УДК 519.872.5

G-СЕТЬ КАК СТОХАСТИЧЕСКАЯ МОДЕЛЬ СЕТИ ПЕРЕДАЧИ ДАННЫХ

Т. В. РУСИЛКО[1)]

[1)]*Гродненский государственный университет им. Янки Купалы, ул. Ожешко, 22, 230023, г. Гродно, Беларусь*

Целью статьи является математическое моделирование сети передачи данных, состоящей из оконечных устройств, соединенных устройствами маршрутизации и каналами передачи данных. В качестве стохастической модели предлагается использовать замкнутую экспоненциальную G-сеть массового обслуживания с однолинейными узлами, в которой циркулируют положительные заявки и сигналы. Модель исследуется в асимптотическом случае при большом числе обрабатываемых заявок. Применяемый математический подход позволяет рассчитать основные статистические характеристики марковского процесса, описывающего состояние модели, а также аналитически восстановить его нормальную функцию плотности распределения вероятностей на основе метода гауссова приближения. Результаты исследования могут быть полезны для расчета показателей производительности сети передачи данных как в переходном, так и в стационарном режиме, а также для проектирования и оптимизации сетей передачи данных.

Ключевые слова: G-сеть; сеть передачи данных; сеть массового обслуживания; асимптотический анализ; гауссово приближение; математическое моделирование.

Благодарность. Работа выполнена в рамках государственной программы научных исследований «Конвергенция-2025» (подпрограмма «Математические модели и методы», задание 1.6.01).

THE G-NETWORK AS A STOCHASTIC DATA NETWORK MODEL

T. V. RUSILKO[a]

[a]*Yanka Kupala State University of Grodno, 22 Azheshka Street, Grodna 230023, Belarus*

The primary objective of this paper is the mathematical modelling of a data network consisting of terminal devices connected by routing devices and data links. A closed exponential G-network of single-server queueing nodes with positive requests and signals is used as a stochastic model. The model is studied in the asymptotic case of a large number of requests being processed. The mathematical approach used makes it possible to calculate the main statistical characteristics of a Markov process describing the model state, as well as to reconstruct analytically its normal probability density function based on the Gaussian approximation method. The results of the study allow us to analyse the data network performance in both transient and steady state. The areas of implementation of the research results are the pre-design of data networks and solving problems of their optimisation.

Keywords: G-network; data network; queueing network; asymptotic analysis; Gaussian approximation; mathematical modelling.

Acknowledgements. The research was supported by the state programme of scientific research «Convergence-2025» (subprogramme «Mathematical models and methods», assignment 1.6.01).

Образец цитирования:
Русилко ТВ. G-сеть как стохастическая модель сети передачи данных. *Журнал Белорусского государственного университета. Математика. Информатика.* 2023;2:45–54 (на англ.).
https://doi.org/10.33581/2520-6508-2023-2-45-54
EDN: QCCPXH

For citation:
Rusilko TV. The G-network as a stochastic data network model. *Journal of the Belarusian State University. Mathematics and Informatics.* 2023;2:45–54.
https://doi.org/10.33581/2520-6508-2023-2-45-54
EDN: QCCPXH

Автор:
Татьяна Владимировна Русилко – кандидат физико-математических наук, доцент; доцент кафедры фундаментальной и прикладной математики факультета математики и информатики.

Author:
Tatiana V. Rusilko, PhD (physics and mathematics), docent; associate professor at the department of fundamental and applied mathematics, faculty of mathematics and informatics.
tatiana.rusilko@gmail.com
https://orcid.org/0000-0002-4880-0619

Introduction

To date, the development of technology has led to the widespread use of systems that provide parallel and decentralised data processing. Examples of such systems are multiprocessor devices, distributed databases, grid systems and data networks. A characteristic feature of these systems is the set of incoming tasks that are quite simple to process. These tasks come to the system nodes, requesting resources for processing. The required transformations are performed using resources, after which the task is considered completed and the resources are released. Due to the peculiarities of such systems, it is necessary to create new and modify existing methods for their analysis and for solving the problems of increasing their efficiency.

The transfer and processing of data is the field of activity of a large number of companies. Network systems and data processing applications are ubiquitous, so the study of the functioning of these systems is relevant. Various mathematical models can be used for research, including models belonging to the queueing theory. Queueing networks are effective mathematical models for studying discrete probabilistic systems with a network-like structure. A queueing network is a collection of interdependent queueing systems (nodes) that provides transfer and processing of requests. The object of investigation in this paper is a stochastic data network model in the form of a G-network.

G-networks are generalised queueing networks of queueing nodes with several types of requests: positive requests, negative requests and, in some cases, triggers. Negative requests and triggers are not serviced, so they are identified as signals. When a negative request arrives at a node, one or a group of positive requests is removed or «killed» in a non-empty queue, while the queued trigger displaces requests and moves positive requests from one node to some other node. G-networks were first introduced by E. Gelenbe and have been studied in a steady state since the 1990s [1–3]. Their field of application is modelling computing systems and networks, evaluating their performance, modelling biophysical neural networks, pattern recognition tasks and etc. [4–7]. More details on the practical use of G-networks with signals are described in work [8].

The purpose of this paper is the mathematical modelling and efficiency analysis of the data network using a closed exponential G-network with signals. An asymptotic analysis of the model is carried out, which implies an approximation method of the queueing network study under the assumption of a large but limited number of requests [9–11]. The mathematical approach used in this article is based on a discrete model of a continuous Markov process and the theory of diffusion approximation of a Markov process [12; 13].

Model description. Formulation of the problem

The focus of this paper is the data network consisting of terminal devices, connected by routing devices and communication channels (data links). The function of terminal devices is the transfer and reception of data, as they are communication endpoints. Each terminal or routing device has many inputs and outputs. Each of the communication channels has one input and one output, which are connected to the inputs and outputs of the devices: they provide data transfer. Data are transmitted over the network in the form of discrete packets. The bandwidth of data links is limited. Network devices and channels process data packets at a limited rate.

