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REVIEW

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Extremely asymmetric sawtooth potential in the ratchet theory

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Abstract
We present a review of analytical approaches involved in developing the ratchet theory, which are based on the model of extremely asymmetric sawtooth potential. Analytical expressions are given for the average velocity of ratchets which operate in various motion modes, namely, motion induced by dichotomous half-period shifts of potential profiles, adiabatic and high-temperature modes, and motion induced by small fluctuations of an arbitrary type. The presence of jumps in the periodic extremely asymmetric sawtooth potential profile leads to a number of features of the obtained solutions which follow from the competition of the reverse sliding time tending to infinity with high fluctuation frequencies. The resulting dependences of the average velocity on the ratchet parameters clearly demonstrate that the motion direction can be controlled by tuning the frequency and temperature. The heuristic value of the presented models for controlling nanoparticle transport is discussed.

KEYWORDS
Brownian motors, diffusion transport, ratchet systems, sawtooth potential, symmetry

1 | INTRODUCTION

Professor S. H. Lin was a bright person always open to the latest trends in modern physics and chemistry, who urged his students not to stagnate within the scientific areas already known, but to be actively engaged in new promising research areas. One of such new areas, which Professor S. H. Lin began to study in 2004, was the theory of ratchet systems.[1] Interest in such systems is based on several factors, among which are attempts to understand the operation principles of protein motors that convert the energy of chemical transformations into the directional motion of biological objects, as well as developing systems for particle segregation in solutions and controlling directional motion of nano-objects.[2–8] From a theoretical point of view, the interest in ratchet systems is induced by the fact that directional motion in asymmetric media can be described by the including fluctuations of various nature in a theoretical model; to do that, the methods of the modern theory of nonequilibrium processes are used within the framework of the well-developed diffusion dynamics.[9]

It should be emphasized that ratchet mechanisms differ from other methods of creating particle motion[10–12] in that, in a ratchet system, the average forces acting on the particles or the particle concentration gradients are equal to zero, and the directional motion arises from various unbiased non-equilibrium perturbations, when the spatial and/or temporal symmetry of the system is broken. The concept of ratchet systems implies systems that initiate exactly unidirectional translational or rotational motion. At the same time, there are nanomachines in which non-equilibrium perturbations can cause reciprocating motion[13]; they include, for example, light-driven reciprocating host–guest molecular machines,[14] light-activated molecular catchers,[15] or rotation-inversion dual-mode molecular systems.[16]

Most ratchet models are based on a dichotomous process of switching periodic potential profiles, among which sawtooth profiles occupy a special place. This special place is because of the fact that piecewise linear models lead to analytical expressions for the desired quantities, and these expressions are greatly simplified if potential profiles are extremely asymmetric sawtooth. It is this shape of the potential profile that was proposed in[17] to explain the high efficiency of energy conversion in a ratchet with half-period shifted periodic potentials.

The operation scheme of a highly efficient ratchet with an extremely asymmetric sawtooth potential profile is given in Figure 1. When a particle moves down the potential profile, this profile is shifted by half a period with a certain frequency, potential barriers then do not impede the motion and it can proceed continuously in the same direction. It turns out that the efficiency of such a process is high when the potential barrier is much greater than the thermal energy. Section 2 presents the solution of this model.

The main parameters affecting the properties of a ratchet system are the frequency of nonequilibrium fluctuations and temperature (equilibrium noise intensity). The smallness of the fluctuation frequency with respect to the inverse characteristic times of the system makes it possible to obtain a number of analytical expressions for the particle current in sawtooth potentials fluctuating in amplitude (Section 3). In these expressions, the temperature can vary over a wide range. If the temperature is assumed to be sufficiently high, that is the ratios of the energy barriers to the thermal energy be sufficiently small, then a wide range of analytical expressions for the particle current can be also derived, and they have very simple forms for the sawtooth barriers (Section 4). In these expressions, on the contrary, the fluctuation frequency can vary over a wide range. Finally, one more approximation exists which is a source of analytical results, in which only the fluctuation energies (but not the potential profile as a whole) are assumed to be small with respect to the thermal energy while both the temperature and the fluctuation frequency are allowed to vary within a wide range of values. This small fluctuation case for an extremely asymmetric sawtooth potential is discussed in Section 5. In Section 6, a number of concluding remarks are made on the role of jumps in potential profiles inherent to extremely asymmetric sawtooth ones as well as on the practically important features which they lead to.

