}&(n-1)(n-2)\cdot\ldots\cdot(n-m)\alpha^{n-m-1}\end{vmatrix}C_m^0\widetilde{b}_m\right)z^0=$$

$$=VC_m^m\widetilde{b}_m z^m+\left(VC_{m-1}^{m-1}\widetilde{b}_{m-1}+V'C_m^{m-1}\widetilde{b}_m\right)z^{m-1}+\left(VC_{m-2}^{m-2}\widetilde{b}_{m-2}+V'C_{m-1}^{m-2}\widetilde{b}_{m-1}+V''C_m^{m-2}\widetilde{b}_m\right)z^{m-2}+$$

$$+\dots+\left(VC_0^0\widetilde{b}_0+V'C_1^0\widetilde{b}_1+V''C_2^0\widetilde{b}_2+\dots+V^{(m)}C_m^0\widetilde{b}_m\right)z^0=$$

$$=\frac{V}{G(\alpha)}\Big(G(\alpha)C_m^m\widetilde{b}_m z^m+\left(G(\alpha)C_{m-1}^{m-1}\widetilde{b}_{m-1}+G'(\alpha)C_m^{m-1}\widetilde{b}_m\right)z^{m-1}+$$

$$+\left(G(\alpha)C_{m-2}^{m-2}\widetilde{b}_{m-2}+G'(\alpha)C_{m-1}^{m-2}\widetilde{b}_{m-1}+G''(\alpha)C_m^{m-2}\widetilde{b}_m\right)z^{m-2}+\dots+$$

$$+\left(G(\alpha)C_0^0\widetilde{b}_0+G'(\alpha)C_1^0\widetilde{b}_1+G''(\alpha)C_2^0\widetilde{b}_2+\dots+G^{(m)}(\alpha)C_m^0\widetilde{b}_m\right)z^0\Big).$$

Finally, due to (15)

$$W^*(z)=\frac{V}{G(\alpha)}\left(b_m z^m+b_{m-1}z^{m-1}+b_{m-2}z^{m-2}+\dots+b_0\right)=\frac{VP(z)}{G(\alpha)},$$

which completes the proof of the lemma. □

According to the method of variation of arbitrary constants (e.g., [12]), the solution of Equation (8) is written according to the following formula:

$$\Phi_+(z)=\sum_{j=1}^{n-1}e^{\lambda_j z}\left(C_j^+ +(-1)^{n+j}\int_{z_0^+}^{z}\frac{F_+(\zeta)W_j(\zeta)d\zeta}{((A_n-\alpha A_{n+1})P(\zeta)-A_{n+1}P'(\zeta))W(\zeta)}\right)+$$
$$+e^{\alpha z}\widetilde{P}(z)\left(C_n^+ +\int_{z_0^+}^{z}\frac{F_+(\zeta)W_n(\zeta)d\zeta}{((A_n-\alpha A_{n+1})P(\zeta)-A_{n+1}P'(\zeta))W(\zeta)}\right), \quad (19)$$

where $z_0^+ \in D_+$, C_j^+ are arbitrary complex constants, $W_j(\zeta)$ are the minors of the Wronskian $W(\zeta)$, obtained by removing the last row and j-th column from it, $j = \overline{1,n}$. Taking into account Formula (18), as well as the values of A_n, A_{n+1}, Formula (19) can be present in the form:

$$\Phi_+(z) = \sum_{j=1}^{n-1} e^{\lambda_j z}\left(C_j^+ + \frac{(-1)^{n+j}G(\alpha)}{V}\int_{z_0^+}^{z}\frac{e^{-\lambda_j\zeta}F_+(\zeta)W_j^*(\zeta)d\zeta}{P^2(\zeta)}\right) + \\ + e^{\alpha z}\widetilde{P}(z)\left(C_n^+ + \frac{G(\alpha)}{V}\int_{z_0^+}^{z}\frac{e^{-\alpha\zeta}F_+(\zeta)W_n^*(\zeta)d\zeta}{P^2(\zeta)}\right), \tag{20}$$

where $W_j^*(\zeta)$, $j = \overline{1,n}$, are minors of the determinant $W^*(\zeta)$ obtained from it by removing the last row and the j-th column.

Generally speaking, Formula (20) does not give a single-valued analytical function in the domain D_+. From now on, we assume that the polynomial $P(z)$ has simple zeros in the domain D_+ at the points $z_k \neq z_0^+$, $k = \overline{1,n_+}$, and the remaining zeros of this polynomial lie in the domain D_- and can have arbitrary multiplicity. For a function $\Phi_+(z)$ to be single-valued, the following conditions are necessary and sufficient:

$$\operatorname*{res}_{z=z_k}\frac{e^{-\lambda_j z}F_+(z)W_j^*(z)}{P^2(z)} = 0,\ \ j = \overline{1,n-1},\ \ k = \overline{1,n_+}, \tag{21}$$

$$\operatorname*{res}_{z=z_k}\frac{e^{-\alpha z}F_+(z)W_n^*(z)}{P^2(z)} = 0,\ \ k = \overline{1,n_+}, \tag{22}$$

which we further consider to be satisfied. Let us also clarify that the integration in Formulas (19) and (20), taking into account conditions (21) and (22), is carried out along any curves connecting the points z_0^+ and z and not passing through the points z_k in the domain D_+.