In general, a payload (information useful to the user), a malicious code (malware) and a service information can be transmitted over data networks. By service information we mean commands that provide load balancing between devices. The load balancing is the process of distributing a set of packets over a set of network units, with the aim of making their overall processing more efficient and avoiding overloading some units.

The problem of mathematical modelling of such a data network can be solved using a G-network with signals. As a model of a data network, we will use a closed exponential G-network, consisting of n queueing nodes S_i, $i = \overline{1, n}$, and a fictitious request source S_0. The node S_0 plays the role of an external environment. Requests in the G-network correspond to data packets transmitted over the data network, positive requests are assigned to payload, signals are assigned to malware and service information. Assume that K homogeneous requests circulate in the G-network. Exponential single-server nodes S_i, $i = \overline{1, n}$, correspond to the terminal and routing devices, as well as network data links. The fictitious system S_0 has K servers.

Each data packet can be in one of the following states corresponding to G-network nodes with the same number:

- S_0 – the data packet is in an external environment outside the data network;
- S_i – the data packet is in one of the devices or data links, $i = \overline{1, n}$.

The transition of a request from the node S_0 to the node S_i, $i = \overline{1, n}$, corresponds to the arrival of a packet in the network. The arrival request flow is divided into a flow of positive requests and signals. Requests arrive from the outside following a Poisson process with the rate $\lambda_0 k_0$, λ_0 is the parameter, k_0 is the number of requests in the node S_0. The probability of payload packet arriving at the time interval $[t, t + \Delta t]$ is $\lambda_0 k_0 p_{0i}^{+} \Delta t + o(\Delta t)$, the arrival probability of packet containing malicious code or service information is $\lambda_0 k_0 p_{0i}^- \Delta t + o(\Delta t),\ i = \overline{1, n}$, $\sum_{i=1}^{n} \left(p_{0i}^+ + p_{0i}^- \right) = 1$. A payload packet transfers from S_i to S_j without modification with the probability p_{ij}^+, transfers from S_i to S_j as a packet containing malicious code or service information with the probability p_{ij}^-, or leaves the network with the probability $p_{i0} = 1 - \sum_{j=1}^{n} \left(p_{ij}^+ + p_{ij}^- \right),\ i,\ j = \overline{1, n}$.

All queueing nodes S_i, $i = \overline{1, n}$, are single-server, the waiting buffer is unlimited. The service time of positive requests is exponentially distributed with the service rate μ_i, $i = \overline{1, n}$. Requests are served according to the FIFO rule (first in first out). Signals arriving at a node are not served by the node servers. A signal arriving at the node S_i either instantly moves a positive request from the system S_i to the system S_j with the probability q_{ij}, note that in this case the signal is called a trigger, or destroys a positive request located at the same node S_i with the probability $q_{i0} = 1 - \sum_{j=1}^{n} q_{ij}$ and immediately leaves the network.

The state of this G-network at the time t is represented by a random process

$$k(t) = \left(k_1(t),\ k_2(t),\ \ldots,\ k_n(t) \right),$$

where $k_i(t)$ is the number of requests (data packets) in the node S_i at the time t, $0 \le k_i(t) \le K, i = \overline{1, n}, t \in [0, +\infty)$. It is obvious that the number of requests serving in the G-network at the time t is $\sum_{i=1}^{n} k_i(t) = K - k_0(t)$. The allocation of data packets according to possible states at the time t fully describes the state of the data network at that time. Accordingly, the allocation of requests by queueing nodes completely determines the state of the G-network used as the data network model. Taking into account the above-described, the process $k(t)$ is a continuous-time Markov process on the finite state space.

Using the technique described in works [9–16], it is possible to derive a set of differential equations for the main statistical characteristics of a random process $k(t)$ in the asymptotic case of a large number of requests.

Asymptotic analysis of the network model

The discrete (discontinuous-state) Markov process $k(t)$ is used to determine the state of the G-network under study. In this paper, the passage to the limit from a Markov chain $k(t)$ to a continuous-state Markov process $\xi(t)$ is considered. In contrast to discontinuous processes, continuous processes in any small time interval $\Delta t \to 0$ have some small change in the state $\Delta x \to 0$. The mathematical approach used in this paper is based on a discrete model of a continuous Markov process described in many books on the theory of diffusion Markov processes (see, for example, [13]).

Theorem. *In the asymptotic case of a large number of requests K the probability density function* $p(x,\ t)$ *of the random process* $\xi(t) = \frac{k(t)}{K} = \left(\frac{k_1(t)}{K},\ \frac{k_2(t)}{K},\ \ldots,\ \frac{k_n(t)}{K} \right)$ *provides that it is differentiable with respect to t and twice continuously differentiable with respect to* x_i, $i = \overline{1, n}$, *satisfies up to* $\mathrm{O}\left(\varepsilon^2\right)$, *where* $\varepsilon = \frac{1}{K}$, *the multidimensional Fokker – Planck – Kolmogorov equation*

$$\frac{\partial p(x,\ t)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(A_i(x,\ t)\, p(x,\ t) \right) + \frac{\varepsilon}{2} \sum_{i,\ j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left(B_{ij}(x,\ t)\, p(x,\ t) \right) \tag{1}$$

with drift coefficients

$$A_i(x,\ t) = \lambda_0 \left(1 - \sum_{i=1}^{n} x_i \right) \left(p_{0i}^+ - p_{0i}^- \right) + \sum_{j=1}^{n} \lambda_0 \left(1 - \sum_{i=1}^{n} x_i \right) p_{0j}^- q_{ji} -$$

