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НАЧАЛЬНО-КРАЕВАЯ ЗАДАЧА С НЕЛОКАЛЬНЫМ ГРАНИЧНЫМ УСЛОВИЕМ ДЛЯ НЕЛИНЕЙНОГО ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С ПАМЯТЬЮ

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Рассмотрено нелинейное параболическое уравнение с памятью $u_t = \Delta u + au^p \int_0^t u^q(x, \tau)d\tau - bu^m$ для $(x, t) \in \Omega \times (0, +\infty)$ с нелинейным нелокальным граничным условием $\left.\frac{\partial u(x, t)}{\partial \nu}\right|_{\partial\Omega \times (0, +\infty)} = \int_\Omega k(x, y, t)u^l(y, t)dy$ и начальными данными $u(x, 0) = u_0(x)$, $x \in \Omega$, где a, b, q, m, l – положительные постоянные; $p \geq 0$; Ω – ограниченная область в пространстве $\mathbb{R}^n$ с гладкой границей $\partial\Omega$; ν – единичная внешняя нормаль к $\partial\Omega$. Неотрицательная непрерывная функция $k(x, y, t)$ определена при $x \in \partial\Omega$, $y \in \overline{\Omega}$, $t \geq 0$, неотрицательная функция $u_0(x) \in C^1(\overline{\Omega})$, при этом она удовлетворяет условию $\frac{\partial u_0(x)}{\partial \nu} = \int_\Omega k(x, y, 0)u_0^l(y)dy$ при $x \in \partial\Omega$. Рассмотрены классические решения. Установлено существование локального максимального решения исходной задачи. Введены понятия верхнего и нижнего решений. Показано, что при выполнении определенных условий верхнее решение не меньше нижнего решения. Найдены условия положительности решений. Как следствие положительности решений и принципа сравнения решений доказана теорема единственности решения.

Ключевые слова: нелинейное параболическое уравнение; нелокальное граничное условие; существование решения; принцип сравнения.

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INITIAL BOUNDARY VALUE PROBLEM WITH NONLOCAL BOUNDARY CONDITION FOR A NONLINEAR PARABOLIC EQUATION WITH MEMORY

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We consider a nonlinear parabolic equation with memory $u_t = \Delta u + au^p \int_0^t u^q(x, \tau)\,d\tau - bu^m$ for $(x, t) \in \Omega \times (0, +\infty)$ under nonlinear nonlocal boundary condition $\left.\frac{\partial u(x, t)}{\partial \nu}\right|_{\partial\Omega \times (0, +\infty)} = \int_\Omega k(x, y, t)u^l(y, t)\,dy$ and initial data $u(x, 0) = u_0(x)$, $x \in \Omega$, where a, b, q, m, l are positive constants; $p \geq 0$; Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$; ν is unit outward normal on $\partial\Omega$. Nonnegative continuous function $k(x, y, t)$ is defined for $x \in \partial\Omega$, $y \in \bar{\Omega}$, $t \geq 0$, nonnegative function $u_0(x) \in C^1(\bar{\Omega})$, while it satisfies the condition $\frac{\partial u_0(x)}{\partial \nu} = \int_\Omega k(x, y, 0)u_0^l(y)\,dy$ for $x \in \partial\Omega$. In this paper we study classical solutions. We establish the existence of a local maximal solution of the original problem. We introduce definitions of a supersolution and a subsolution. It is shown that under some conditions a supersolution is not less than a subsolution. We find conditions for the positiveness of solutions. As a consequence of the positiveness of solutions and the comparison principle of solutions, we prove the uniqueness theorem.

Keywords: nonlinear parabolic equation; nonlocal boundary condition; existence of a solution; comparison principle.

Introduction

In this paper we consider the initial boundary value problem for the nonlinear parabolic equation

$$u_t = \Delta u + au^p \int_0^t u^q(x, \tau)\,d\tau - bu^m, \quad x \in \Omega, \ t > 0, \tag{1}$$

with nonlinear nonlocal boundary condition

$$\frac{\partial u(x, t)}{\partial \nu} = \int_\Omega k(x, y, t)u^l(y, t)\,dy, \quad x \in \partial\Omega, \ t > 0, \tag{2}$$

and initial datum

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{3}$$

where a, b, q, m, l are positive constants; $p \geq 0$; Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$; ν is unit outward normal on $\partial\Omega$.

