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Proc. R. Soc. Lond. A 1998 **454**, doi: 10.1098/rspa.1998.0153, published online 8 January 1998

A Khintchine-type version of Schmidt's theorem for planar curves

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Received 26 July 1996; accepted 19 June 1997

An analogue of the convergence part of the Khintchine–Groshev theorem is proved for planar curves obeying certain curvature conditions.

Keywords: Diophantine approximation; extremal planar curves; Khintchine's theorem

1. Introduction

Mahler's fundamental studies in the theory of transcendental numbers led him to conjecture in 1932 that for any $\varepsilon > 0$ the inequality

$$|P(x)| < H(P)^{-n-\varepsilon} \tag{1.1}$$

has at most a finite number of solutions in integer polynomials P of degree n for almost all $x \in \mathbb{R}$, where the height $H(P)$ is the maximum of the moduli of the coefficients. About 30 years later this was proved by Sprindžuk (1967). Baker (1966) subsequently proved the more general result that if the function ψ is monotonic, decreasing, strictly positive and such that $\sum_{r=1}^{\infty} \psi(r) < \infty$ then the set of those x for which

$$|P(x)| < \psi^n(H(P)) \tag{1.2}$$

holds for infinitely many P has Lebesgue measure zero. Bernik (1989) extended Baker's result by replacing the right-hand side of (1.2) with $H^{-n+1}\psi(H)$. Now by Groshev's theorem (1938) the set of points $\boldsymbol{x} \in \mathbb{R}^n$ which satisfy the inequality

$$|\boldsymbol{q} \cdot \boldsymbol{x} - p| < \psi(|\boldsymbol{q}|)/|\boldsymbol{q}|^{n-1}$$

where $|\boldsymbol{q}| = \max\{|q_1|, \ldots, |q_n|\}$, for infinitely many $\boldsymbol{q} \in \mathbb{Z}^n$ has full or zero Lebesgue measure depending on whether $\sum_r \psi(r)$ converges or diverges. Evidently Bernik (1989) is an analogue of the convergence case of Groshev's theorem and indeed is the best possible as the complementary divergence case has been proved recently (Beresnevich *et al.* 1997).

We now consider more general planar curves. Let I be an interval $[a, b]$, and let

$$\Gamma = \{(f_1(x), f_2(x)) : x \in I\},$$

be a planar curve such that f_1''', f_2''' are continuous, with Wronskian $W(f_1', f_2')$ given by

$$W(f_1', f_2') = \det \begin{pmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{pmatrix}.$$

Further, let

$$F(x) = q_0 + q_1 f_1(x) + q_2 f_2(x),$$

where $\boldsymbol{q} = (q_0, q_1, q_2) \in \mathbb{Z}^3$.

Schmidt (1964) proved the remarkable result that if the Wronskian $W(f_1', f_2')$ is non-zero almost everywhere (equivalent to the curvature being non-zero almost everywhere), then for any positive ε, the inequality

$$\|q_1 f_1(x) + q_2 f_2(x)\| < q^{-2-\varepsilon}, \tag{1.3}$$

where for each real θ, $||\theta|| = \min_{k\in\mathbb{Z}} |\theta - k|$ and $q = \max\{|q_1|, |q_2|\}$, has only finitely many solutions $(q_1, q_2) \in \mathbb{Z}^2$ for almost all $x \in \mathbb{R}$. This inequality can also be expressed in the essentially equivalent form in which

$$|F(x)| < H(F)^{-2-\varepsilon}, \tag{1.4}$$

where $H(F) = \max\{|q_0|, |q_1|, |q_2|\}$, holds for infinitely many F for almost no $x \in I$. In other words, the planar curve Γ is extremal (see Sprindzuk 1979). For $\varepsilon = 0$ it follows from Dirichlet's theorem that every point satisfies the inequality. Subsequently, Baker (1978) generalized this result by replacing inequality (1.4) by

$$|F(x)| < \psi(H(F))^2,$$

where the function ψ is monotone decreasing and $\sum_{q=1}^{\infty} \psi(q) < \infty$. By combining some estimates for differentiable functions and using Schmidt's theorem (1964) (for step 3 below), we obtain the following Khintchine-type result which is sharper than Baker (1978).