$$- \mu_i \min(x_i,\ \varepsilon) \sum_{j=1}^{n} p_{ij}^- \left(1 - \theta(x_j) \right) + \sum_{j=1}^{n} \mu_j \min(x_j,\ \varepsilon) \left(p_{ji}^+ - p_{ji}^- - \delta_{ji} \right) +$$

$$+ \sum_{j,\ s=1}^{n} \mu_j \min(x_j,\ \varepsilon)\, p_{js}^- q_{si}$$

and diffusion coefficients

$$B_{ii}(x,\, t) = \lambda_0\left(1 - \sum_{i=1}^{n} x_i\right)\left(p_{0i}^{+} + p_{0i}^{-}\right) + \sum_{j=1}^{n}\lambda_0\left(1 - \sum_{i=1}^{n} x_i\right)p_{0j}^{-}q_{ji} +$$

$$+ \mu_i \min(x_i,\, \varepsilon)\sum_{j=1}^{n} p_{ij}^{-}\left(1 - \theta(x_j)\right) + \sum_{j=1}^{n}\mu_j \min(x_j,\, \varepsilon)\left(p_{ji}^{+} + p_{ji}^{-} + \delta_{ji}\right) +$$

$$+ \sum_{j,\, s=1}^{n}\mu_j \min(x_j,\, \varepsilon)\, p_{js}^{-}q_{si},$$

$$B_{ij}(x,\, t) = -\mu_i \min(x_i,\, \varepsilon)\, p_{ij}^{+} - \lambda_0\left(1 - \sum_{i=1}^{n} x_i\right)p_{0i}^{-}q_{ij} +$$

$$+ \mu_i \min(x_i,\, \varepsilon)\, p_{ij}^{-}q_{j0} + \mu_i \min(x_i,\, \varepsilon)\, p_{ij}^{-}\sum_{s=1}^{n} q_{js} -$$

$$- \mu_i \min(x_i,\, \varepsilon)\sum_{s=1}^{n} p_{is}^{-}q_{sj} - \sum_{s=1}^{n}\mu_s \min(x_s,\, \varepsilon)\, p_{sj}^{-}q_{ji},\ i \neq j,$$

where δ_{ji} *is the Kronecker delta,* $i,\ j = \overline{1,\, n}$.

P r o o f. First of all, we consider all possible ways of changing the state of the Markov random process $k(t)$ in a small time Δt. Let us introduce a n-vector of the form $I_i = \left(\overbrace{0,\, 0,\, \ldots,\, 0,\, 1}^{i},\, 0,\, 0,\, \ldots,\, 0\right)$ and the Heaviside function

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

As mentioned above, the process $k(t)$ is a continuous-time Markov process on the finite state space. The assumptions made in the model description determine that in the short time Δt the Markov process $k(t) = (k,\, t)$ can make one of the following transitions:

- from the state $(k - I_i,\, t)$ to the state $(k,\, t + \Delta t)$ with the probability

$$\lambda_0\left(K - \sum_{i=1}^{n} k_i(t) + 1\right)p_{0i}^{+}\Delta t + o(\Delta t),$$

that corresponds to a payload packet arrival from the node S_0 to the node S_{n+1};

- from the state $(k + I_i,\, t)$ to the state $(k,\, t + \Delta t)$ with the probability

$$\lambda_0\left(K - \sum_{i=1}^{n} k_i(t) - 1\right)p_{0i}^{-}q_{i0}\Delta t +$$

$$+ \mu_i \min\left(k_i(t) + 1,\, 1\right)\left(p_{i0} + p_{ij}^{-}\left(1 - \theta\left(k_j(t)\right)\right)\right)\Delta t + o(\Delta t),$$

which is possible when a packet containing malicious code arrives from the external environment S_0, when a payload packet is routed from S_i to the external environment S_0, or when a payload packet is transmitted as a signal from S_i to the empty node S_j, $i,\ j = \overline{1,\, n}$;

- from the state $(k + I_i - I_j,\, t)$ to the state $(k,\, t + \Delta t)$ with the probability

$$\left(\mu_i \min\left(k_i(t) + 1,\, 1\right)p_{ij}^{+} + \lambda_0\left(K - \sum_{i=1}^{n} k_i(t)\right)p_{0i}^{-}q_{ij}\right)\Delta t + o(\Delta t),$$

which is possible when a payload packet is transferred from the system S_i to the system S_j without modification or when a signal (trigger) arrives from the external environment S_0 to S_i and this trigger moves the payload packet from S_i to S_j, $i,\ j = \overline{1,\, n}$;

• from the state $\left(k + I_i + I_j,\ t\right)$ to the state $\left(k,\ t + \Delta t\right)$ with the probability

$$\mu_i \min\left(k_i(t) + 1, 1\right) p_{ij}^{-} q_{j0} \Delta t + o(\Delta t),$$

which corresponds to the transfer of a payload packet from the node S_i to the node S_j, $i,\ j = \overline{1,\ n}$, as malware;

• from the state $\left(k + I_i + I_j - I_s,\ t\right)$ to the state $\left(k,\ t + \Delta t\right)$ with the probability

$$\mu_i \min\left(k_i(t) + 1, 1\right) p_{ij}^{-} q_{js} \Delta t + o(\Delta t),$$

when a payload packet moves from the node S_i to the node S_j as a signal (trigger) that moves the packet from S_j to S_s, $i,\ j,\ s = \overline{1,\ n}$;

• from the state $(k,\ t)$ to the state $(k,\ t + \Delta t)$ with the probability

$$1 - \left(\lambda_0 \left(K - \sum_{i=1}^{n} k_i(t)\right) + \sum_{i=1}^{n} \mu_i \min\left(k_i(t), 1\right)\left(1 + \sum_{j=1}^{n} p_{ij}^{-}\left(1 - \theta\left(k_j(t)\right)\right)\right)\right)\Delta t + o(\Delta t),$$

which corresponds to no packets transfer;

• from other states to the state $(k,\ t + \Delta t)$ with the probability $o(\Delta t)$.