Each of the integrals in Formula (20) generally gives poles of the first order at the points z_k. Let us justify that Formula (20) nevertheless leads to an analytical function at these points due to the fact that when added, the poles in it cancel each other out. Let us denote the non-zero constants s_k from the representation $P^2(z) = s_k(z - z_k)^2$, $k = \overline{1,n_+}$. Then, the main parts of the Laurent series of the function $\Phi_+(z)$ in the neighborhood of the points z_k are equal to

$$-\frac{G(\alpha)F_+(z_k)}{V\,s_k}\left(\sum_{j=1}^{n-1}(-1)^{n+j}W_j^*(z_k) + \widetilde{P}(z_k)W_n^*(z_k)\right)\frac{1}{z - z_k},\ \ k = \overline{1,n_+}. \tag{23}$$

Sums

$$\sum_{j=1}^{n-1}(-1)^{n+j}W_j^*(z_k) + \widetilde{P}(z_k)W_n^*(z_k)$$

can be expressed as the results of expansions over the elements of the last row of determinants

$$\begin{vmatrix} 1 & \dots & 1 & \sum_{j=0}^{0}\alpha^{0-j}C_0^j\widetilde{P}^{(j)}(z_k) \\ \lambda_1 & \dots & \lambda_{n-1} & \sum_{j=0}^{1}\alpha^{1-j}C_1^j\widetilde{P}^{(j)}(z_k) \\ \lambda_1^2 & \dots & \lambda_{n-1}^2 & \sum_{j=0}^{2}\alpha^{2-j}C_2^j\widetilde{P}^{(j)}(z_k) \\ \dots & \dots & \dots & \dots \\ \lambda_1^{n-2} & \dots & \lambda_{n-1}^{n-2} & \sum_{j=0}^{n-2}\alpha^{n-2-j}C_{n-2}^j\widetilde{P}^{(j)}(z_k) \\ 1 & \dots & 1 & \widetilde{P}(z_k) \end{vmatrix}.$$

But such determinants are equal to zero since each of them has the same first and last rows, and along with them, the main parts of (23) are also equal to zero. So, if conditions (21) and (22) are satisfied, Formula (20) gives an analytic solution to Equation (8) in the domain D_+.

4. Solution of the Differential Equation for the Exterior Domain

The formula for the solution of Equation (9), to which the method of varying arbitrary constants leads, is similar to Formula (20) and has the form

$$\begin{aligned}\Phi_-(z) = \sum_{j=1}^{n-1} e^{\mu_j z}\left(C_j^- + \frac{(-1)^{n+j}H(\beta)}{U}\int_{z_0^-}^{z}\frac{e^{-\mu_j\zeta}F_-(\zeta)E_j(\zeta)d\zeta}{Q^2(\zeta)}\right)+\\ +e^{\beta z}\widetilde{Q}(z)\left(C_n^- + \frac{H(\beta)}{U}\int_{z_0^-}^{z}\frac{e^{-\beta\zeta}F_-(\zeta)E_n(\zeta)d\zeta}{Q^2(\zeta)}\right),\end{aligned} \tag{24}$$

where $z_0^- \in D_-$, C_j^-, $j = \overline{1,n}$, are arbitrary complex constants, U is Vandermonde's determinant of numbers $\mu_1, \mu_2, \ldots, \mu_{n-1}, \beta$, $\widetilde{Q}(z) = \widetilde{d}_l z^l + \widetilde{d}_{l-1}z^{l-1} + \ldots + \widetilde{d}_0$ is a polynomial with coefficients that are solutions to the system

$$\begin{cases} H(\beta)C_l^l\widetilde{d}_l = d_l, \\ H'(\beta)C_l^{l-1}\widetilde{d}_l + H(\beta)C_{l-1}^{l-1}\widetilde{d}_{l-1} = d_{l-1}, \\ H''(\beta)C_l^{l-2}\widetilde{d}_l + H'(\beta)C_{l-1}^{l-2}\widetilde{d}_{l-1} + H(\beta)C_{l-2}^{l-2}\widetilde{d}_{l-2} = d_{l-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ H^{(l)}(\beta)C_l^0\widetilde{d}_l + H^{(l-1)}(\beta)C_{l-1}^0\widetilde{d}_{l-1} + H^{(l-2)}(\beta)C_{l-2}^0\widetilde{d}_{l-2} + \ldots + H(\beta)C_0^0\widetilde{d}_0 = d_0, \end{cases} \tag{25}$$

$E_j(\zeta)$ are minors of the determinant

$$\begin{vmatrix} 1 & \ldots & 1 & \sum\limits_{j=0}^{0}\beta^{0-j}C_0^j\widetilde{Q}^{(j)}(\zeta) \\ \mu_1 & \ldots & \mu_{n-1} & \sum\limits_{j=0}^{1}\beta^{1-j}C_1^j\widetilde{Q}^{(j)}(\zeta) \\ \mu_1^2 & \ldots & \mu_{n-1}^2 & \sum\limits_{j=0}^{2}\beta^{2-j}C_2^j\widetilde{Q}^{(j)}(\zeta) \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots \\ \mu_1^{n-1} & \ldots & \mu_{n-1}^{n-1} & \sum\limits_{j=0}^{n-1}\beta^{n-1-j}C_{n-1}^j\widetilde{Q}^{(j)}(\zeta) \end{vmatrix},$$

obtained from it by removing the last row and j-th column, $j = \overline{1,n}$.