Theorem 1.1. *Let f_1, f_2 be two functions with continuous third derivatives and such that $W(f_1', f_2')$ is non-zero almost everywhere. Suppose that ψ is a decreasing positive function and $\sum_r \psi(r) < \infty$. Then the set $\mathcal{S}(\psi)$ of points $x \in \mathbb{R}$ for which the inequality*

$$|F(x)| = |q_0 + q_1 f_1(x) + q_2 f_2(x)| < H(F)^{-1}\psi(H(F)), \tag{1.5}$$

holds for infinitely many F (or infinitely many $(q_0, q_1, q_2) \in \mathbb{Z}^3$) has Lebesgue measure zero.

In view of Beresnevich *et al.* (1997), this theorem is the best possible. Note that if the curvature vanishes on an interval, the measure of the set of points for which $|F(x)| < H^{-n+1}\psi(H(F))$ can be positive.

To avoid expressing all the different constants (which do not affect the results) the Vinogradov symbols ($\ll$ and $\gg$) will be used ($A \ll B$ means that there exists a positive constant c such that $A \leqslant cB$ with a similar definition for $A \gg B$). If $A \ll B$ and $A \gg B$ then A is comparable to B, written $A \asymp B$. As $\sum_r \psi(r)$ converges, we can assume without loss of generality that $\psi(N) \ll N^{-1}$, where N is a sufficiently large integer. We now state some technical lemmas. The first is due to Pyartli (1970). The Lebesgue measure of a set X in $\mathbb{R}$ is denoted by $|X|$.

Lemma 1.2. *Given positive real numbers δ, ε and a natural number n, let $f : [a, b] \to \mathbb{R}$ be a function with $|f^{(n)}(x)| \geqslant \delta$ for each $x \in [a, b]$. Then*

$$|\{x \in [a, b] : |f(x)| < \varepsilon\}| < C(\varepsilon/\delta)^{1/n},$$

where C is an absolute constant.

The next lemma is proposition 4 in Beresnevich & Bernik (1994).

Lemma 1.3. *When $\gamma > 0$, the two inequalities*

$$|F(x)| < H(F)^{-1-\gamma}, \quad |F'(x)| < H(F)^{-\gamma/2}$$

hold simultaneously for infinitely many F for almost no real x.

Finally, we state lemma 7 in Baker (1966).

Lemma 1.4. *For each positive integer N, denote by $U(N)$ a finite set of closed intervals. Let $K(N)$ denote a subset of $U(N)$ such that for each interval $I \in K(N)$ there exists an interval $J \neq I$, $J \in U(N)$, for which $|I \cap J| \geqslant \frac{1}{2}|I|$. Let $V(N)$ denote the union of the points of the intervals I of $K(N)$ and let $v(N)$ denote the union of the intervals $I \cap J$. Further, let W and w denote the set of points contained in infinitely many $V(N)$ and in infinitely many $v(N)$ respectively. Then if w has measure zero so does W.*

2. The proof

Let $\sigma(F)$ be the set of x in the interval I such that (1.5) holds, i.e.

$$\sigma(F) = \{x \in I : |F(x)| < H(F)^{-1}\psi(H(F))\}.$$

Then $\sigma(F)$ is the union of a finite number of open intervals and we can suppose without loss of generality that $\sigma(F)$ is an interval. Using this notation $\mathcal{S}(\psi)$ can be written in the form

$$\mathcal{S}(\psi) = \bigcap_{N=1}^{\infty} \bigcup_{F:H(F)\geqslant N} \sigma(F) \subset \bigcup_{F:H(F)\geqslant N} \sigma(F).$$

This is a standard covering of the set; four different cases of F will be considered.

Let $\varepsilon > 0$ be a sufficiently small real number. We proceed to make some simplifications. Clearly we may assume that $H(F)$ is sufficiently large. By the implicit function theorem, we can take F to be of the form

$$F(x) = q_2 f(x) + q_1 x + q_0$$

on a suitable interval $[a, b]$. As in Baker (1978) the problem can be reduced to considering those x for which

$$|f''(x)| > c \tag{2.1}$$

for $c > 0$ since the inequality complementary to (2.1) holds only for sets with small measure (this follows from the inequality $|W(f_1', f_2')| > 0$ holding almost everywhere). Further, we can assume that for x in $\sigma(F)$