With regard to the transitions listed above in the short time Δt, using the law of total probability, the following set of equations is valid for the probability $P(k,\ t) = P\left(k(t) = k\right)$:

$$\begin{aligned}
\frac{dP(k,\ t)}{dt} &= \sum_{i=1}^{n} \lambda_0 \left(K - \sum_{i=1}^{n} k_i(t)\right) p_{0i}^{+} \left(P\left(k - I_i,\ t\right) - P(k,\ t)\right) + \\
&+ \sum_{i=1}^{n} \lambda_0 p_{0i}^{+} P\left(k - I_i,\ t\right) + \left(\sum_{i=1}^{n} \lambda_0 \left(K - \sum_{i=1}^{n} k_i(t)\right) p_{0i}^{-} q_{i0} + \right. \\
&+ \left. \sum_{i,\ j=1}^{n} \mu_i \min\left(k_i(t),\ 1\right)\left(p_{i0} + p_{ij}^{-}\left(1 - \theta\left(k_j(t)\right)\right)\right)\right)\left(P\left(k + I_i,\ t\right) - P(k,\ t)\right) + \\
&+ \left(\sum_{i=1}^{n} \lambda_0 p_{0i}^{-} q_{i0} + \sum_{i,\ j=1}^{n} \mu_i \left(\min\left(k_i(t) + 1,\ 1\right) - \min\left(k_i(t),\ 1\right)\right)\left(p_{i0} + p_{ij}^{-}\left(1 - \theta\left(k_j(t)\right)\right)\right)\right) \times \\
&\times P\left(k + I_i,\ t\right) + \sum_{i,\ j=1}^{n} \mu_i \min\left(k_i(t),\ 1\right) p_{ij}^{+} \left(P\left(k + I_i - I_j,\ t\right) - P(k,\ t)\right) + \\
&+ \sum_{i,\ j=1}^{n} \mu_i \left(\min\left(k_i(t) + 1,\ 1\right) - \min\left(k_i(t),\ 1\right)\right) p_{ij}^{+} P\left(k + I_i - I_j,\ t\right) + \\
&+ \sum_{i,\ j=1}^{n} \lambda_0 \left(K - \sum_{i=1}^{n} k_i(t)\right) p_{0i}^{-} q_{ij} \left(P\left(k + I_i - I_j,\ t\right) - P(k,\ t)\right) + \\
&+ \sum_{i,\ j=1}^{n} \mu_i \min\left(k_i(t), 1\right) p_{ij}^{-} q_{j0} \left(P\left(k + I_i + I_j,\ t\right) - P(k,\ t)\right) + \\
&+ \sum_{i,\ j=1}^{n} \mu_i \left(\min\left(k_i(t) + 1,\ 1\right) - \min\left(k_i(t),\ 1\right)\right) p_{ij}^{-} q_{j0} P\left(k + I_i + I_j,\ t\right) + \\
&+ \sum_{i,\ j,\ s=1}^{n} \mu_i \min\left(k_i(t), 1\right) p_{ij}^{-} q_{js} \left(P\left(k + I_i + I_j - I_s,\ t\right) - P(k,\ t)\right) + \\
&+ \sum_{i,\ j,\ s=1}^{n} \mu_i \left(\min\left(k_i(t) + 1,\ 1\right) - \min\left(k_i(t),\ 1\right)\right) p_{ij}^{-} q_{js} P\left(k + I_i + I_j - I_s,\ t\right).
\end{aligned} \quad (2)$$

Data networks typically handle a large number of data packets. In connection with this, we proceed to the limit from the Markov chain $k(t)$ to the continuous-state Markov process $\xi(t)=\frac{k(t)}{K}=\left(\frac{k_1(t)}{K}, \frac{k_2(t)}{K}, \ldots, \frac{k_n(t)}{K}\right)$ when K tends to be very large number. The state space of the relative vector $\xi(t)=\frac{k(t)}{K}$ is

$$X=\left\{x=(x_1, x_2, \ldots, x_n): x_i \geq 0, i=\overline{1, n}, \sum_{i=1}^{n} x_i \leq 1\right\}.$$

The increment of $\xi_i(t)$ in the short time $\Delta t \to 0$ is $\Delta x_i = \varepsilon$, where $\varepsilon=\frac{1}{K}$. As $K \to \infty$, the increment of $\xi_i(t)$ decreases, and in any small time interval $\Delta t \to 0$ the process $\xi_i(t)$ has some small change in the state $\Delta x_i \to 0$. We can assume that the limiting distribution of $\xi_i(t)$ is continuous. The vector $\xi(t)$ will be continuous-time continuous-state Markov processes with a probability density function $p(x, t)$. The probability density function satisfies the asymptotic relation

$$K^n P(k, t)=K^n P(xK, t) \xrightarrow[K \to \infty]{} p(x, t),\ x \in X. \tag{3}$$

Realising the passage to limit (3) for equation (2), assuming, $\varepsilon=\frac{1}{K}$ and $e_i = I_i\varepsilon=\frac{I_i}{K}$, we obtain the following partial differential equation:

$$\begin{aligned}
\frac{\partial p(x, t)}{\partial t} &= K\sum_{i=1}^{n}\lambda_0\left(1-\sum_{i=1}^{n}x_i\right)p_{0i}^{+}\left(p(x-e_i, t)-p(x, t)\right)+\\
&+\sum_{i=1}^{n}\lambda_0 p_{0i}^{+}p(x-e_i, t)+K\left(\sum_{i=1}^{n}\lambda_0\left(1-\sum_{i=1}^{n}x_i\right)p_{0i}^{-}q_{i0}+\right.\\
&\left.+\sum_{i, j=1}^{n}\mu_i\min(x_i, 1)\left(p_{i0}+p_{ij}^{-}\left(1-\theta(x_j)\right)\right)\right)\left(p(x+e_i, t)-p(x, t)\right)+\\
&+\sum_{i=1}^{n}\lambda_0 p_{0i}^{-}q_{i0}+\sum_{i, j=1}^{n}\mu_i\frac{\partial\min(x_i, \varepsilon)}{\partial x_i}\left(p_{i0}+p_{ij}^{-}\left(1-\theta(x_j)\right)\right)p(x+e_i, t)+\\
&+K\sum_{i, j=1}^{n}\mu_i\min(x_i, 1)p_{ij}^{+}\left(p(x+e_i-e_j, t)-p(x, t)\right)+\\
&+\sum_{i, j=1}^{n}\mu_i\frac{\partial\min(x_i, \varepsilon)}{\partial x_i}p_{ij}^{+}p(x+e_i-e_j, t)+K\sum_{i, j=1}^{n}\lambda_0\left(1-\sum_{i=1}^{n}x_i\right)p_{0i}^{-}q_{ij}\left(p(x+e_i-e_j, t)-p(x, t)\right)+\\
&+\sum_{i, j=1}^{n}\mu_i\frac{\partial\min(x_i, \varepsilon)}{\partial x_i}p_{ij}^{-}q_{j0}p(x+e_i+e_j, t)+\\
&+K\sum_{i, j=1}^{n}\mu_i\min(x_i, 1)p_{ij}^{-}q_{js}\left(p(x+e_i+e_j-e_s, t)-p(x, t)\right)+\\
&+\sum_{i, j, s=1}^{n}\mu_i\frac{\partial\min(x_i, \varepsilon)}{\partial x_i}p_{ij}^{-}q_{js}p(x+e_i+e_j-e_s, t).
\end{aligned} \tag{4}$$

If $p(x, t)$ is a twice continuously differentiable function with respect to x, then we can use the second degree Taylor series of functions $p(x \pm e_i, t)$, $p(x+e_i-e_j, t)$, $p(x+e_i+e_j, t)$ and $p(x+e_i+e_j-e_s, t)$ at a point x [9; 11]. Substituting the above-mentioned Taylor series into equation (4), having grouped the terms in the resulting equation, we conclude that compact mathematical expression (1) is valid. The theorem is proven.

Equation (1) is known as the multidimensional Fokker – Planck – Kolmogorov equation. The drift coefficients $A_i(x, t)$ characterise the rate of change of random process $\xi(t)$. The diffusion coefficients $B_{ij}(x, t)$ characterise the rate of change in the variance of the considered process $\xi(t)$. Note that the drift and diffusion coefficients depend linearly on x.

The main statistical characteristics of a Markov process describing the model state

The probability distribution of the vector $\xi(t)$ given by the probability density function $p(x, t)$ is a complete and exhaustive characteristic of the G-network state at the time t. However, such an exhaustive characteristic cannot be found, since equation (1) is not explicitly solvable. Therefore, instead of the probability density function $p(x, t)$, we will use an incomplete approximate description of a random process $\xi(t)$ using its moments. The probability distribution of a random process is usually characterised by a small number of parameters, which also have a practical interpretation. It is often enough to know what «average value» of $\xi(t)$ is, how far from this average value the values of $\xi(t)$ typically are, and how the statistical relationship between its components $\xi_i(t)$ and $\xi_j(t)$ is characterised.

The minimum set of parameters by which an n-dimensional random process can be characterised is as follows.

1. The expected values $E\xi_1(t)=\nu_1^{(1)}(t)$, $E\xi_2(t)=\nu_2^{(1)}(t)$, …, $E\xi_n(t)=\nu_n^{(1)}(t)$. Expectations are non-random functions of the time that characterise the mean trajectories of the process components around which they are grouped.

2. The variances $D\xi_1(t)$, $D\xi_2(t)$, …, $D\xi_n(t)$. Variances are non-random functions of the time that characterise the spread or dispersion of process realisations relative to the expectations.

3. The correlation moments

$$K_{ij}(t)=E\Big(\big(\xi_i(t)-E\xi_i(t)\big)\big(\xi_j(t)-E\xi_j(t)\big)\Big)=$$
$$=E\big(\xi_i(t)\xi_j(t)\big)-E\xi_i(t)M\xi_j(t)=\nu_{ij}^{(1,\,1)}(t)-\nu_i^{(1)}(t)\nu_j^{(1)}(t).$$

They characterise the pairwise correlation of the components included in the vector $\xi(t)$. The notation $\nu_{ij}^{(1,\,1)}(t)=E\big(\xi_i(t)\xi_j(t)\big)$ is the mixed raw moment of the second order, $i, j=\overline{1, n}$.

It was found [16] that the set of ordinary differential equations for the first-order and second-order raw moments of the state vector elements $\xi_i(t)$ is

$$\frac{d\nu_i^{(1)}(t)}{dt}=\frac{dM\big(\xi_i(t)\big)}{dt}=A_i\Big(\nu^{(1)}(t)\Big),$$
$$\frac{d\nu_{ij}^{(1,\,1)}(t)}{dt}=\frac{dM\big(\xi_i(t)\xi_j(t)\big)}{dt}=M\Big(\xi_i(t)A_j\big(\xi(t)\big)\Big)+ \tag{5}$$
$$+M\Big(\xi_j(t)A_i\big(\xi(t)\big)\Big)+\varepsilon B_{ij}\Big(\nu^{(1)}(t)\Big),\ i, j=\overline{1, n}.$$

It is proven that the moments are determined with an accuracy of $\mathrm{O}\big(\varepsilon^2\big)$, where $\varepsilon=\frac{1}{K}$, from the set of ordinary differential equations (5). The solution of set (5) with a certain initial condition, firstly, makes it possible to predict the mean and the dispersion of the number of data packets at each model state with time, and, secondly, draw a conclusion about the correlation of the number of packets at different data network units with time. These results are useful in decision making and network load analysis. They are applicable with a specified accuracy in both transient and steady state, this is a fundamental advantage of the used asymptotic method.