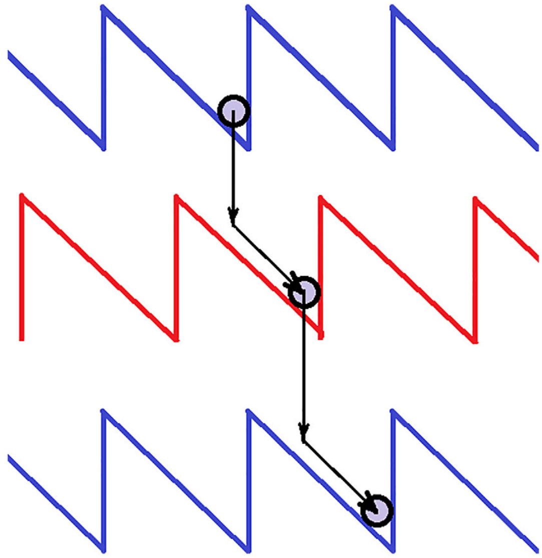

FIGURE 1 Schematic representation of an extremely asymmetric sawtooth potential fluctuating in the "shift by half a period" way, which can initiate a directional motion characterized by high efficiency

2 | POTENTIAL FLUCTUATIONS FOR HALF A PERIOD

Following Ref. [17], we assume that there are two periodic potential profiles $V_+(x)$ and $V_-(x)$ alternating with frequency γ. To describe the efficiency of energy conversion in the ratchet system, we define the output power (the useful work done against a load F per unit time) as the product $W_{\text{out}} = F\langle v\rangle$, where $\langle v\rangle$ is the average ratchet velocity. The total potential energy in "±" states is written as $U_\pm(x) = V_\pm(x) + Fx$. To find the velocity $\langle v\rangle$ and the input power, we need the probability density $\rho_\pm(x, t)$ of the particle being in the "±" state at the point x and time t, which is a solution of the Smoluchowski Equation[9] with sources and sinks:

$$\frac{\partial \rho_\pm(x,t)}{\partial t} = -\frac{\partial j_\pm(x,t)}{\partial x} - \gamma\left[\rho_\pm(x,\ t) - \rho_\mp(x,\ t)\right], \quad (1)$$

with the currents $j_\pm(x, t)$ given by the expression

$$j_\pm(x,t) = -De^{-\beta U_\pm(x)}\frac{\partial}{\partial x}\left[e^{\beta U_\pm(x)}\rho_\pm(x,\ t)\right]. \quad (2)$$

Here, D is the diffusion coefficient, $\beta = (k_B T)^{-1}$ is the inverse thermal energy (k_B is the Boltzmann constant and T is the absolute temperature). We will be interested in the stationary state in which $\partial\rho_\pm(x,t)/\partial t = 0$, and the argument t of the functions $\rho_\pm(x, t)$ and $j_\pm(x, t)$ can then be omitted. In the stationary state, the total current $J \equiv j_+(x) + j_-(x)$ is a constant, and the average velocity $\langle v\rangle$ is equal to the product of this current and the spatial period L of the potential profiles, so that

$$W_{\text{out}} = FJL. \quad (3)$$

The power required to switch the potential profiles $V_+(x)$ and $V_-(x)$ with frequency γ is determined by the following expression[18]:

$$W_{\text{in}} = \gamma\int_0^L \left[V_+(x) - V_-(x)\right]\left[\rho_-(x) - \rho_+(x)\right]dx. \quad (4)$$

Relations (3) and (4) determine the efficiency of the ratchet: $\eta = W_{\text{out}}/W_{\text{in}}$.

An extremely asymmetric sawtooth potential with a period L and a barrier ΔV can be specified on its period $(\varepsilon, L+\varepsilon)$, $\varepsilon \to 0$, as follows:

$$V(x) = \Delta V[x - L\theta(x-L)],\ \theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \quad (5)$$

If we consider that, in the "+" state, $V_+(x) = V(x)$, then, in the "−"state, the potential be $V_-(x) = V(x + L/2)$. The solution of this model has been obtained in[17]:

$$J = \frac{DA}{2(Z_1B - Z_2A)},\ \eta = \frac{FL}{4\Delta V}\frac{f}{1-e^{-fL/2}}\frac{A}{\tilde{\gamma}\sinh\beta\Delta V}, \quad (6)$$