Throughout this paper we suppose that the functions $k(x, y, t)$ and $u_0(x)$ satisfy the following conditions:

$$k(x, y, t) \in C(\partial\Omega \times \bar{\Omega} \times [0, +\infty)), \ k(x, y, t) \geq 0,$$

$$u_0(x) \in C^1(\bar{\Omega}), \ u_0(x) \geq 0 \text{ in } \Omega, \ \frac{\partial u_0(x)}{\partial \nu} = \int_\Omega k(x, y, 0)u_0^l(y)\,dy \text{ on } \partial\Omega.$$

Initial boundary value problems with nonlocal terms in parabolic equations or in boundary conditions have been considered in many papers (see, for example, [1–17] and the references therein). In particular, the initial boundary value problem (1)–(3) with $a = 0$ was considered for $b = b(x, t) \geq 0$ and $b = b(x, t) \leq 0$ in publications [18; 19] and [20; 21] respectively. Problem (1)–(3) with $p = 0$ and nonlocal boundary condition

$$u(x, t) = \int_\Omega k(x, y, t)u^l(y, t)\,dy, \quad x \in \partial\Omega, \ t > 0, \tag{4}$$

was investigated in work [22].

The aim of this paper is to study problem (1)–(3) for any positive p, q, m and l. We prove existence of a local solution of problem (1)–(3). Comparison principle and the uniqueness of a solution are established. We show the nonuniqueness of solution of problem (1)–(3) with $u_0(x) \equiv 0$ also.

Local existence

In this section a local existence theorem for problem (1)–(3) will be proved. We begin with definitions of a supersolution, a subsolution and a maximal solution of problem (1)–(3). Let $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$, $\Gamma_T = S_T \cup \bar{\Omega} \times \{0\}$, $T > 0$.

Definition 1. We say that a nonnegative function $u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a supersolution of problem (1)–(3) in Q_T if

$$u_t \geq \Delta u + au^p \int_0^t u^q(x, \tau)d\tau - bu^m, \ (x, t) \in Q_T, \tag{5}$$

$$\frac{\partial u(x, t)}{\partial \nu} \geq \int_\Omega k(x, y, t)u^l(y, t)dy, \ (x, t) \in S_T, \tag{6}$$

$$u(x, 0) \geq u_0(x), \ x \in \Omega, \tag{7}$$

and a nonnegative function $u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a subsolution of problem (1)–(3) in Q_T if $u \geq 0$ and it satisfies inequalities (5)–(7) in the reverse order. We say that $u(x, t)$ is a solution of problem (1)–(3) in Q_T if $u(x, t)$ is both a subsolution and a supersolution of problem (1)–(3) in Q_T.

Definition 2. We say that $u(x, t)$ is a maximal solution of problem (1)–(3) in Q_T if for any other solution $w(x, t)$ of problem (1)–(3) in Q_T the inequality $w(x, t) \leq u(x, t)$ is satisfied for $(x, t) \in Q_T \cup \Gamma_T$.

Let $\{\varepsilon_m\}$ be decreasing to zero a sequence such that $0 < \varepsilon_m < 1$ and $\varepsilon_m \to 0$ as $m \to \infty$. For $\varepsilon = \varepsilon_m$, $m = 1, 2, \ldots$, let $u_{0\varepsilon}(x)$ be the functions with the following properties:

$$\begin{gathered} u_{0\varepsilon}(x) \in C^1(\bar{\Omega}), \ u_{0\varepsilon}(x) \geq \varepsilon, \ u_{0\varepsilon_i}(x) \geq u_{0\varepsilon_j}(x) \text{ for } \varepsilon_i \geq \varepsilon_j, \\ u_{0\varepsilon}(x) \to u_0(x) \text{ as } \varepsilon \to 0 \text{ uniformly in } \bar{\Omega}, \\ \frac{\partial u_{0\varepsilon}(x)}{\partial \nu} = \int_\Omega k(x, y, 0)u_{0\varepsilon}^l(y)dy, \ x \in \partial\Omega. \end{gathered} \tag{8}$$