$$c_1 H(F) < |F''(x)| < c_2 H(F). \tag{2.2}$$

From (1.5), one of q_1, q_2 must be $\gg H(F)$. The upper bound for $|F''(x)| = |q_2 f''(x)|$ in (2.2) follows from the continuity of f'' on $[a, b]$. Using this fact and $|F''(x)| = |q_2 f''(x)| \gg q_2$, it is not difficult to show that if the lower bound of (2.2) does not hold, then $|F'(x)| \gg H(F)$. The case of large first derivative is the 'transverse' case, that is the vector (q_1, q_2) is almost parallel to the curve $\Gamma = \{(x, f(x) : x \in I\}$. The set of x for which (1.5) holds for one vector $\boldsymbol{q} \in \mathbb{Z}^3$ is the length of curve which is within a distance $H(F)^{-2}\psi(H(F))$ of the line $\{(x_1, x_2) \in \mathbb{R}^2 : q_1 x_1 + q_2 x_2 + q_o = 0\}$. For the transverse case this line is almost normal to the curve so the intersection is $\ll H(F)^{-2}\psi(H(F))$ and the theorem readily follows from the convergence of $\sum_r \psi(r)$.

We now consider various ranges for $|F'(x)|$. Note that $|F'(x)| = |q_2 f'(x) + q_1| \ll H(F)$. It follows that if $H(F)$ is sufficiently large and $\varepsilon > 0$ then $|F'(x)| < H(F)^{1+\varepsilon}$. The inequality

$$|F'(x)| = |q_2 f'(x) + q_1| \leqslant H(F)^{-2-\varepsilon}$$

holds only for a set of small Lebesgue measure, and the set of those x for which it holds infinitely often is of measure zero by Khintchine's theorem (see Sprindžuk 1979). Thus we can assume without loss of generality that

$$H(F)^{-2-\varepsilon} < |F'(x)| \leqslant H(F)^{1+\varepsilon}. \tag{2.3}$$

For details see remark 5 in Beresnevich & Bernik (1994). Now the exponent of $H(F)$ in (2.3) is divided into ranges of length ε. Suppose that $\ell \in \mathbb{Z}$ and

$$H(F)^{(\ell-1)\varepsilon} < |F'(x)| < H(F)^{\ell\varepsilon}.$$

Evidently, from (2.3) it is only necessary to consider $-2\varepsilon^{-1} < \ell \leqslant 1 + \varepsilon^{-1}$.

It follows from (2.2) that $q_2 \gg q_1$. Denote by $\mathfrak{F}(N)$ the class of functions F with $q_2 = N$ and with

$$H(F) = \max\{|q_0|, |q_1|, |q_2|\} \ll N,$$

so that $H(F) \asymp N$. Let $\mathfrak{F}(N,\ell)$ denote the subclass of functions $F \in \mathfrak{F}(N)$ with fixed ℓ. In what follows, if $\sigma(F)$ is not empty, $\alpha \in \sigma(F)$ will be given by

$$|F'(\alpha)| = \inf_{x\in\sigma(F)} |F'(x)|.$$

The proof now falls into four steps, in each of which a different range for the exponent ℓ of $F'(\alpha)$ is considered.

Step 1, $1/(2\varepsilon)+1 < \ell < 1/\varepsilon+1$. Denote by $\mathfrak{F}(N,\ell,q_1)$ the subclass of functions $F(x) = Nf(x) + q_1 x + q_0$ in $\mathfrak{F}(N,\ell)$ with q_1 fixed. Let $\sigma_1(F)$ be the set of $x \in [a,b]$ satisfying

$$|x-\alpha| \leqslant \frac{1}{4|F'(\alpha)|}.$$

For large N and each F, the interval $\sigma_1(F)$ lies in the interval $[a-1, b+1]$. By Taylor's formula,

$$F(x) = F(\alpha) + F'(\alpha)(x-\alpha) + \tfrac{1}{2}F''(\xi)(x-\alpha)^2,$$

where $\xi \in [x,\alpha]$, whence for large N the inequalities

$$|F(\alpha)| < \tfrac{1}{8}, \quad |\tfrac{1}{2}F''(\xi)(x-\alpha)^2| \ll N^{1-2(\ell-1)\varepsilon} < \tfrac{1}{8},$$

hold (since $\alpha \in \sigma(F)$) and we obtain

$$|F(x)| < \tfrac{1}{2}.$$

It follows that intervals of the form $(\alpha - 1/(4|F'(\alpha)|), \alpha + 1/(4|F'(\alpha)|))$ cannot intersect for different $F \in \mathfrak{F}(N,\ell,q_1)$, since if $x \in \sigma_1(F_1) \cap \sigma_1(F_2)$ where F_1, F_2 are in $\mathfrak{F}(N,\ell,q_1)$, then

$$1 \leqslant |F_1(x) - F_2(x)| < \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