where

$$\begin{aligned} A &= \Psi_0 e^{\beta\Delta V}\left(e^{-\beta FL/2}\cosh\beta\Delta V - 1\right) \\ &\quad + \Psi_1\left(e^{-\beta FL/2} - \cosh\beta\Delta V\right), \\ B &= e^{-\beta FL/2}\left(\Psi_0 e^{\beta\Delta V}\cosh\beta\Delta V + \Psi_1\right), \\ Z_1 &= \frac{4}{f^2}\sinh^2(fL/4),\ Z_2 = \frac{1}{f^2}\left(fL/2 + e^{-fL/2} - 1\right), \\ \Psi_j &= \frac{1}{2\sinh L\Delta/4}\left\{\left[\exp\left((-1)^j fL/4\right) + \cosh(L\Delta/4)\right]\Delta\right. \\ &\quad \left. + (-1)^j f\sinh L\Delta/4\right\}, j = 0,1, \\ f &= \beta F - 2\beta\Delta V/L,\ \tilde{\gamma} = \gamma/D,\ \Delta = \sqrt{f^2 + 8\tilde{\gamma}} \end{aligned} \quad (7)$$

Figure 2 depicts the current and the efficiency as functions of the load force F with the different values of the parameter $\tilde{\gamma}L^2$. While the current is a monotonically decreasing function of the load force F, the F-dependence of the efficiency is nonmonotonic. It was shown in[17] that the maximum of the dependence $\eta(F)$ tends to unity as $\beta\Delta V \to \infty$ according to the law:

$$\eta_m \underset{\beta\Delta V\to\infty}{\to} 1 - \frac{\ln(2\beta\Delta V)}{\beta\Delta V}. \quad (8)$$

Thus, the presented model is indeed very promising to explain the high efficiency of protein motors.

3 | ADIABATIC MODE OF MOTION

Here, we still assume that the process which initiates the directional motion is dichotomous. Then the adiabatic operation mode corresponds to sufficiently low frequencies γ of switching the potential profiles relative to the inverse characteristic times of the system, so that the system has time to reach local thermodynamic equilibrium in each of the states. Depending on temperature, the characteristic times can be the diffusion time over the period of the potential profiles, the sliding time along the linear sections of these profiles, or the time required to overcome potential barriers. The smallness of γ values makes it possible to obtain, using the Parrondo's lemma,[19] an analytical expression for the particle current in a ratchet system operating due to the adiabatic switching of sawtooth potentials, which are not shifted relative to each other and have fluctuating amplitudes ΔV_a and ΔV_b. In the adiabatic motion mode, the

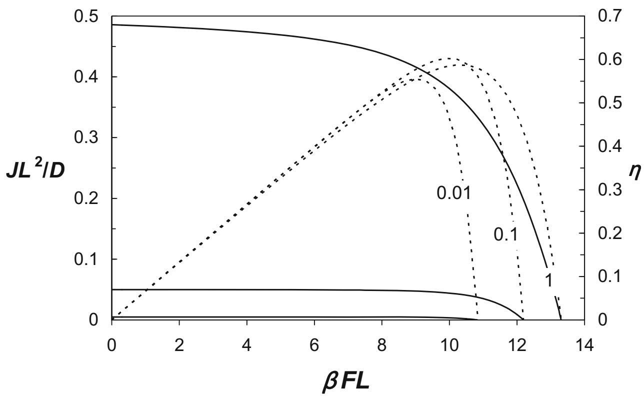

FIGURE 2 Current J (solid lines, the left axis) and the efficiency η (dashed lines, the right axis) versus the load force F for an extremely asymmetric sawtooth potential at several values of $\tilde{\gamma}L^2$ (indicated on the curves) and constant value of temperature (amplitude) $\beta\Delta V = 15$. The dependences are calculated by Equations (6) and (7)

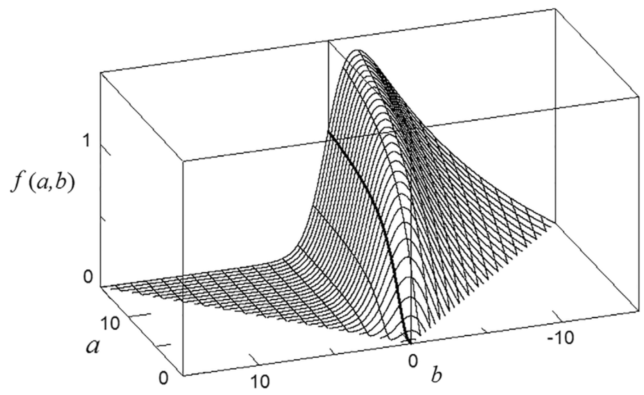

FIGURE 3 Function $f(a, b)$, which determines the average particle velocity in Equation (9) for the sawtooth potentials with fluctuating amplitudes

resulting expression for the current is linear in γ and can be represented as[20,21]:

$$J = \frac{\gamma}{4}\kappa f(a, b),$$

$$f(a, b) = \frac{a}{\sinh^2 a} + \frac{b}{\sinh^2 b} - \frac{a+b}{a-b}\frac{\sinh(a-b)}{\sinh a \sinh b}, \tag{9}$$

where $a = \beta\Delta V_a/2$, $b = \beta\Delta V_b/2$, and $\kappa = 1 - 2l/L$ is the asymmetry parameter (l and $L - l$ are the widths of the links of the sawtooth potential). The parameter κ changes from zero for the symmetric potential ($l = 1/2$), in which there is no ratchet effect, to unity for the extremely asymmetric potential ($l = 0$) having jumps in each period. It is this feature that leads to a singularity when one tries to describe inertial effects in the framework of the adiabatic motion mode[21] and gives a number of interesting consequences, which we will discuss further.

It follows from the expression (9) that the function $f(a, b)$ is positive when $a > |b|$ and it vanishes when either there are no fluctuations ($a = b$) or the sign fluctuates ($a = -b$); in the latter case $V_-(x) = -V_+(x)$. In the high-temperature limit ($a, |b| << 1$), the function $f(a, b)$ takes on the following form:

$$f(a, b) \approx \frac{2}{45}(a+b)(a-b)^2, \tag{10}$$

so that its maximum is reached at $b/a = -1/3$. In the case $b = 0$, one of the states of the dichotomous process corresponds to the zero potential and the free diffusion occurs in it (the system is called on–off ratchet). Then the expression (9) is simplified to

$$f(a, 0) = \coth a + a/\sinh^2 a - 2/a. \tag{11}$$

Figure 3 represents the dependence $f(a, b)$, which, up to a factor $\gamma L/4$, describes the velocity of the ratchet with fluctuating extremely asymmetric sawtooth potentials as a function of the potential amplitudes. One can see that $f(a, b) \geq 0$ at $a \geq |b|$. In addition, in the cross-sections of the surface, corresponding to $a = \text{const}$, the maximum values of $f(a, b)$ as a function of b are reached at $b/a \approx -1/3$ (strict equality is realized in the high-temperature limit in accordance with the formula (10)).

4 | HIGH TEMPERATURE DRIVING

The theory of high-temperature ratchets was proposed in Ref. [22] and successfully developed in Refs. [23–25]. The assumption on smallness of the ratio of the barriers of periodic potentials to the thermal energy made it possible to obtain the explicit solution of the Smoluchowski equation in the most general case via a series expansion in this small parameter. That solution led to a wide range of analytical results on features of ratchets driven by deterministic and stochastic fluctuations,[26] in particular, on ratchet symmetry properties.[27] In this section, we present the result obtained in[25] for the current in the stochastic high-temperature on–off flashing ratchet with the sawtooth potential characterized by the barrier ΔV and the widths l and $L - l$ of the links of the "saw":

$$J = \tau_D^{-1}\frac{(\xi' - \xi)(\beta\Delta V)^3}{128(\xi'\xi z)^2}[6f_1(z, \xi) - 3f_2(z, \xi) + f_1(z, \xi)f_2(z, \xi)], \tag{12}$$

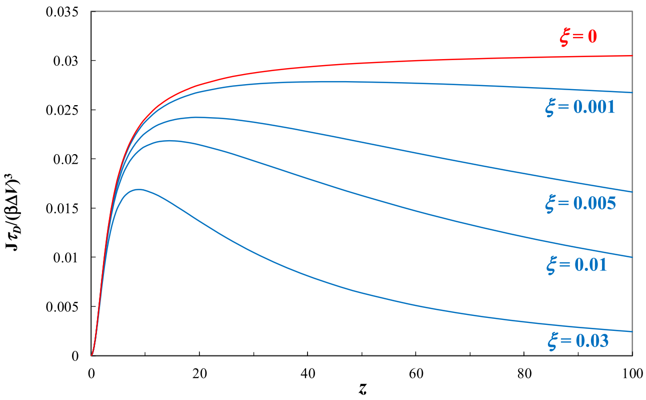

FIGURE 4 Frequency dependence of the normalized current (given by Equations (12) and (13)) in the stochastic high-temperature on–off flashing ratchet with the sawtooth potential. The curves, top-down, are in the order of increasing the parameter ξ (the upper curve, red online, corresponds to the extremely asymmetric sawtooth potential, Equation (14))

where $\tau_D = D/L^2$ is the characteristic diffusion time over the period L, $z = \sqrt{\tau_D\gamma/2}$, $\xi = l/L$, and

$$f_1(z,\xi) = 1 - \frac{\sinh z\xi \sinh z\xi'}{\xi\xi' z \sinh z},\; f_2(z,\xi) = 1 - \frac{\sinh z(\xi-\xi')}{(\xi-\xi')\sinh z},\; \xi' = 1-\xi. \tag{13}$$