Further, we will assume that the polynomial $Q(z)$ has simple zeros in the domain D_- at points $\zeta_k \neq z_0^-$, $k = \overline{1,n_-}$. The remaining zeros of this polynomial lie in the domain D_+ and can have any multiplicity. For the function $\Phi_-(z)$ to be single-valued, the following conditions are necessary and sufficient:

$$\operatorname*{res}_{z=\zeta_k}\frac{e^{-\mu_j z}F_-(z)E_j(z)}{Q^2(z)} = 0,\ \ k = \overline{1,n_-},\ \ j = \overline{1,n-1}, \tag{26}$$

$$\operatorname*{res}_{z=\zeta_k}\frac{e^{-\beta z}F_-(z)E_n(z)}{Q^2(z)} = 0,\ \ k = \overline{1,n_-}, \tag{27}$$

$$\operatorname*{res}_{z=\infty}\frac{e^{-\mu_j z}F_-(z)E_j(z)}{Q^2(z)} = 0,\ \ j = \overline{1,n-1}, \tag{28}$$

$$\operatorname*{res}_{z=\infty} \frac{e^{-\beta z} F_-(z) E_n(z)}{Q^2(z)} = 0, \tag{29}$$

which we further assume to be satisfied. Taking these conditions into account, integration in Formula (24) can be made along any curves connecting the points z_0^- and z and not passing through the points ζ_k in the domain D_-. As in Formula (20), the integrals in Formula (24) can give poles at points ζ_k; however, the function $\Phi_-(z)$ is analytic at these points due to the mutual cancellation of the poles during addition.

Generally speaking, at the point $z = \infty$ the Formula (24) gives essential singularity of the function $\Phi_-(z)$. To obtain analyticity at this point with the condition $\Phi_-(\infty) = 0$, let us expand the function into the corresponding Laurent series and equate to zero the coefficients of this expansion for non-negative powers of z. The result is the following infinite system of linear algebraic equations:

$$\begin{cases} C_1^- + \ldots + C_{n-1}^- + \widetilde{d}_0 C_n^- = \nu_0, \\ \mu_1 C_1^- + \ldots + \mu_{n-1} C_{n-1}^- + \left(\widetilde{d}_0 \beta + \widetilde{d}_1\right) C_n^- = \nu_1, \\ \mu_1^2 C_1^- + \ldots + \mu_{n-1}^2 C_{n-1}^- + \left(\widetilde{d}_0 \beta^2 + 2\widetilde{d}_1 \beta + 2\widetilde{d}_2\right) C_n^- = \nu_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mu_1^l C_1^- + \ldots + \mu_{n-1}^l C_{n-1}^- + \left(\widetilde{d}_0 \beta^l + l\widetilde{d}_1 \beta^{l-1} + (l-1)l\widetilde{d}_2 \beta^{l-2} + \right. \\ \qquad \left. + (l-2)(l-1)l\widetilde{d}_3 \beta^{l-3} + \ldots + 1 \cdot 2 \cdot \ldots \cdot l\widetilde{d}_l \beta^0\right) C_n^- = \nu_l, \\ \mu_1^{l+1} C_1^- + \ldots + \mu_{n-1}^{l+1} C_{n-1}^- + \left(\widetilde{d}_0 \beta^{l+1} + (l+1)\widetilde{d}_1 \beta^l + l(l+1)\widetilde{d}_2 \beta^{l-1} + \right. \\ \qquad \left. + \ldots + 2 \cdot 3 \cdot \ldots \cdot (l+1)\widetilde{d}_l \beta\right) C_n^- = \nu_{l+1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \tag{30}$$

where

$$\nu_s = -\frac{s!}{2\pi i} \int\limits_{\Gamma} \frac{dt}{t^{s+1}} \left(\sum_{j=1}^{n-1} \frac{(-1)^{n+j} e^{\mu_j t} H(\beta)}{U} \int\limits_{z_0^-}^{t} \frac{e^{-\mu_j \zeta} F_-(\zeta) E_j(\zeta) d\zeta}{Q^2(\zeta)} + \frac{e^{\beta t} \widetilde{Q}(t) H(\beta)}{U} \int\limits_{z_0^-}^{t} \frac{e^{-\beta \zeta} F_-(\zeta) E_n(\zeta) d\zeta}{Q^2(\zeta)} \right),$$

$s = 0, 1, 2, \ldots$; Γ is a circle with a center at zero of a sufficiently large radius.

Lemma 2. *The rank of the matrix of system (30) is equal to n.*

Proof. Let us represent the matrix of system (30) in the form

$$\begin{pmatrix} 1 & \ldots & 1 & \widetilde{d}_0 \beta^0 + \widetilde{d}_1 (\beta^0)' + \widetilde{d}_2 (\beta^0)'' + \ldots + \widetilde{d}_l (\beta^0)^{(l)} \\ \mu_1 & \ldots & \mu_{n-1} & \widetilde{d}_0 \beta^1 + \widetilde{d}_1 (\beta^1)' + \widetilde{d}_2 (\beta^1)'' + \ldots + \widetilde{d}_l (\beta^1)^{(l)} \\ \mu_1^2 & \ldots & \mu_{n-1}^2 & \widetilde{d}_0 \beta^2 + \widetilde{d}_1 (\beta^2)' + \widetilde{d}_2 (\beta^2)'' + \ldots + \widetilde{d}_l (\beta^2)^{(l)} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ \mu_1^s & \ldots & \mu_{n-1}^s & \widetilde{d}_0 \beta^s + \widetilde{d}_1 (\beta^s)' + \widetilde{d}_2 (\beta^s)'' + \ldots + \widetilde{d}_l (\beta^s)^{(l)} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \end{pmatrix} \tag{31}$$