Let us consider the following auxiliary problem:

$$\begin{cases} u_t = \Delta u + au^p \int_0^t u^q(x, \tau)d\tau - bu^m + b\varepsilon^m, \ (x, t) \in Q_T, \\ \dfrac{\partial u(x, t)}{\partial \nu} = \int_\Omega k(x, y, t)u^l(y, t)dy, \ (x, t) \in S_T, \\ u(x, 0) = u_{0\varepsilon}(x), \ x \in \Omega, \end{cases} \tag{9}$$

where $\varepsilon = \varepsilon_m$. The notion of a solution u_ε for problem (9) can be defined in a similar way as in the definition 1.

Theorem 1. *Problem (9) has a unique solution in Q_T for small values of $T > 0$.*

P r o o f. Denote $K = \sup\limits_{\partial\Omega \times Q_1} k(x, y, t)$ and introduce an auxiliary function $\psi(x)$ with the following properties:

$$\psi(x) \in C^2(\bar{\Omega}), \ \inf_\Omega \psi(x) \geq \max\left(\sup_\Omega u_{0\varepsilon}(x), 1\right), \ \inf_{\partial\Omega} \frac{\partial \psi(x)}{\partial \nu} \geq K \max(1, \exp(l-1)) \int_\Omega \psi^l(y)dy.$$

We put

$$w(x, t) = \exp(\alpha t)\psi(x),$$

where α will be defined below.

To prove the existence of a solution for problem (9) we introduce the set

$$B = \left\{h(x, t) \in C(\bar{Q}_T) : \varepsilon \leq h(x, t) \leq w(x, t), \ h(x, 0) = u_{0\varepsilon}(x)\right\}$$

and consider the problem

$$\begin{cases} u_t = \Delta u + a\upsilon^p \int\limits_0^t \upsilon^q(x,\tau)\,d\tau - bu^m + b\varepsilon^m, \ (x,t) \in Q_T, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \int\limits_\Omega k(x,y,t)\upsilon^l(y,t)\,dy, \ (x,t) \in S_T, \\ u(x,0) = u_{0\varepsilon}(x), \ x \in \Omega, \end{cases} \tag{10}$$

where $\upsilon \in B$. It is obvious, B is a nonempty convex subset of $C(\bar{Q}_T)$. By classical theory [23] problem (10) has a solution $u \in C^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T)$ for small values of T. Let us call $A(\upsilon) = u$, where $\upsilon \in B$, and u is a solution of problem (10). In order to show that A has a fixed point in B we verify that A is a continuous mapping from B into itself such that AB is relatively compact. Obviously, the function $u(x,t) = \varepsilon$ is a subsolution of problem (10). Let us show that $w(x,t)$ is a supersolution of problem (10) for suitable choice of $\alpha > 0$ and $T > 0$.

Indeed,

$$w_t - \Delta w - a\upsilon^p \int\limits_0^t \upsilon^q(x,\tau)\,d\tau + bw^m - b\varepsilon^m \geq w_t - \Delta w - aw^p \int\limits_0^t w^q(x,\tau)\,d\tau + bw^m - b\varepsilon^m \geq$$

$$\geq \exp(\alpha t)\left[\alpha\psi(x) - \Delta\psi(x)\right] - a\exp(p\alpha t)\frac{\exp(q\alpha t) - 1}{q\alpha}\psi^{p+q} + b\left(\exp(m\alpha t)\psi^m(x) - \varepsilon^m\right) \geq 0$$

for $(x,t) \in Q_T$ if

$$\alpha \geq \max\left\{\frac{1}{q},\ a\exp(1)\sup_\Omega \psi^{p+q-1}(x) + \sup_\Omega \frac{\Delta\psi(x)}{\psi(x)}\right\},\ T \leq \frac{1}{(p+q)\alpha}.$$

On the boundary S_T we have

$$\frac{\partial w(x,t)}{\partial \nu} - \int\limits_\Omega k(x,y,t)\upsilon^l(y,t)\,dy \geq \exp(\alpha t)K\max(1, \exp(l-1))\int\limits_\Omega \psi^l(y)\,dy -$$

$$- K\exp(l\alpha t)\int\limits_\Omega \psi^l(y)\,dy \geq 0$$

for $T \leq \dfrac{1}{\alpha}$. The inequality

$$w(x,0) - u_{0\varepsilon}(x) \geq 0$$

holds for $x \in \Omega$. Then $w(x,t)$ is a supersolution of problem (10) and thanks to a comparison principle for problem (10) A maps B into itself.