In the particular case of the extremely asymmetric sawtooth potential ($\xi \to 0$), the relations (12) and (13) yield

$$J = \tau_D^{-1}\frac{(\beta\Delta V)^3}{16}\left(\frac{\cosh 2z}{4\sinh^2 z} - \frac{5}{4z}\coth z + \frac{1}{z^2}\right). \tag{14}$$

The frequency dependences of the current calculated by the relations (12) and (13) are given in Figure 4. The low-frequency limit of the current (12) leads to a linearly increasing dependence on γ, namely $J = \kappa(\beta\Delta V)^3\gamma/720$, which at $\kappa = 1$ coincides with the low-frequency limit of the expression (14). The high-frequency limits for the current in the extremely asymmetric sawtooth potential and its analogue in the potential of an arbitrary symmetry differ significantly: It follows from Equations (12) that, as $\gamma \to \infty$, $J \to \tau_D^{-1}(\beta\Delta V)^3/32 \neq 0$ ($\xi = 0$) but $J \propto \gamma^{-1}$ ($\xi \neq 0$). These differences are a consequence of the presence of jumps in the potential relief in the extremely asymmetric case.

5 | MODE OF SMALL FLUCTUATIONS

Currently, the adiabatic and high-temperature modes of motion considered in Sections 3 and 4 are the most fruitful approximations that make it possible to obtain analytical expressions for the characteristics of ratchet systems. The disadvantages of these approximations include the limitations of the obtained results with respect to the set of control parameters. For example, within the high-temperature approximation, one cannot predict the possibility of controlling the motion characteristics by changing the temperature, while within the low- and high-frequency approximations, by the fluctuation frequency.

One can avoid the above difficulties in analytical analysis of nanotransport control when using the approximation of small potential energy fluctuations.[28,29] In this approximation, the potential energy in the "±" states is written as $V_\pm(x) = u(x) \pm w(x)$, and only the fluctuations $w(x)$ relative to thermal energy is considered small, not the potential profile $V_\pm(x)$ as a whole. This allows using the perturbation theory for the Smoluchowski equation with respect to a small quantity $w'(x)$; as a result, the current can be written as[28]:

$$J = \beta^2 D^2 \int_0^L dx \rho_+(x) w'(x) \int_0^L dy S(x,y) \frac{\partial}{\partial y} w'(y)\rho_-(y),$$
$$\rho_\pm(x) = e^{\pm\beta u(x)} \Big/ \int_0^L dx e^{\pm\beta u(x)}. \tag{15}$$

Here, $\rho_-(x)$ is the equilibrium Boltzmann distribution in the stationary potential $u(x)$, and the function $S(x, y)$ is the Laplace image of the retarded Green's function of diffusion in the stationary potential relief, which satisfies the equation

$$\left[\frac{d}{dx}\hat{J}(x) + 2\gamma\right] S(x,y) = -\delta(x-y), \tag{16}$$

where

$$\hat{J}(x) = -De^{-\beta u(x)}(\partial/\partial x)e^{\beta u(x)} \tag{17}$$

is the current operator. The physical meaning of the function $S(x, y)$ in the case of a stochastic dichotomous process is that the value $-2\gamma S(x, y)$ specifies the probability density of finding a particle at the point x in the potential $u(x)$ with the lifetime $(2\gamma)^{-1}$, provided that the particle was originally placed at the point y.

An explicit analytical expression for the function $S(x, y)$ can only be obtained in the case of a piecewise linear potential relief; the solution, quite cumbersome for the case of an arbitrary asymmetry of this relief, was derived in Ref. [29]. Here, we present the result obtained in Ref. [30] for its limiting case – the extremely asymmetric sawtooth stationary component $u(x)$, defined analogously to the expression (5). The function $S(x, y)$ is then written as

