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APPLICATION OF FRACTIONAL CALCULUS TO MODELLING AND ANALYSIS OF ECONOMIC PROCESSES

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It is proposed a short survey of modern generalizations of the classical economic models based on the application of fractional calculus. Three main types of such generalizations are considered, namely, continuous deterministic models in form of the fractional differential equations, stochastic models using fractional analogs of the Brownian motion, and discrete models handled by applying such methodology of analysis of time series as ARFIMA.

***Keywords*:** economic models with fractional derivatives; differential equations of fractional order; fractional Brownian motion; ARFIMA.

ПРИМЕНЕНИЕ ДРОБНОГО ИСЧИСЛЕНИЯ ДЛЯ МОДЕЛИРОВАНИЯ И АНАЛИЗА ЭКОНОМИЧЕСКИХ ПРОЦЕССОВ

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В статье предлагается краткий обзор современных обобщений классических экономических моделей, основанных на применении дробного исчисления. Выделены три основных типа таких обобщений, а именно, непрерывные детерминированные модели в форме дифференциальных уравнений дробного порядка, стохастические модели, связанные с использованием дробных аналогов броуновского движения, а также дискретные модели, изучаемые с использованием такой методологии анализа временных рядов как ARFIMA.

***Ключевые слова*:** экономические модели с дробными производными; дифференциальные уравнения дробного порядка; дробное броуновское движение; ARFIMA

This article presents a survey of the recent results of application of the fractional type approaches for the modeling and analysis of various economical phenomenon. From one side, the modern application of the fractional calculus (FC) in the study of economical process is due to the recent rapid development of the technique of the derivatives of non-integer order. The concept of Fractional Calculus is an old one and developed in the works by Leibniz, Fourier, Laplace, Liouville, Riemann, Sonine, Letnikov, Grünwald, Marchaud, Weyl and Riesz (see, e. g. [1; 2]). However during the last three decades the majority of FC publications become very important for modelling of a broad class of systems and processes using either the FC operators or the so-called fractional ordinary or partial differential equations. From the other side, a successful application of the fractional-order derivatives is due to their special features, in particular, non-locality and memory of power-law type (cf. [3–4]). These properties are applicable at the study of economic processes with non-locality and economic processes with a long or short memory [5].

Formally speaking most of the fractional models in economics can be of two types, namely, continuous fractional economic models and discrete fractional economic models. Continuous fractional economic models are formulated either in the deterministic form as certain fractional differential equations containing main types of fractional derivatives or in the stochastic form using e. g. fractional Brownian motion approach.

As an example of continuous fractional economic model we consider a generalization of the growth model which in the standard setting describes the dependent of the volume of production (the output) $Y(t)$ on the net investment $I(t)$. In this generalization [5, Ch. 10] a power-law memory is taken into account. To do this it is used the left-sided Dzherbashian-Caputo fractional derivative of order $\alpha > 0$ with respect to time (see, e. g., [6]):

$$\left({}^{DC}D_{0+}^{\alpha}Y\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{Y^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \qquad (1)$$

where $n-1 < \alpha \leq n$, $Y^{(n)}(\tau)$ is the derivative of integer order $n \in \mathbb{N}$ of the function $Y(\tau)$ with respect to τ such that $0 < \tau < t < T$.

In order for the operator to exist, it is necessary that the function $Y(\tau)$ should have integer-order derivatives up to $(n-1)$-th order, which are continuous functions on the interval [0, T], and the derivative $Y^{(n)}(\tau)$ is integrable on the interval [0, T]. Assuming that the amount of investment $I(t)$ is a fixed part of income $PY(t)$, we obtain

$$I(t) = mPY(t), \qquad (2)$$

where m is the norm of the net investment ($0 < m < 1$), which describes a part of income that is spent on the net investment.

Using the equation of the accelerator with power-law memory

$$\left({}^{DC}D_{0+}^{\alpha}Y\right)(t) = \frac{1}{v(\alpha)}I(t), \qquad (3)$$

we get by accounting (2) a fractional generalization of the standard growth model in the form

$$\left({}^{DC}D_{0+}^{\alpha}Y\right)(t) = \frac{mP}{v(\alpha)}Y(t). \qquad (4)$$

Fractional equation (4) takes into account one-parameter memory with power-law fading.

Several continuous fractional models accounting different aspects of memory effects are presented in [5]. We can mention the fractional generalization of the Harrod-Domar growth model which can be considered as one-sectoral form of the dynamic Leontief model; a generalization of Leontief dynamic inter-sectoral model which takes into account the memory effects; a new economic model of the dynamics of the market prices, which is a generalization of the Evans models dealing with an equilibrium price in a market (see also [7]); a new Cagan type model of inflation with memory; an economic model of natural growth in a competitive environment with power-law memory which is a generalization of the logistic growth model; a generalization of the Kaldor-type model of business cycles and its special cases based on the Van der Pol equation; a generalization of the standard Solow-Lucas growth model for closed economy without capital depreciation that is considered by R. E. Lucas (the suggested model with memory is described by nonlinear fractional differential equation); a generalized Lucas model of learning with memory. Several other fractional type economic models are presented in books and articles from the extended list of references in [5]. Recent years several attempts were done to understand other economic phenomenon, e. g. in [8] (see also references therein) it is considered the linear fractional differential equation for bank resource allocation and financial risk management.

An interest to these subjects as well as too many other situations in financial mathematics and actuarial science are mostly modelling by using stochastic approaches (see, e. g. [9–11]). Thus in [9] it is proposed a mixed fractional Brownian motion version of a well-known credit risk pricing structural model: the Merton model. In this model it is assumed that the firm has only issued zero coupon bonds with maturity T and total face value L, that default may happen only at maturity. It is denoted by $M(V_T)$ and $N(V_T)$ the prices in T of a defaultable zero coupon bond and the equity respectively. In case of default bondholders are assumed to have absolute priority, i. e. bond value at time T is $M(V_T) = min(L, V_T)$ and the equity is simply a call option, $N(V_T) = max(V_T - L, 0)$. Whereas the original model assumes a Geometric Brownian motion for the firm value, in [9] it is considered the following dynamics for Vt:

$$dV_t = \mu V_t dt + dX_t, \tag{5}$$

where Xt denotes a mixed fractional Brownian motion ($mfBm$) and the stochastic integration is divergence-type.

The so called mixed fractional Brownian motion Xt is linear combination of a Brownian motion W_t and an independent fractional Brownian motion B_t^H with Hurst parameter $0 < H < 1$ defined on the same probability space (Ω, F, P), i. e.,

$$X_t = \sigma B_t^H + \varepsilon W_t, \tag{6}$$

where σ and ε are two real constants such that $(\sigma, \varepsilon) \neq (0, 0)$.

Mixed fractional Brownian motions form a special class of long memory processes when Hurst parameter $H > 1/2$.

The discrete models (which are characteristic for many economic situations) are mostly handled by using specialized methodologies which are generalizations of the classical approaches. Thus in [12] it is described a special phenomenon appeared in many time series, namely a long memory feature. It is introduced a new model ARFIMA (Auto Regressive Fractional Moving Average) – FIGARCH (Fractionally Integrated Generalized Auto Regressive Conditionally Heteroscedastic)/FIAPARCH, which allows the presence of a long memory as in the return series of financial assets as in the volatility series. Different reasons for presence of the long memory are considered and their influence on the hypothesis of the effective market (see also [5, Ch. 29]).

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