We will prove the lemma if we find for this matrix a non-zero minor formed by its n rows. Let first none of the numbers $\mu_1, \mu_2, \ldots, \mu_{n-1}$ equal zero and let $\Lambda = \mu_1 \mu_2 \cdot \ldots \cdot \mu_{n-1}$. For $k = \overline{0, l}$ we denote M_k the minors of the matrix formed by its rows from the k-th to the $(k + n - 1)$-th. In this case, we start numbering the rows from zero. Then,

$$M_k = \sum_{j=1}^{l} \begin{vmatrix} \mu_1^k & \dots & \mu_{n-1}^k & \widetilde{d}_j \left(\beta^k\right)^{(j)} \\ \mu_1^{k+1} & \dots & \mu_{n-1}^{k+1} & \widetilde{d}_j \left(\beta^{k+1}\right)^{(j)} \\ \dots & \dots & \dots & \dots\dots\dots\dots \\ \mu_1^{k+n-1} & \dots & \mu_{n-1}^{k+n-1} & \widetilde{d}_j \left(\beta^{k+n-1}\right)^{(j)} \end{vmatrix}$$

Further, we understand via differentiation of the determinant U its differentiation with respect to β. Then, we obtain

$$\begin{aligned} M_k &= \Lambda^k \sum_{j=0}^{l} \widetilde{d}_j \left(U\,\beta^k\right)^{(j)} = \Lambda^k \sum_{j=0}^{l} \widetilde{d}_j \sum_{q=0}^{j} C_j^q U^{(j-q)} \left(\beta^k\right)^{(q)} = \tfrac{\Lambda^k U}{H(\beta)} \sum_{j=0}^{l} \widetilde{d}_j \sum_{q=0}^{j} C_j^q H^{(j-q)}(\beta) \left(\beta^k\right)^{(q)} = \\ &= \tfrac{\Lambda^k U}{H(\beta)} \Big(\beta^k \Big(C_0^0 H(\beta)\widetilde{d}_0 + C_1^0 H'(\beta)\widetilde{d}_1 + C_2^0 H''(\beta)\widetilde{d}_2 + \ldots + C_l^0 H^{(l)}(\beta)\widetilde{d}_l \Big) + \left(\beta^k\right)' \Big(C_1^1 H(\beta)\widetilde{d}_1 + \\ &+ C_2^1 H'(\beta)\widetilde{d}_2 + C_3^1 H''(\beta)\widetilde{d}_3 + \ldots + C_m^1 H^{(l-1)}(\beta)\widetilde{d}_l \Big) + \left(\beta^k\right)'' \Big(C_2^2 H(\beta)\widetilde{d}_2 + C_3^2 H'(\beta)\widetilde{d}_3 + \ldots + \\ &+ C_l^2 H^{(l-2)}(\beta)\widetilde{d}_l \Big) + \ldots + \left(\beta^k\right)^{(l-1)} \Big(C_{l-1}^{l-1} H(\beta)\widetilde{d}_{l-1} + C_l^{l-1} H'(\beta)\widetilde{d}_l \Big) + \left(\beta^k\right)^{(l)} \Big(C_l^l H(\beta)\widetilde{d}_l \Big) \Big) \end{aligned} \tag{32}$$

In this case, we used expressions of the form

$$U^{(k)} = \frac{U\,H^{(k)}(\beta)}{H(\beta)},$$

quite similar to Equalities (17). Finally, due to system (25), we obtain

$$M_k = \frac{\Lambda^k U}{H(\beta)} \left(\beta^k d_0 + \left(\beta^k\right)' d_1 + \left(\beta^k\right)'' d_2 + \ldots + \left(\beta^k\right)^{(l-1)} d_{l-1} + \left(\beta^k\right)^{(l)} d_l \right).$$

If $k = 0$ then $M_0 = \frac{U}{H(\beta)} d_0$. If $d_0 \neq 0$, the minor M_0 is the required one. If $d_0 = 0$, then we take $M_1 = \frac{\Lambda U}{H(\beta)} (\beta d_0 + d_1) = \frac{\Lambda U}{H(\beta)} d_1$. And also, if $d_1 = 0$, then we take $M_2 = \frac{\Lambda^2 U}{H(\beta)} (\beta^2 d_0 + 2\beta d_1 + 2d_2) = \frac{\Lambda^2 U}{H(\beta)} 2d_2$ etc. If there is no non-zero minor M_k with $k < l$, then $M_l = \frac{\Lambda^l U}{H(\beta)} l! d_l \neq 0$.

Now, let one of the numbers μ_j be equal to zero. For definiteness, let $\mu_1 = 0$. Then, $H(\mu) = \mu\, H_1(\mu)$, where $H_1(\mu)$ is a polynomial with roots $\mu_2, \mu_3, \ldots, \mu_{n-1}$, and $H_1(\beta) \neq 0$. In this case, the first column of matrix (31) has the form $\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \ldots \end{pmatrix}^T$ (symbol T means transposition). Obviously, the rank of such a matrix is one greater than the rank of the matrix

$$\begin{pmatrix} \mu_2 & \dots & \mu_{n-1} & \widetilde{d}_0 \beta^1 + \widetilde{d}_1 (\beta^1)' + \widetilde{d}_2 (\beta^1)'' + \ldots . + \widetilde{d}_l (\beta^1)^{(l)} \\ \mu_2^2 & \dots & \mu_{n-1}^2 & \widetilde{d}_0 \beta^2 + \widetilde{d}_1 (\beta^2)' + \widetilde{d}_2 (\beta^2)'' + \ldots . + \widetilde{d}_l (\beta^2)^{(l)} \\ \dots & \dots & \dots & \dots\dots\dots\dots\dots\dots \\ \mu_2^s & \dots & \mu_{n-1}^s & \widetilde{d}_0 \beta^s + \widetilde{d}_1 (\beta^s)' + \widetilde{d}_2 (\beta^s)'' + \ldots . + \widetilde{d}_l (\beta^s)^{(l)} \\ \dots & \dots & \dots & \dots\dots\dots\dots\dots\dots \end{pmatrix}. \tag{33}$$