In this paper, we restrict ourselves to considering only the set of differential equations for expected values $\nu_i^{(1)}(t)$, $i=\overline{1, n}$, of the defined form

$$\frac{d\nu_i^{(1)}(t)}{dt}=\lambda_0\left(1-\sum_{i=1}^{n}\nu_i^{(1)}(t)\right)\left(p_{0i}^{+}-p_{0i}^{-}\right)+\sum_{j=1}^{n}\lambda_0\left(1-\sum_{i=1}^{n}\nu_i^{(1)}(t)\right)p_{0j}^{-}q_{ji}-$$
$$-\mu_i\min\left(\nu_i^{(1)}(t),\ \varepsilon\right)\sum_{j=1}^{n}p_{ij}^{-}\left(1-\theta\left(\nu_j^{(1)}(t)\right)\right)+$$

$$+\sum_{j=1}^{n}\mu_j \min\left(\nu_j^{(1)}(t),\ \varepsilon\right)\left(p_{ji}^{+}-p_{ji}^{-}-\delta_{ji}\right)+\sum_{j,\ s=1}^{n}\mu_j \min\left(\nu_j^{(1)}(t),\ \varepsilon\right)p_{js}^{-}q_{si}.$$

In the asymptotic case of large K, the Gaussian approximation method [13; 17] can be used to analytically reconstruct the normal probability density function

$$p(x,\ t)=(2\pi)^{-\frac{n}{2}}\left(\det K(t)\right)^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\left(x-\nu^{(1)}(t)\right)^{T}K^{-1}(t)\left(x-\nu^{(1)}(t)\right)\right]$$

from the found moments of the process $\xi(t)$ and to analyse this process using the normal density properties, $K^{-1}(t)$ being the inverse covariance matrix [18].

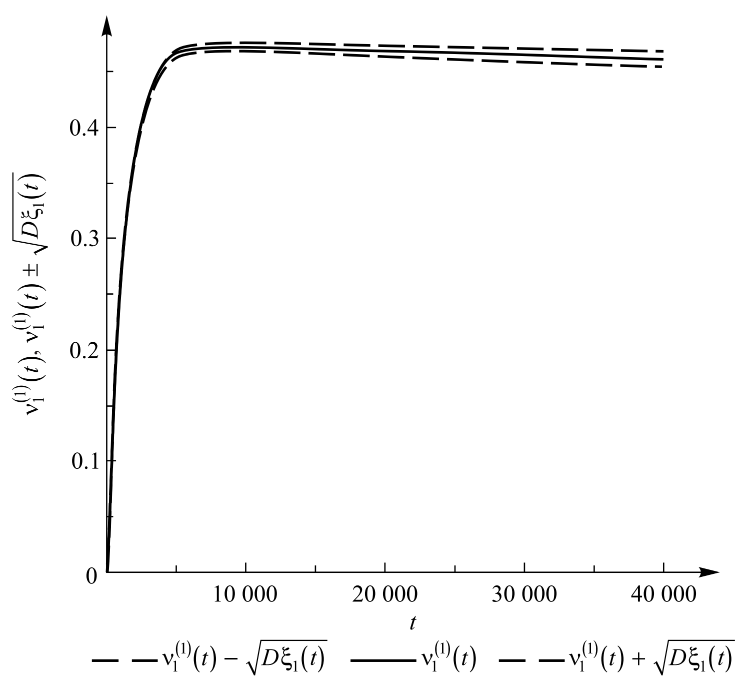
Numerical example

Consider the data network with a router to which two terminal devices are connected via two data links. The mathematical model of this network will be the above-described G-network of four nodes ($n = 4$). Nodes S_1 and S_2 are data links, nodes S_3 and S_4 are terminal devices, the external environment S_0 is a router. The structure of the G-network is set by the following non-zero elements of the transition matrices: $p_{01}^{+} = 0.6$, $p_{01}^{-} = 0.2$, $p_{02}^{+} = 0.19$, $p_{02}^{-} = 0.01$, $p_{13}^{+} = 0.65$, $p_{13}^{-} = 0.01$, $p_{10} = 0.34$, $p_{24}^{+} = 0.02$, $p_{24}^{-} = 0.7$, $p_{20} = 0.28$, $p_{31}^{+} = 0.99$, $p_{31}^{-} = 0.01$, $p_{42}^{+} = 0.01$, $p_{42}^{-} = 0.99$, $q_{12} = 0.97$, $q_{10} = 0.03$, $q_{21} = 0.95$, $q_{20} = 0.05$, $q_{30} = 1$, $q_{40} = 1$.

Let the number of data packets not exceed $K = 100\ 000$, and the network operation be specified by the following parameters: the arrival rate is $\lambda_0 = 0.001$; the number of node servers are $m_1 = 1$, $m_2 = 1$, $m_3 = 1$, $m_4 = 1$; the service rates are $\mu_1 = 10$, $\mu_2 = 10$, $\mu_3 = 100$, $\mu_4 = 100$; the initial placement of packets is $\nu_i^{(1)}(t) = 0$, $\nu_{ij}^{(1,\ 1)}(t) = 0$, $i,\ j = \overline{1,\ 4}$.

Let us solve set (5) by numerical methods under the above initial condition. The figure shows a graphical solution of set (5) for $\nu_1^{(1)}(t)$ and $\nu_1^{(1)}(t) \pm \sqrt{D\xi_1(t)}$, which allows us to observe the dynamics of the average relative number of packets at the node S_1 ant its variation.