Let $G(x,y;t-\tau)$ denote the Green's function for a heat equation with homogeneous Neumann boundary condition. The Green's function has the following properties (see, for example, [24]):

$$\begin{gathered} G(x,y;t-\tau) \geq 0,\ x,y \in \Omega,\ 0 \leq \tau < t, \\ \int\limits_\Omega G(x,y;t-\tau)\,dy = 1,\ x \in \Omega,\ 0 \leq \tau < t. \end{gathered} \tag{11}$$

It is well known that $u(x,t)$ is a solution of problem (10) in Q_T if and only if for $(x,t) \in Q_T$

$$u(x,t) = \int\limits_\Omega G(x,y;t)u_{0\varepsilon}(y)\,dy +$$

$$+ \int\limits_0^t \int\limits_\Omega G(x,y;t-\tau)\left(a\upsilon^p(y,\tau)\int\limits_0^\tau \upsilon^q(y,\sigma)\,d\sigma + b\left(\varepsilon^m - u^m(y,\tau)\right)\right)dy\,d\tau +$$

$$+ \int\limits_0^t \int\limits_{\partial\Omega} G(x,\xi;t-\tau)\int\limits_\Omega k(\xi,y,\tau)\upsilon^l(y,\tau)\,dy\,dS_\xi\,d\tau. \tag{12}$$

We claim that A is continuous. In fact let υ_k be a sequence in B converging to $\upsilon \in B$ in $C(\bar{Q}_T)$. Denote $u_k = A\upsilon_k$. Then by (11) and (12) we see that

$$|u-u_k| = \left| \int_0^t \int_\Omega G(x, y; t-\tau) \left\{ a\left(\upsilon^p(y, \tau) - \upsilon_k^p(y, \tau)\right) \int_0^\tau \upsilon^q(y, \sigma) d\sigma + \right.\right.$$

$$\left. + a\upsilon_k^p(y, \tau) \int_0^\tau \left(\upsilon^q(y, \sigma) - \upsilon_k^q(y, \sigma)\right) d\sigma - b\left(u^m(y, \tau) - u_k^m(y, \tau)\right) \right\} dy\, d\tau +$$

$$\left. + \int_0^t \int_{\partial\Omega} G(x, \xi; t-\tau) \int_\Omega k(\xi, y, \tau) \left(\upsilon^l(y, \tau) - \upsilon_k^l(y, \tau)\right) dy dS_\xi d\tau \right| \le$$

$$\le aT^2 \sup_{Q_T} \left|\upsilon^p - \upsilon_k^p\right| \sup_{Q_T} w^q + aT^2 \sup_{Q_T} \left|\upsilon^q - \upsilon_k^q\right| \sup_{Q_T} w^p +$$

$$+ \theta T \sup_{Q_T} |u - u_k| + KT|\Omega| \sup_{Q_T} \left|\upsilon^l - \upsilon_k^l\right|,$$

where $\theta = mb \max\left(\varepsilon^{m-1}, \sup_{Q_T} w^{m-1}(x, t)\right)$; $T \le \min\left\{1, \frac{1}{2\theta}\right\}$. Now we can conclude that u_k converges to u in $C\left(\bar{Q}_T\right)$ as $k \to \infty$.

The equicontinuity of AB follows from equation (12) and the properties of the Green's function (see, for example, [25]). The Ascoli – Arzelà theorem guarantees the relative compactness of AB. Thus we are able to apply the Schauder – Tychonoff fixed point theorem and conclude that A has a fixed point in B if T is small. Now if u_ε is a fixed point of A, $u_\varepsilon \in C^{2,1}(Q_T) \cap C^{1,0}\left(\bar{Q}_T\right)$ and it is a solution of problem (9) in Q_T. Uniqueness of the solution follows from a comparison principle for problem (9) which can be proved in a similar way as in the next section. Theorem 1 is proved.