Let us show that the rank of matrix (33) is equal to $n - 1$, that is, equal to, the number of its columns. To see this, we justify the existence of the corresponding non-zero minor. Let for $k = \overline{1, m+1}$ N_k be the minors of the matrix (33), formed by their rows from k-th to $(k + n - 2)$-th. These minors have the same structure as the minors M_k; therefore, their values are found using a formula similar to Formula (32):

$$N_k = \frac{\Lambda_1^k U_1}{H_1(\beta)}\Big(\beta^k\Big(C_0^0 H_1(\beta)\widetilde{d}_0 + C_1^0 H'_1(\beta)\widetilde{d}_1 + C_2^0 H''_1(\beta)\widetilde{d}_2 + \ldots + C_l^0 H_1{}^{(l)}(\beta)\widetilde{d}_l\Big) + \Big(\beta^k\Big)'\Big(C_1^1 H_1(\beta)\widetilde{d}_1 + \\ + C_2^1 H'_1(\beta)\widetilde{d}_2 + C_3^1 H''_1(\beta)\widetilde{d}_3 + \ldots + C_l^1 H_1{}^{(l-1)}(\beta)\widetilde{d}_l\Big) + \Big(\beta^k\Big)''\Big(C_2^2 H_1(\beta)\widetilde{d}_2 + C_3^2 H'_1(\beta)\widetilde{d}_3 + C_4^2 H''_1(\beta)\widetilde{d}_4 + \\ + \ldots + C_l^2 H_1{}^{(l-1)}(\beta)\widetilde{d}_l\Big) + \ldots + \Big(\beta^k\Big)^{(l-1)}\Big(C_{l-1}^{l-1} H_1(\beta)\widetilde{d}_{l-1} + C_l^{l-1} H'_1(\beta)\widetilde{d}_l\Big) + \Big(\beta^k\Big)^{(l)}\Big(C_l^l H_1(\beta)\widetilde{d}_l\Big)\Big),$$

where $\Lambda_1 = \mu_2\mu_3\cdot\ldots\cdot\mu_{n-1}$, U_1 is Vandermonde's determinant of numbers $\mu_2, \mu_3, \ldots, \mu_{n-1}, \beta$.

Let us take a system (25) and put $H(\beta) = \beta\, H_1(\beta)$ in it. Further, it will be easy to transform this system into the form

$$\begin{cases} C_l^l H_1(\beta)\widetilde{d}_l = \frac{d_l}{\beta}, \\ C_l^{l-1} H'_1(\beta)\widetilde{d}_l + C_{l-1}^{l-1} H_1(\beta)\widetilde{d}_{l-1} = \frac{d_{l-1}}{\beta} - \frac{l d_l}{\beta^2}, \\ C_l^{l-2} H''_1(\beta)\widetilde{d}_l + C_{l-1}^{l-2} H'_1(\beta)\widetilde{d}_{l-1} + C_{l-2}^{l-2} H_1(\beta)\widetilde{d}_{l-2} = \frac{d_{l-2}}{\beta} - \frac{(l-1)d_{l-1}}{\beta^2} + \frac{(l-1)l d_l}{\beta^3}, \\ C_l^{l-3} H'''_1(\beta)\widetilde{d}_l + C_{l-1}^{l-3} H''_1(\beta)\widetilde{d}_{l-1} + C_{l-2}^{l-3} H'_1(\beta)\widetilde{d}_{l-2} + C_{l-3}^{l-3} H_1(\beta)\widetilde{d}_{l-3} = \\ \qquad = \frac{d_{l-3}}{\beta} - \frac{(l-2)d_{l-2}}{\beta^2} + \frac{(l-2)(l-1)d_{l-1}}{\beta^3} - \frac{(l-2)(l-1)l d_l}{\beta^4}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_l^0 H_1^{(l)}(\beta)\widetilde{d}_l + C_{l-1}^0 H_1^{(l-1)}(\beta)\widetilde{d}_{l-1} + \ldots + C_1^0 H'_1(\beta)\widetilde{d}_1 + C_0^0 H_1(\beta)\widetilde{d}_0 = \\ \qquad = \frac{d_0}{\beta} - \frac{1!d_1}{\beta^2} + \frac{2!d_2}{\beta^3} - \frac{3!d_3}{\beta^4} + \ldots + (-1)^l \frac{l!d_l}{\beta^{l+1}}. \end{cases}$$

Thus, we will create a system suitable to continue calculating the minors N_k.

$$N_1 = \frac{\Lambda_1 U_1}{H_1(\beta)}\Big(\beta\Big(\frac{d_0}{\beta} - \frac{1!d_1}{\beta^2} + \frac{2!d_2}{\beta^3} - \frac{3!d_3}{\beta^4} + \ldots + (-1)^l\frac{l!d_l}{\beta^{l+1}}\Big) + \Big(\frac{d_1}{\beta} - \frac{2d_2}{\beta^2} + \frac{2\cdot 3d_3}{\beta^3} - \frac{2\cdot 3\cdot 4d_4}{\beta^4} + \\ + \ldots + (-1)^{l-1}\frac{2\cdot 3\cdot\ldots\cdot l d_l}{\beta^l}\Big)\Big) = \frac{\Lambda_1 U_1}{H_1(\beta)} d_0.$$