Thus

$$\sum_{F\in\mathfrak{F}(N,\ell,q_1)} |\sigma_1(F)| \leqslant (b-a)+2$$

and

$$\sum_{F\in\mathfrak{F}(N,\ell)} |\sigma(F)| \ll \sum_{|q_1|\ll N}\ \sum_{F\in\mathfrak{F}(N,\ell,q_1)} |\sigma(F)|$$
$$\ll \sum_{|q_1|\ll N}\ \sum_{F\in\mathfrak{F}(N,\ell,q_1)} |\sigma_1(F)|\psi(N)N^{-1} \ll \psi(N),$$

since from lemma 1.2 (with $n=1$) we have

$$\frac{|\sigma(F)|}{|\sigma_1(F)|} \ll \frac{N^{-1}\psi(N)|F'(\alpha)|^{-1}}{|F'(\alpha)|^{-1}} \ll N^{-1}\psi(N).$$

Hence

$$\sum_F |\sigma(F)| = \sum_{N=1}^{\infty}\ \sum_{\ell=[1+1/(2\varepsilon)]}^{[1+1/\varepsilon]}\ \sum_{F\in\mathfrak{F}(N,\ell)} |\sigma(F)| \ll \sum_{N=1}^{\infty}\psi(N) < \infty.$$

The proof of step 1 now follows from the Borel–Cantelli lemma.

Step 2, $1<\ell\leqslant 1/(2\varepsilon)+1$.
Let

$$\sigma_2(F) = \{x : |x-\alpha| < N^{-1}|F'(\alpha)|^{-1}\}. \tag{2.4}$$

It is evident (by lemma 1.2 with $n=1$) that $\sigma(F)\subset\sigma_2(F)$. The Taylor series expansion for $F(x)$ at α is

$$F(x) = F(\alpha) + F'(\alpha)(x-\alpha) + \tfrac{1}{2}F''(\xi)(x-\alpha)^2.$$

But $|F(\alpha)| \ll N^{-1}$ (as $\alpha\in\sigma(F)$); by the definition of $\sigma_2(F)$ the second term is also $\ll N^{-1}$. The last term $F''(\xi)(x-\alpha)^2$ can be shown to be $\ll N^{-1}$ using (2.4) and the range of ℓ. Therefore

$$|F(x)| \ll N^{-1}.$$

In order to estimate the sum $\sum_F |\sigma(F)|$ more efficiently, essential and inessential domains, introduced by Sprindžuk (1967), are used. The interval $\sigma_2(F)$ is called *inessential* if there exists a function $G\in\mathfrak{F}(N,\ell)$ such that

$$|\sigma_2(F)\cap\sigma_2(G)| \geqslant \tfrac{1}{2}|\sigma_2(F)|$$

and *essential* otherwise.

If the interval $\sigma_2(F)$ is inessential then on the interval $I_1=\sigma_2(F)\cap\sigma_2(G)$ the difference

$$R(x) = F(x) - G(x) = b_1x + b_0$$

satisfies

$$|R(x)| \ll N^{-1}, \quad H(R) \ll N.$$

As the length of the interval I_1 is $\asymp N^{-1}|F'(\alpha)|^{-1}$ and R is linear, the height $H(R)$ of R satisfies

$$H(R) \ll |F'(\alpha)|$$

and

$$|R(x)| \ll H(R)^{-1-\varepsilon'},$$

for some $\varepsilon' > 0$. By the one-dimensional case of Khintchine's theorem the last inequality holds for infinitely many R only on a set of measure zero. At the moment we have only proved that the set of x in the intersections of inessential intervals is of measure zero. However, lemma 1.4 extends the result to the whole interval.

If $\sigma_2(F)$ is an essential interval, then every point $x \in [a-1, b+1]$ belongs to no more than three essential intervals and hence

$$\sum_{F \in \mathfrak{F}(N,\ell)} |\sigma_2(F)| < 3(b-a+2).$$

Since $|\sigma(F)| \ll |\sigma_2(F)|\psi(N)$ (by lemma 1.2 with $n=1$) it follows that

$$\sum_{F \in \mathfrak{F}(N,\ell)} |\sigma(F)| \ll \psi(N) \sum_{F \in \mathfrak{F}(N,\ell)} |\sigma_2(F)| \ll \psi(N).$$

But the series $\sum_{N=1}^{\infty} \psi(N)$ converges, whence by the Borel–Cantelli lemma, the set of x falling into infinitely many essential intervals $\sigma_2(F)$ has zero measure.