Now, let $\varepsilon_2 > \varepsilon_1$. Then it is easy to see that $u_{\varepsilon_2}(x, t)$ is a supersolution of problem (9) with $\varepsilon = \varepsilon_1$. Applying to problem (9) a comparison principle we have $u_{\varepsilon_1}(x, t) \le u_{\varepsilon_2}(x, t)$. Using the last inequality and the continuation principle of solutions we deduce that the existence time of u_ε does not decrease as $\varepsilon \to 0$. Taking $\varepsilon \to 0$, we get $u_M(x, t) = \lim_{\varepsilon \to 0} u_\varepsilon(x, t) \ge 0$ and $u_M(x, t)$ exists in Q_T for some $T > 0$. We know that $u_\varepsilon(x, t)$ is a solution of problem (9) in Q_T if and only if for $(x, t) \in Q_T$

$$u_\varepsilon(x, t) = \int_\Omega G(x, y; t) u_{0\varepsilon}(y) dy +$$

$$+ \int_0^t \int_\Omega G(x, y; t-\tau) \left(a u_\varepsilon^p(y, \tau) \int_0^\tau u_\varepsilon^q(y, \sigma) d\sigma + b\left(\varepsilon^m - u_\varepsilon^m(y, \tau)\right) \right) dy\, d\tau +$$

$$+ \int_0^t \int_{\partial\Omega} G(x, \xi; t-\tau) \int_\Omega k(\xi, y, \tau) u_\varepsilon^l(y, \tau) dy dS_\xi d\tau. \quad (13)$$

Passing to the limit as $\varepsilon \to 0$ in equation (13), we obtain by dominated convergence theorem

$$u_M(x, t) = \int_\Omega G(x, y; t) u_0(y) dy +$$

$$+ \int_0^t \int_\Omega G(x, y; t-\tau) \left(a u_M^p(y, \tau) \int_0^\tau u_M^q(y, \sigma) d\sigma - b u_M^m(y, \tau) \right) dy\, d\tau +$$

$$+ \int_0^t \int_{\partial\Omega} G(x, \xi; t-\tau) \int_\Omega k(\xi, y, \tau) u_M^l(y, \tau) dy dS_\xi d\tau$$

for $(x, t) \in Q_T$. Therefore, $u_M(x, t)$ is a solution of problem (1)–(3). Let $u(x, t)$ be any other solution of problem (1)–(3). Then by comparison principle from the next section $u_\varepsilon(x, t) \ge u(x, t)$. Taking $\varepsilon \to 0$, we conclude $u_M(x, t) \ge u(x, t)$. Now we proved the following local existence theorem.

Theorem 2. *Problem (1)–(3) has a maximal solution in Q_T for small values of T.*

Comparison principle

Theorem 3. *Let* $\overline{u}(x,t)$ *and* $\underline{u}(x,t)$ *be a supersolution and a subsolution of problem (1)–(3) in* Q_T *respectively. Suppose that* $\underline{u}(x,t)>0$ *or* $\overline{u}(x,t)>0$ *in* $Q_T\cup\Gamma_T$ *if either* $\min(q,l)<1$ *or* $0<p<1$. *Then* $\overline{u}(x,t)\ge\underline{u}(x,t)$ *in* $Q_T\cup\Gamma_T$.

Proof. Suppose that $\min(p,q,l)\ge1$. Let $T_0\in(0,T)$ and $u_{0\varepsilon}(x)$ have the same properties as in (8) but only $u_{0\varepsilon}(x)\to\underline{u}(x,0)$ as $\varepsilon\to0$ uniformly in $\overline{\Omega}$. We can construct a solution $u_M(x,t)$ of problem (1)–(3) with $u_0(x)=\underline{u}(x,0)$ in the following way: $u_M(x,t)=\lim\limits_{\varepsilon\to0}u_\varepsilon(x,t)$, where $u_\varepsilon(x,t)$ is a solution of problem (9). To establish the theorem we will show that

$$\underline{u}(x,t)\le u_M(x,t)\le\overline{u}(x,t),\ (x,t)\in\overline{Q}_{T_0}. \tag{14}$$