If $d_0 \neq 0$, then N_1 is the required minor. If $d_0 = 0$, then we calculate

$$N_2 = \frac{\Lambda_1^2 U_1}{H_1(\beta)}\Big(\beta^2\Big(-\frac{1!d_1}{\beta^2} + \frac{2!d_2}{\beta^3} - \frac{3!d_3}{\beta^4} + \frac{4!d_4}{\beta^5} + \ldots + (-1)^l\frac{l!d_l}{\beta^{l+1}}\Big) + 2\beta\Big(\frac{d_1}{\beta} - \frac{2d_2}{\beta^2} + \frac{2\cdot 3d_3}{\beta^3} - \frac{2\cdot 3\cdot 4d_4}{\beta^4} + \\ + \ldots + (-1)^{l-1}\frac{2\cdot 3\cdot\ldots\cdot l d_l}{\beta^l}\Big) + 2\Big(\frac{d_2}{\beta} - \frac{3d_3}{\beta^2} + \frac{3\cdot 4d_4}{\beta^3} + \ldots + (-1)^{l-1}\frac{3\cdot\ldots\cdot(l-1)\cdot l d_l}{\beta^{l-1}}\Big)\Big) = \frac{\Lambda_1^2 U_1}{H_1(\beta)} d_1.$$

Further reasoning for minors N_k is similar to those for minors M_k. When $d_0 = d_1 = \ldots = d_{k-2} = 0$, we have $N_1 = N_2 = \ldots = N_{k-1} = 0$,

$$N_k = \frac{\Lambda_1^k U_1}{H_1(\beta)}(k-1)!d_{k-1},\ k = \overline{2, l+1}. \tag{34}$$

Consequently, if $k < l+1$, the non-zero minor N_k, does not occur, then, in any case, there will be $N_{l+1} \neq 0$. Formula (34) is derived using direct calculations similar to the case $k = 1, 2$. These calculations, which are somewhat cumbersome, are not presented here. This completes the proof of the lemma. It follows that system (30) has only one solution. □

5. The Main Result—An Example

Since the solutions $F_\pm(z)$ of the Riemann problem (10) are H-continuously extendable to the curve L, then from Formulas (20) and (24), it is obvious that the solutions of Equations (8) and (9) (if they exist) H-continuously extendable onto the curve L. Also, derivatives $\Phi_\pm^{(k)}(z)$, $k = \overline{1, n}$, are H-continuously extendable onto the curve L. This follows from the formulas for derivatives in the method of variation of arbitrary constants. Let us limit ourselves by referring to quite similar formulas for derivatives in [10] (p. 366). And then, the desired function, which is found using Formula (6) with $k = 0$, belongs to the class in which the solution to the original equation was determined. Let us formulate the final result.

Theorem. *For Equation (5) to be solvable, it is necessary and sufficient that conditions (11) are satisfied (when $\gamma < -l$), (21), (22), (26)–(29) and the system (30) are solvable. If Equation (5) turns out to be solvable, then its solution has the form*

$$\varphi(t) = \sum_{j=1}^{n-1} e^{\lambda_j t}\left(C_j^+ + \frac{(-1)^{n+j}G(\alpha)}{V}\int_{z_0^+}^{t} \frac{e^{-\lambda_j\zeta}F_+(\zeta)W_j^*(\zeta)d\zeta}{P^2(\zeta)}\right)+$$
$$t\ +e^{\alpha\, t}\widetilde{P}(t)\left(C_n^+ + \frac{G(\alpha)}{V}\int_{z_0^+}^{t}\frac{e^{-\alpha\zeta}F_+(\zeta)W_n^*(\zeta)d\zeta}{P^2(\zeta)}\right) - \sum_{j=1}^{n-1} e^{\mu_j t}\left(C_j^- + \frac{(-1)^{n+j}H(\beta)}{U}\int_{z_0^-}^{t}\frac{e^{-\mu_j\zeta}F_-(\zeta)E_j(\zeta)d\zeta}{Q^2(\zeta)}\right)-$$
$$-e^{\beta\, t}\widetilde{Q}(t)\left(C_n^- + \frac{H(\beta)}{U}\int_{z_0^-}^{t}\frac{e^{-\beta\,\zeta}F_-(\zeta)E_n(\zeta)d\zeta}{Q^2(\zeta)}\right),$$

where the constants C_j^+ are arbitrary, and the constants C_j^- are a solution to the system (30), $j = \overline{1, n}$.

We note that conditions (21), (22), and (26)–(29) with $\gamma + l - 1 \geq 0$, in expanded form, express the solvability of the system of linear algebraic equations for the coefficients of the polynomial $R(z)$. The explicit form of this system can be easily obtained and is not written here. Let us solve an example.

$$t^2(t^2+t-11)\,\varphi'''(t) - 2t(4t^3+5t^2+49t+1)\,\varphi''(t) + t^2(17t^2+31t-201)\,\varphi\prime(t)-$$
$$-2t(5t^3+15t^2-57t-4)\varphi(t) + \frac{6t^2(t^2+t-13)}{\pi i}\int_{|\tau|=5}\frac{\varphi(\tau)d\tau}{(\tau-t)^4} - \frac{4t(4t^3+5t^2+48t-1)}{\pi i}\int_{|\tau|=5}\frac{\varphi(\tau)d\tau}{(\tau-t)^3}+$$
$$+\frac{t^2(17t^2+31t-193)}{\pi i}\int_{|\tau|=5}\frac{\varphi(\tau)d\tau}{(\tau-t)^2} - \frac{2t(5t^3+10t^2-58t+4)}{\pi i}\int_{|\tau|=5}\frac{\varphi(\tau)d\tau}{\tau-t} =$$
$$= 2\left(t^2(t+4)^2(t-2) + \frac{10}{t^2} + \frac{2}{t} - 12 - 4t\right), \quad |t| = 5.$$