The figure demonstrates that the process does not reach the steady state in the considered time interval. At time $t = 30\ 000$, the average number of packets at the node S_1 is $K\nu_1^{(1)}(t) = 100\ 000 \cdot 0.457\,5 = 45\ 750$. It can be concluded that the efficiency of the data network is limited by the data link capacity modelled by the queueing system S_1. It is recommended to expand this data link, which is the network bottleneck. Similarly, we can get the results for the rest of the network nodes.



Dynamics of $\nu_1^{(1)}(t)$ and $\nu_1^{(1)}(t) \pm \sqrt{D\xi_1(t)}$

The second-order moments found from set (5) allow us to investigate the correlation between the number of requests in different network nodes with time:

$$r_{ij}(t) = r\left(\xi_i(t),\ \xi_j(t)\right) = \frac{\nu_{ij}^{(1,1)}(t) - \nu_i^{(1)}(t)\nu_j^{(1)}(t)}{\sqrt{D\xi_i(t)}\sqrt{D\xi_j(t)}},\ i,\ j = \overline{1,\ n}.$$

Thus, the calculation results can be useful in analysing and making decisions regarding the operation of the data network with different parameters. Data network performance indicators and some revenues can be found using mathematical methods for calculating the nodal characteristics of queueing networks [19; 20].

Conclusions

In this paper, the queueing G-network with signals was presented as a stochastic data network model. Obviously, both payload and malware, as well as service information, can be transmitted over a data network. Thus, a closed Markov queueing G-network is an appropriate mathematical model for a data network. Requests in the G-network correspond to data packets transmitted over the data network, positive requests are assigned to payload, signals are assigned to malware and service information. The model was studied in the asymptotic case of a large number of requests. As a result, the main statistical characteristics of the number of requests at each network unit were found in both transient and steady state. In particular, it is possible to investigate the correlation between the number of requests in different network nodes with time. The presented technique allows us to reconstruct the normal probability density function of the state process $\xi(t)$ based on the Gaussian approximation method. These results allow us to analyse the network efficiency and load balancing, i. e. distribute incoming traffic between several devices to improve the stability of their operation.

Библиографические ссылки

1. Gelenbe E. Product-form queueing networks with negative and positive customers. *Journal of Applied Probability.* 1991;28(3): 656–663. DOI: 10.2307/3214499.

2. Gelenbe E. G-networks by triggered customer movement. *Journal of Applied Probability.* 1993;30(3):742–748. DOI: 10.2307/3214781.

3. Gelenbe E. G-networks with signals and batch removal. *Probability in the Engineering and Informational Sciences.* 1993;7(3): 335–342. DOI: 10.1017/S0269964800002953.

4. Caglayan MU. G-networks and their applications to machine learning, energy packet networks and routing: introduction to the special issue. *Probability in the Engineering and Informational Sciences.* 2017;31(4):381–395. DOI: 10.1017/S0269964817000171.

5. Zhang Yunxiao. Optimal energy distribution with energy packet networks. *Probability in the Engineering and Informational Sciences.* 2021;35(1):75–91. DOI: 10.1017/S0269964818000566.

6. Gelenbe E. Steps toward self-aware networks. *Communications of the ACM.* 2009;52(7):66–75. DOI: 10.1145/1538788.1538809.

7. Matalytski M, Naumenko V. Investigation of G-network with signals at transient behavior. *Journal of Applied Mathematics and Computational Mechanics.* 2014;13(1):75–86. DOI: 10.17512/jamcm.2014.1.08.

8. Gelenbe E, Pujolle G. *Introduction to queueing networks.* 2nd edition. Chichester: John Wiley & Sons; 1998. XIII, 244 p.

9. Rusilko TV. Network stochastic call centre model. In: Petrenko SA, Tarkhov DA, editors. *Distance learning technologies – 2021 (DLT-2021). Selected papers of the VI International scientific and practical conference; 2021 September 20–22; Yalta, Crimea* [Internet]. [S. l.]: [s. n.]; 2021 [cited 2022 March 21]. p. 91–101 (CEUR workshop proceedings; volume 3057). Available from: https://ceur-ws.org/Vol-3057/paper9.pdf.

10. Rusilko T. Asymptotic analysis of a closed G-network of unreliable nodes. *Journal of Applied Mathematics and Computational Mechanics*. 2022;21(2):91–102. DOI: 10.17512/jamcm.2022.2.08.

11. Маталыцкий МА, Романюк ТВ. *Приближенные методы анализа сетей с центральной системой обслуживания и их применения.* Гродно: ГрГУ; 2003. 200 с.

12. Медведев ГА. Замкнутые системы массового обслуживания и их оптимизация. *Известия Академии наук СССР. Техническая кибернетика.* 1978;6:199–203.

13. Тихонов ВИ, Миронов МА. Марковские процессы. Москва: Советское радио; 1977. 488 с.

14. Параев ЮИ. *Введение в статистическую динамику процессов управления и фильтрации.* Москва: Советское радио; 1976. 184 с. (Библиотека технической кибернетики).

15. Гардинер КВ. *Стохастические методы в естественных науках.* Доброславский АС, Коломенский АА, Толстопятенко АВ, переводчики; Стратонович РЛ, редактор. Москва: Мир; 1986. 526 с.

16. Русилко ТВ. Метод определения моментов первых двух порядков для вектора состояния сети массового обслуживания в асимптотическом случае. *Веснік Гродзенскага дзяржаўнага ўніверсітэта імя Янкі Купалы. Серыя 2, Матэматыка. Фізіка. Інфарматыка, вылічальная тэхніка і кіраванне.* 2021;11(2):152–161.

17. Русилко ТВ. Метод гауссова приближения для определения плотности вероятности вектора состояния сети массового обслуживания в асимптотическом случае. *Веснік Гродзенскага дзяржаўнага ўніверсітэта імя Янкі Купалы. Серыя 2, Матэматыка. Фізіка. Інфарматыка, вылічальная тэхніка і кіраванне.* 2022;12(3):151–161.