$$S^{l,r}(x,y)=\frac{L}{D(\Lambda_1-\Lambda_2)\mathrm{Det}\left(\hat{1}-\hat{C}\right)}\sum_{k,m=1,2}(-1)^{m-1}a_{km}^{l,r}e^{\Lambda_k x/L-\Lambda_m y/L},$$
$$\hat{a}^l=\hat{C}-\hat{1},\ \hat{C}=\frac{1}{\Lambda_1-\Lambda_2}\begin{pmatrix}\Lambda_1e^{-\Lambda_2}-\Lambda_2e^{\Lambda_1} & \Lambda_1\left(e^{-\Lambda_1}-e^{\Lambda_2}\right)\\ \Lambda_2\left(e^{\Lambda_1}-e^{-\Lambda_2}\right) & \Lambda_1e^{\Lambda_2}-\Lambda_2e^{-\Lambda_1}\end{pmatrix},$$
$$\hat{a}^r=\hat{1}-\hat{C}^{-1},$$
$$\hat{C}^{-1}=\frac{1}{\Lambda_1-\Lambda_2}\begin{pmatrix}\Lambda_1e^{\Lambda_2}-\Lambda_2e^{-\Lambda_1} & -\Lambda_1\left(e^{-\Lambda_1}-e^{\Lambda_2}\right)\\ -\Lambda_2\left(e^{\Lambda_1}-e^{-\Lambda_2}\right) & \Lambda_1e^{-\Lambda_2}-\Lambda_2e^{\Lambda_1}\end{pmatrix},$$
$$\mathrm{Det}\left(1-\hat{C}\right)=\frac{4}{\Lambda_1-\Lambda_2}\left(\Lambda_2\sinh^2\frac{\Lambda_1}{2}-\Lambda_1\sinh^2\frac{\Lambda_2}{2}\right), \tag{18}$$

where the superscripts l and r correspond to x on the intervals $x\in(+0, y)$ and $x\in(y, L-0)$, respectively, and

$$\Lambda_j=-\frac{\alpha}{2}\left[1+(-1)^j\sqrt{1+16(z/\alpha)^2}\right],\ j=1,2,$$
$$\alpha=\beta\Delta V,\ z=\sqrt{\tau_D\gamma/2}. \tag{19}$$

The surface in Figure 5 represents the function $S(x, y)$ in the basic region of values of its arguments x and y. If the lifetime of the state with the potential $u(x)$ tends to zero ($\gamma\to\infty$), then $-2\gamma LS(x,y)\to L\delta(x-y)$. Therefore, at finite value of γ, the surface contains the line of cusp points corresponding to $x=y$. The $S(x, y)$-surface also demonstrates both the jumps at the boundaries $x=0$ and $x=L$ of the region when changing x values and continuity at the boundaries $y=0$ and $y=L$ when changing y values. Note that in the long-lived potentials ($\gamma\to 0$) the dependence on the initial particle position disappears, and the function $-2\gamma S(x, y)$ tends to the equilibrium Boltzmann distribution in the stationary potential $u(x)$ (see the expression for $\rho_-(x)$ in Equation (15)).

To illustrate the usefulness of the Green's function of the diffusion in an extremely asymmetric sawtooth potential, we present the result for the current in the flashing ratchet controlled by spatially harmonic fluctuations

$$w(x)=w\cos\left[2\pi(x/L-\lambda_0)\right]. \tag{20}$$

For the Green's function (18) of the extremely asymmetric sawtooth potential and the coordinate dependence (20) of its slight perturbation, the integrals in Equation (15) are reduced to ones of the products of exponential and trigonometric functions and can be

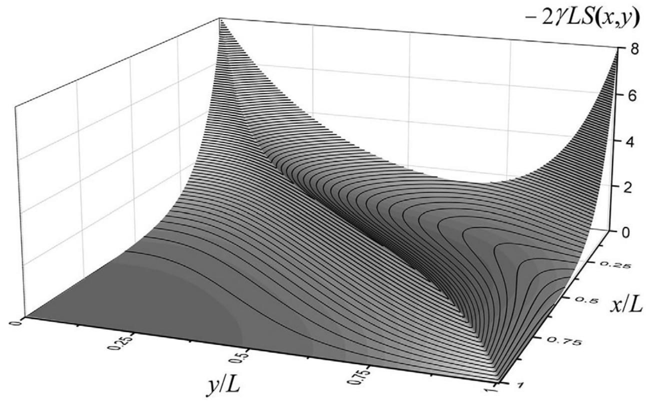

FIGURE 5 Surface plot for the dimensionless probability density $-2\gamma LS(x, y)$, Equations (18)–(19), at fixed values of dimensionless parameters α (inverse temperature) and z (inverse correlation time): $\alpha=\beta\Delta V=5$, $z=\sqrt{\tau_D\gamma/2}=2.5$

taken analytically. Simplifying the cumbersome result of the double integration leads us to the following expression:

$$J=J_0\frac{8\pi^2\alpha^2z^2}{\varphi_1^2(\alpha)\varphi_2(\alpha z)}\left\{\frac{\cosh\delta-\cosh(\alpha/2)}{\Delta(\alpha\gamma)}\varphi_1(\alpha)\left[\cos4\pi\lambda_0-1+\frac{8\pi^2\alpha^2}{\varphi_2(\alpha z)}\right]+1\right\},$$
$$\varphi_1(\alpha)=\frac{\sinh(\alpha/2)}{\alpha/2},\quad \varphi_2(\alpha,z)=\left(\Lambda_1^2+(2\pi)^2\right)\left(\Lambda_2^2+(2\pi)^2\right),$$
$$\Delta(\alpha,\gamma)=\cosh\delta\cosh\frac{\alpha}{2}-1-\frac{\alpha}{2\delta}\sinh\delta\sinh\frac{\alpha}{2},$$
$$\Lambda_{1,2}=-\alpha/2\pm\delta,\ \delta=(1/2)\sqrt{\alpha^2+16z^2}, \tag{21}$$

where $J_0=(w/\Delta V)^2\tau_L^{-1}$, $\tau_L=\zeta L^2/\Delta V$ is the characteristic sliding time over the period of the extremely asymmetric sawtooth potential, $\zeta=(\beta D)^{-1}$ is the friction coefficient.

Figure 6 depicts the frequency and temperature dependences of the current at several values of the phase shift λ_0 of the fluctuating harmonic signal relative to the stationary extremely asymmetric sawtooth profile. The frequency dependence at the fixed temperature (corresponding to $\alpha=5$) and for $\lambda_0=0.19$ is a sign-constant function, while for $\lambda_0=0.21$ and $\lambda_0=0.25$ it becomes sign-alternating with one zero point (called the stopping point). The temperature dependence at the fixed frequency $\gamma=2\tau_L^{-1}$ with the same values of the phase shift λ_0 always turns out to be sign- alternating with one stopping point. Thus, these dependencies clearly show the possibilities of controlling the motion direction by tuning the frequency and temperature. It is noteworthy that the analytical expression (21) obtained for the extremely asymmetric sawtooth potential can describe

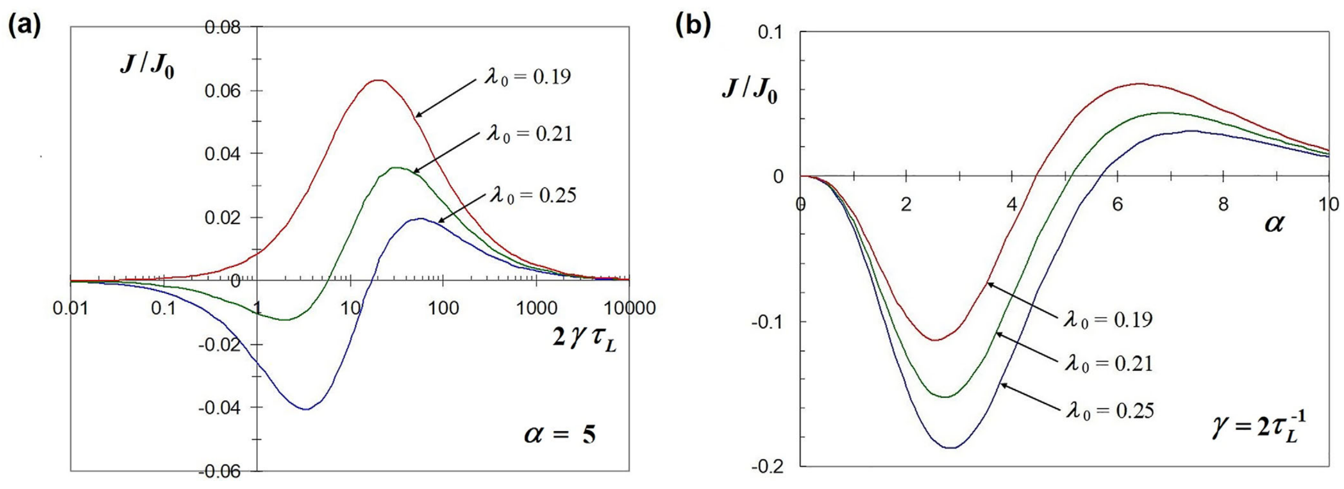

FIGURE 6 Analytical dependences (21) of the current in the ratchet with the extremely asymmetric sawtooth potential $u(x)$ on the frequency γ at the fixed value of the temperature parameter ($\alpha = 5$ (a) and on the temperature parameter α at the fixed γ value $\gamma = 2\tau_L^{-1}$ (b))

such a rich behavior of the ratchet system under consideration.