This form can be given to Equation (5) with $n = 3$, $a(t) = t^2$, $b(t) = 1$, $A_1 = 10$, $A_2 = -7$, $B_1 = -4$, $B_2 = 0$, $P(t) = t^2 + t - 12$, $Q(t) = t^2$, $\alpha = \beta = 1$. The curve L and expression for function $f(t)$ are obvious. The corresponding Riemann problem

$$F_+(t) = \frac{1}{t^2}F_-(t) + (t+4)^2(t-2) + \frac{10}{t^4} + \frac{2}{t^3} - \frac{12}{t^2} - \frac{4}{t}, \quad |t| = 5,$$

must be solved in the class of functions that admit a pole of the first order at infinity. Such a solution exists, is unique, and has the form

$$F_+(z) = (z+4)^2(z-2), \quad |z| < 5,$$

$$F_-(z) = -\frac{10}{z^2} - \frac{2}{z} + 12 + 4z, \quad |z| > 5.$$

Now, let us write the corresponding Equation (8)

$$(z^2+z-12)\Phi'''_+(z) - (8z^2+10z-95)\Phi''_+(z)+$$
$$+(17z^2+31z-197)\Phi'_+(z) - 10(z^2+3z-11)\Phi_+(z) = (z+4)^2(z-2), \quad |z| < 5. \tag{35}$$

Here $G(\lambda) = \lambda^2 - 7\lambda + 10$, $\lambda_1 = 2$, $\lambda_2 = 5$, $G(\alpha) = 4$, $V = 12$, $\widetilde{P}(z) = \frac{1}{4}z^2 + \frac{7}{8}z - \frac{65}{32}$, and the fundamental system of solutions to a homogeneous equation is formed using the functions

$$e^{2z},\ \ e^{5z},\ \ e^{z}\left(\frac{1}{4}z^2 + \frac{7}{8}z - \frac{65}{32}\right).$$

From the equation $P(z) = 0$, we find two points $z_1 = -4$ and $z_2 = 3$, at which we should verify conditions of the form (21) and (22) for the functions

$$\frac{e^{-\lambda_1 z} F_+(z) W_1^*(z)}{P^2(z)} = -\frac{e^{-2z}(z^2+3z-9)(z-2)}{(z-3)^2}, \tag{36}$$

$$\frac{e^{-\lambda_2 z} F_+(z) W_2^*(z)}{P^2(z)} = -\frac{e^{-5z}(8z^2+12z-93)(z-2)}{32(z-3)^2}, \tag{37}$$

$$\frac{e^{-\alpha z} F_+(z) W_3^*(z)}{P^2(z)} = \frac{3e^{-z}(z-2)}{(z-3)^2}. \tag{38}$$

At point $z_1 = -4$ these functions do not have a pole, so at this point, conditions (21) and (22) are satisfied. At point $z_2 = 3$, the corresponding conditions

$$\operatorname*{res}_{z=3}\left(-\frac{e^{-2z}(z^2+3z-9)(z-2)}{(z-3)^2}\right) = \operatorname*{res}_{z=3}\left(-\frac{e^{-5z}(8z^2+12z-93)(z-2)}{32(z-3)^2}\right) = \operatorname*{res}_{z=3}\frac{3e^{-z}(z-2)}{(z-3)^2} = 0$$

are satisfied, which is not entirely obvious and can be established using simple calculations.

Fulfillment of the specified conditions means that the functions (36)–(38) have primitives $\widetilde{C}_1^+(z)$, $\widetilde{C}_2^+(z)$, $\widetilde{C}_3^+(z)$, respectively, in the domain $\{z : |z| < 5\} \setminus \{3\}$. Calculations give, for example,

$$\widetilde{C}_1^+(z) = \frac{e^{-2z}(2z^2+9z-9)}{4(z-3)}, \quad \widetilde{C}_2^+(z) = \frac{3e^{-5z}(40z^2+108z-309)}{32(z-3)}, \quad \widetilde{C}_3^+(z) = -\frac{3e^{-z}}{z-3}.$$

The formula for solving Equation (35) can be given in the form

$$\Phi_+(z) = e^{2z}\left(C_1^+ + \frac{1}{3}\widetilde{C}_1^+(z)\right) + e^{5z}\left(C_2^+ - \frac{1}{3}\widetilde{C}_2^+(z)\right) + e^{z}\left(\frac{1}{4}z^2 + \frac{7}{8}z - \frac{65}{32}\right)\left(C_3^+ + \frac{1}{3}\widetilde{C}_3^+(z)\right).$$

Substituting here the expressions for primitives and making simplifications, we obtain

$$\Phi_+(z) = C_1^+ e^{2z} + C_2^+ e^{5z} + C_3^+ e^{z}\left(\frac{1}{4}z^2 + \frac{7}{8}z - \frac{65}{32}\right) - \frac{1}{100}(10z+47).$$

Let us deal with Equation (9) in the considered case.

$$z^2\Phi'''_-(z) - z(z+2)\Phi''_-(z) - 4z^2\Phi'_-(z) + 4z(z+2)\Phi_-(z) = -\frac{10}{z^2} - \frac{2}{z} + 12 + 4z, \quad |z| > 5. \tag{39}$$

Here, $H(\mu) = \mu^2 - 4$, $\mu_1 = 2$, $\mu_2 = -2$, $H(\beta) = -3$, $U = 12$, $\widetilde{Q}(z) = -\left(\frac{1}{3}z^2 + \frac{4}{9}z + \frac{14}{27}\right)$, and the fundamental system of solutions of the corresponding homogeneous equation is formed using the functions

$$e^{2z}, \quad e^{-2z}, \quad -e^{z}\left(\frac{1}{3}z^2 + \frac{4}{9}z + \frac{14}{27}\right).$$