18. Tu TT, Kharin AYu. Sequential probability ratio test for many simple hypotheses on parameters of time series with trend. *Journal of the Belarusian State University. Mathematics and Informatics*. 2019;1:35–45. DOI: 10.33581/2520-6508-2019-1-35-45.

19. Rusilko TV. Application of queueing network models in insurance. *Izvestiya of Saratov University. New series. Series: Mathematics. Mechanics. Informatics.* 2022;22(3):315–321. DOI: 10.18500/1816-9791-2022-22-3-315-321.

20. Matalytski MA. Forecasting anticipated incomes in the Markov networks with positive and negative customers. *Automation and Remote Control.* 2017;78(5):815–825. DOI: 10.1134/S0005117917050046.

References

1. Gelenbe E. Product-form queueing networks with negative and positive customers. *Journal of Applied Probability.* 1991;28(3): 656–663. DOI: 10.2307/3214499.

2. Gelenbe E. G-networks by triggered customer movement. *Journal of Applied Probability.* 1993;30(3):742–748. DOI: 10.2307/3214781.

3. Gelenbe E. G-networks with signals and batch removal. *Probability in the Engineering and Informational Sciences.* 1993;7(3): 335–342. DOI: 10.1017/S0269964800002953.

4. Caglayan MU. G-networks and their applications to machine learning, energy packet networks and routing: introduction to the special issue. *Probability in the Engineering and Informational Sciences.* 2017;31(4):381–395. DOI: 10.1017/S0269964817000171.

5. Zhang Yunxiao. Optimal energy distribution with energy packet networks. *Probability in the Engineering and Informational Sciences.* 2021;35(1):75–91. DOI: 10.1017/S0269964818000566.

6. Gelenbe E. Steps toward self-aware networks. *Communications of the ACM.* 2009;52(7):66–75. DOI: 10.1145/1538788.1538809.

7. Matalytski M, Naumenko V. Investigation of G-network with signals at transient behavior. *Journal of Applied Mathematics and Computational Mechanics.* 2014;13(1):75–86. DOI: 10.17512/jamcm.2014.1.08.

8. Gelenbe E, Pujolle G. *Introduction to queueing networks.* 2nd edition. Chichester: John Wiley & Sons; 1998. XIII, 244 p.

9. Rusilko TV. Network stochastic call centre model. In: Petrenko SA, Tarkhov DA, editors. *Distance learning technologies – 2021 (DLT-2021). Selected papers of the VI International scientific and practical conference; 2021 September 20–22; Yalta, Crimea* [Internet]. [S. l.]: [s. n.]; 2021 [cited 2022 March 21]. p. 91–101 (CEUR workshop proceedings; volume 3057). Available from: https://ceur-ws.org/Vol-3057/paper9.pdf.

10. Rusilko T. Asymptotic analysis of a closed G-network of unreliable nodes. *Journal of Applied Mathematics and Computational Mechanics*. 2022;21(2):91–102. DOI: 10.17512/jamcm.2022.2.08.

11. Matalytski MA, Romaniuk TV. *Priblizhennye metody analiza setei s tsentral'noi sistemoi obsluzhivaniya i ikh primeneniya* [Approximate methods for analysis of networks with a central queueing system and their applications]. Grodna: Yanka Kupala State University of Grodno; 2003. 200 p. Russian.

12. Medvedev GA. [Closed queueing systems and their optimisation]. *Izvestiya Akademii nauk SSSR. Tekhnicheskaya kibernetika.* 1978;6:199–203. Russian.

13. Tikhonov VI, Mironov MA. *Markovskie protsessy* [Markov processes]. Moscow: Sovetskoe radio; 1977. 488 p. Russian.

14. Paraev YuI. *Vvedenie v statisticheskuyu dinamiku protsessov upravleniya i fil'tratsii* [Introduction to statistical dynamics of management and filtering]. Moscow: Sovetskoe radio; 1976. 184 p. (Biblioteka tekhnicheskoi kibernetiki). Russian.

15. Gardiner CW. *Handbook of stochastic methods for physics, chemistry and the natural sciences.* 2nd edition. Berlin: Springer-Verlag; 1985. 442 p. (Springer series in synergetics; volume 13).

Russian edition: Gardiner CW. *Stokhasticheskie metody v estestvennykh naukakh.* Dobroslavskii AS, Kolomenskii AA, Tolstopyatenko AV, translators; Stratonovich RL, editor. Moscow: Mir; 1986. 526 p.

16. Rusilko TV. The first two orders moments of determination method for the state vector of the queueing network in the asymptotic case. *Vesnik of Yanka Kupala State University of Grodno. Series 2, Mathematics. Physics. Informatics, Computer Technology and Control.* 2021;11(2):152–161. Russian.

17. Rusilko TV. Gaussian approximation method for determining the probability density of the state vector of the queueing network in the asymptotic case. *Vesnik of Yanka Kupala State University of Grodno. Series 2, Mathematics. Physics. Informatics, Computer Technology and Control.* 2022;12(3):151–161. Russian.

18. Tu TT, Kharin AYu. Sequential probability ratio test for many simple hypotheses on parameters of time series with trend. *Journal of the Belarusian State University. Mathematics and Informatics*. 2019;1:35–45. DOI: 10.33581/2520-6508-2019-1-35-45.

19. Rusilko TV. Application of queueing network models in insurance. *Izvestiya of Saratov University. New series. Series: Mathematics. Mechanics. Informatics.* 2022;22(3):315–321. DOI: 10.18500/1816-9791-2022-22-3-315-321.

20. Matalytski MA. Forecasting anticipated incomes in the Markov networks with positive and negative customers. *Automation and Remote Control.* 2017;78(5):815–825. DOI: 10.1134/S0005117917050046.

Received 15.03.2023 / revised 22.06.2023 / accepted 22.06.2023.