6 | DISCUSSION AND CONCLUSIONS

The piecewise linear form of the sawtooth potential led to a number of analytical results in the presence of simplifying factors which are the dichotomous process of shifting the potential by half a period, adiabatic and high-temperature modes of the motion, and the mode of slight fluctuations of the potential energy (see Sections 2–5). An important simplifying factor was the choice of the extremely asymmetric sawtooth potential (5), having only one linear section and one jump on its period (instead of two linear sections for a sawtooth potential of arbitrary symmetry). The jump leads to a number of features of the obtained solutions, one of which is clearly illustrated in Figure 4. The high-frequency asymptotics of the current has a nonzero value in the presence of jumps in the potential profile and is equal to zero when the profile is smooth. A similar regularity took place for the model with the potential shifted by half a period.[17] For the so-called rocking ratchets (they are not of interest in our review), there are even more striking manifestations of potential-profile jumps in the ratchet characteristics.[25]

To explain the appearance of the mentioned features, let us consider the limiting transition from a sawtooth potential of arbitrary asymmetry to the extremely asymmetric one. To do this, we introduce the parameter l (the sawtooth width), which tends to zero in the extremely asymmetric case. This width is associated with the time of sliding along this "saw", equal to $\tau_l = \zeta l^2/\Delta V$. As $l \to 0$, the value τ_l^{-1} tends to infinity and begins to compete with high fluctuation frequencies γ. Therefore, understanding which of the limits, $l \to 0$ or $\gamma \to \infty$, is taken first becomes utterly important. For the extremely asymmetric sawtooth potential, $l = 0$ and the high-frequency limit for the current can give a result different from the result obtained in the high-frequency limit for the potential with $l \neq 0$. Note that the sawtooth potential of arbitrary asymmetry has cusp points with jumps in the derivatives. The jumps in the applied forces lead to the features known for rocking ratchets.[25] Smoothing the cusp points eliminates these features, as was shown in particular in Ref. [21] using the example of eliminating the divergence of inertial corrections.

The short sliding time τ_l can also compete with the characteristic time $\tau_v = m/\zeta$ (m is particle mass) of relaxation in the velocity phase space. Therefore, the results obtained from the Smoluchowski equation (that is, without accounting for particle inertia) and from the Klein-Kramers equation (taking inertia into account) will differ for $l = 0$ and $l \neq 0$.[31]

There are several sources of a prominent role which a sawtooth potential plays in the ratchet theory (see, for example, Refs. [2,17,21,31–33]). First of all, only two parameters are used to define its shape: the energy barrier ΔV and the ratio l/L characterizing the potential asymmetry. Such small number of parameters is very convenient in analyzing experimental data and predicting new features of ratchet systems. Moreover, a sawtooth potential is not only a theoretical idealization, but can be realized experimentally; examples can be found in Ref. [34]. In numerous experiments on directional motion of colloidal particles, sawtooth shapes of the ratchet potential are created by means of interdigitated electrodes, deposited on the glass slides with photolithographic techniques (see, for example, Chap. 7 in Ref. [4]). In experiments for manipulating charged components within supported lipid bilayers,[35] such shape of the ratchet potential is created by a patterned bilayer.

Potential profiles having near jump-like sections can also be created experimentally. Small times of sliding over such sections can compete with other characteristic times in the system; this gives new prerequisites for controlling ratchet parameters. The possibility of the controlling has been demonstrated in Ref. [30]: Additional stopping points can appear with the fluctuation frequency changes as a result of the presence of short sliding times in the ratchet system. Summarizing, the models of ratchet systems involving extremely asymmetric sawtooth potential profiles, considered in this paper, not only make it possible to obtain the dependences of the characteristics of the ratchet systems on their parameters, but also have heuristic value for controlling directional motion of nanoparticles.

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NOMENCLATURE

"±"	Alternating states
$V_{\pm}(x)$	Periodic potential energies
$U_{\pm}(x)$	Total potential energies
F	Load force
$\rho_{\pm}(x, t)$	Probability densities
$j_{\pm}(x, t)$	Probability currents
J	Total current
D	Diffusion coefficient
$\beta = (k_{\mathrm{B}}T)^{-1}$	Inverse thermal energy
γ	Alternating frequency
W_{in}	Input power
W_{out}	Output power
η	Motor efficiency
L	Period of the potential
ΔV	Barrier of the sawtooth potential
l	Width of a link of the "saw"
κ	Asymmetry parameter
τ_D	Diffusion time
τ_l	Sliding time on length l
$u(x)$	Static part of the potential
$w(x)$	Fluctuating part of the potential

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