Further, we obtain

$$\frac{e^{-\mu_1 z} F_-(z) E_1(z)}{Q^2(z)} = 4e^{-2z}\left(\frac{5}{z^6} + \frac{6}{z^5} - \frac{5}{2z^4} - \frac{15}{2z^3} - \frac{5}{z^2} - \frac{1}{z}\right), \tag{40}$$

$$\frac{e^{-\mu_2 z} F_-(z) E_2(z)}{Q^2(z)} = 4e^{2z}\left(-\frac{5}{27z^6} + \frac{14}{27z^5} - \frac{1}{2z^4} - \frac{41}{54z^3} + \frac{9}{7z^2} + \frac{1}{3z}\right), \tag{41}$$

$$\frac{e^{-\beta z} F_-(z) E_3(z)}{Q^2(z)} = 8e^{-z}\left(\frac{5}{z^6} + \frac{1}{z^5} - \frac{6}{z^4} - \frac{2}{z^3}\right). \tag{42}$$

The polynomial $Q(z)$ has no zero in the domain $|z| > 5$, so conditions like (26) and (27) do not arise. Only conditions of the form (28) and (29) arise, the validity of which for functions (40)–(42) can be established using simple calculations. Now, Formula (20) for Equation (35) can be given in the form

$$\Phi_-(z) = e^{2z}\left(C_1^- - \frac{1}{4}\widetilde{C}_1^-(z)\right) + e^{-2z}\left(C_2^- + \frac{1}{4}\widetilde{C}_2^-(z)\right) - e^z\left(\frac{1}{3}z^2 + \frac{4}{9}z + \frac{14}{27}\right)\left(C_3^- - \frac{1}{4}\widetilde{C}_3^-(z)\right), \tag{43}$$

where $\widetilde{C}_1^-(z)$, $\widetilde{C}_2^-(z)$, $\widetilde{C}_3^-(z)$ are some of the primitives in the domain $|z| > 5$ of functions (40)–(42), respectively. For example, the functions

$$\widetilde{C}_1^-(z) = 4e^{-2z}\left(-\frac{1}{z^5} - \frac{1}{z^4} + \frac{3}{2z^3} + \frac{9}{4z^2} + \frac{1}{2z}\right), \quad \widetilde{C}_2^-(z) = 4e^{2z}\left(\frac{1}{27z^5} - \frac{1}{9z^4} + \frac{5}{54z^3} + \frac{17}{36z^2} + \frac{1}{6z}\right),$$

$$\widetilde{C}_3^-(z) = 8e^{-z}\left(-\frac{1}{z^5} + \frac{2}{z^3}\right)$$

can be such primitives. Substituting these primitives into Formula (43) and making simplifications, we obtain

$$\Phi_-(z) = C_1^- e^{2z} + C_2^- e^{-2z} - C_3^- e^z\left(\frac{1}{3}z^2 + \frac{4}{9}z + \frac{14}{27}\right) + \frac{1}{z}. \tag{44}$$

Let us show system (30) for solving (44)

$$\begin{cases} C_1^- + C_2^- - \frac{14}{27}C_3^- = 0, \\ 2C_1^- - 2C_2^- - \frac{26}{27}C_3^- = 0, \\ 4C_1^- + 4C_2^- - \frac{56}{27}C_3^- = 0, \\ 8C_1^- - 8C_2^- - \frac{104}{27}C_3^- = 0, \\ 16C_1^- + 16C_2^- - \frac{170}{27}C_3^- = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}.$$

Since $d_0 = d_1 = 0$, $d_2 \neq 0$, then in accordance with the theory, it turns out

$$M_0 = \begin{vmatrix} 1 & 1 & -\frac{14}{27} \\ 2 & -2 & -\frac{26}{27} \\ 4 & 4 & -\frac{56}{27} \end{vmatrix} = 0, \; M_1 = \begin{vmatrix} 2 & -2 & -\frac{26}{27} \\ 4 & 4 & -\frac{56}{27} \\ 8 & -8 & -\frac{104}{27} \end{vmatrix} = 0, \; M_2 = \begin{vmatrix} 4 & 4 & -\frac{56}{27} \\ 8 & -8 & -\frac{104}{27} \\ 16 & 16 & -\frac{170}{27} \end{vmatrix} \neq 0,$$

which leads to the uniqueness of the obvious solution $C_1^- = C_2^- = C_3^- = 0$ of this system.

System (45) could not be written down since, from Formula (44), it is immediately clear that when $C_1^- = C_2^- = C_3^- = 0$ we obtain the only possible solution $\Phi_-(z) = \frac{1}{z}$ to Equation (39), analytical in the domain $|z| > 5$ with the condition $\Phi_-(\infty) = 0$.

Finally, let us write down the solution to the example

$$\varphi(t) = C_1^+ e^{2t} + C_2^+ e^{5t} + C_3^+ e^t\left(\frac{1}{4}t^2 + \frac{7}{8}t - \frac{65}{32}\right) - \frac{1}{100}(10t + 47) - \frac{1}{t} \tag{45}$$

6. Concluding Remarks

Apparently, if the zeros of the polynomials $P(z)$ and $Q(z)$ have a higher multiplicity than 1 at points z_k and ζ_k, Formulas (20) and (24) will still lead to functions that are analytic at these points, but justifying this fact will require cumbersome calculations. Such calculations can be made in subsequent studies. In subsequent studies, we will also try to find applications of Equation (5) to mechanics and other fields of science.

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