Step 3, $1 - 1/(4\varepsilon) < \ell \leqslant 1$.

Let $\mathfrak{F}_n(N, \ell, k)$ be the set of polynomials with $k^2 < q_2 = N \leqslant (k+1)^2$, $|q_1|$, $|q_o| \ll |q_2|$. Thus the cardinality of $\mathfrak{F}_n(N, \ell, k)$ is $\asymp k^5$. Let

$$\sigma_3(F) = \{x : |x - \alpha| < k^{-3}|F'(\alpha)|^{-1}\}.$$

Then $\sigma(F) \subset \sigma_3(F)$. As in step 2 the Taylor expansion of F at α is

$$F(x) = F(\alpha) + F'(\alpha)(x-\alpha) + \tfrac{1}{2}F''(\xi)(x-\alpha)^2$$

and for $x \in \sigma_3(F)$,

$$|F(x)| \ll k^{-3}.$$

Similarly, the sets $\sigma_3(F)$ will be divided into essential and inessential intervals. First assume that $\sigma_3(F)$ is inessential, that is there exist F_1, F_2 in $\mathfrak{F}_n(N, \ell, k)$ such that

$$|\sigma_3(F_1) \cap \sigma_3(F_2)| \geqslant (\tfrac{1}{2})|\sigma_3(F_1)|.$$

Let $R(x) = F_1(x) - F_2(x) = b_2 f(x) + b_1 x + b_0$. Then it is evident that $|b_2| \ll k$ and on the interval $\sigma_3(F_1) \cap \sigma_3(F_2)$ that $|R(x)| \ll k^{-3}$. Also $R'(x) = b_2 f'(x) + b_1$ and as $|F'(x)| < N^{\varepsilon}$ this implies that $|b_1| \ll k$. This and the fact that $|R(x)|$ is small implies that $|b_0| \ll k$. Thus $H(R) \ll k$. It follows from Schmidt's theorem that the set of x for which these results hold for infinitely many R is of measure zero. Again lemma 1.4 can be used to extend this to the whole interval $\sigma_3(F)$.

Now assume that $\sigma_3(F)$ is an essential interval. As before

$$\sum_{F \in \mathfrak{F}_n(N,\ell,k)} |\sigma_3(F)| \ll 1.$$

Also $|\sigma(F)| \leqslant N^{1/2}\psi(N)|\sigma_3(F)|$. Therefore

$$\begin{aligned}\sum_{F \in \mathfrak{F}_n(N,\ell,k)} |\sigma(F)| &\ll \sum_{F \in \mathfrak{F}_n(N,\ell,k)} N^{1/2}\psi(N)|\sigma_3(F)| \\ &\ll \sum_{F \in \mathfrak{F}_n(N,\ell,k)} (k+1)\psi(k^2)|\sigma_3(F)| \ll (k+1)\psi(k^2).\end{aligned}$$

Thus it remains to prove that $\sum_{k=1}^{\infty}(k+1)\psi(k^2)$ converges. Obviously for $k \geqslant 2$

$$(k+1)\psi(k^2) \leqslant 2k\psi(k^2) \leqslant \sum_{r=(k-1)^2+1}^{k^2} \psi(r) + \psi((k-1)^2),$$

giving

$$\sum_{k=1}^{\infty}(k+1)\psi(k^2) \leqslant \sum_{r=1}^{\infty}\psi(r) + \sum_{r=1}^{\infty}\psi(r^2),$$

which is convergent by hypothesis. As before the proof of this step follows from the Borel–Cantelli lemma.

Step 4, $-2\varepsilon^{-1} < \ell \leqslant 1 - 1/(4\varepsilon)$.

Since $\psi(N) \ll 1/N$, it follows from (1.5) that the system of inequalities

$$|F(x)| < N^{-2}, \quad |F'(x)| < N^{-1/4+\varepsilon},$$

holds. But the set of x which satisfy this system is contained in the set of x satisfying the system in lemma 1.3 and thus $|\mathcal{S}(\psi)| = 0$ proving theorem 1.1.

Note that the measure is not estimated explicitly. Thus we cannot use regular systems to obtain the Hausdorff dimension of $\mathcal{S}(\psi)$ in contrast with Baker (1978).

Much of the work for this paper was carried out while the first author was visiting York last year and he is grateful to the department for its hospitality and to the Royal Society for an ex-quota award to fund the visit. The second author is supported by the EPSRC. Finally, thanks are due to the referees for their helpful comments.

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