We prove the second inequality in relations (14) only since the proof of the first one is similar. Let $\varphi(x,\tau)\in C^{2,1}\left(\overline{Q}_{T_0}\right)$ be a nonnegative function such that

$$\frac{\partial\varphi(x,t)}{\partial\nu}=0$$

for $(x,t)\in S_{T_0}$. If we multiply the first equation in problem (9) by $\varphi(x,t)$ and then integrate over Q_t for $t\in(0,T_0)$, we obtain

$$\int_0^t\int_\Omega u_{\varepsilon\tau}(x,\tau)\varphi(x,\tau)\,dx\,d\tau=$$
$$=\int_0^t\int_\Omega\left(\Delta u_\varepsilon(x,\tau)+au_\varepsilon^p(x,\tau)\int_0^\tau u_\varepsilon^q(x,\sigma)\,d\sigma+b\left(\varepsilon^m-u_\varepsilon^m(x,\tau)\right)\right)\varphi(x,\tau)\,dx\,d\tau.$$

Integrating by parts and using Green's identity, we have

$$\int_\Omega u_\varepsilon(x,t)\varphi(x,t)\,dx\le\int_\Omega u_\varepsilon(x,0)\varphi(x,0)\,dx+$$
$$+\int_0^t\int_\Omega\left(u_\varepsilon(x,\tau)\varphi_\tau(x,\tau)+u_\varepsilon(x,\tau)\Delta\varphi(x,\tau)\right)dx\,d\tau+$$
$$+\int_0^t\int_\Omega\left(au_\varepsilon^p(x,\tau)\int_0^\tau u_\varepsilon^q(x,\sigma)\,d\sigma+b\left(\varepsilon^m-u_\varepsilon^m(x,\tau)\right)\right)\varphi(x,\tau)\,dx\,d\tau+$$
$$+\int_0^t\int_{\partial\Omega}\varphi(x,\tau)\int_\Omega k(x,y,\tau)u_\varepsilon^l(y,\tau)\,dy\,dS_x\,d\tau. \tag{15}$$

On the other hand, $\overline{u}$ satisfies (15) with reversed inequality and with $\varepsilon=0$. Set $w(x,t)=u_\varepsilon(x,t)-\overline{u}(x,t)$. Then $w(x,t)$ satisfies

$$\int_\Omega w(x,t)\varphi(x,t)\,dx\le\int_\Omega w(x,0)\varphi(x,0)\,dx+\varepsilon^m b\int_0^t\int_\Omega\varphi(x,\tau)\,dx\,d\tau+$$
$$+\int_0^t\int_\Omega w(x,\tau)\left(\varphi_\tau(x,\tau)+\Delta\varphi(x,\tau)-mb\theta_1^{m-1}(x,\tau)\right)\varphi(x,\tau)\,dx\,d\tau+$$
$$+\int_0^t\int_\Omega\left(a\overline{u}^p(x,\tau)\varphi(x,\tau)\int_0^\tau q\theta_2^{q-1}(x,\sigma)w(x,\sigma)\,d\sigma\right)dx\,d\tau+$$
$$+\int_0^t\int_\Omega\left(ap\theta_3^{p-1}(x,\tau)w(x,\tau)\varphi(x,\tau)\int_0^\tau u_\varepsilon^q(x,\sigma)\,d\sigma\right)dx\,d\tau+$$
$$+\int_0^t\int_{\partial\Omega}\varphi(x,\tau)\int_\Omega k(x,y,\tau)l\theta_4^{l-1}(y,\tau)w(y,\tau)\,dy\,dS_x\,d\tau, \tag{16}$$

where $\theta_i(x, \tau)$, $i = 1, 2, 3, 4$, are some continuous functions between $u_\varepsilon(x, t)$ and $\bar{u}(x, t)$. Note here that by hypotheses for $k(x, y, \tau)$, $u_\varepsilon(x, t)$ and $\bar{u}(x, t)$, we have

$$\begin{aligned} &0 \le \bar{u}(x, t) \le M,\ \varepsilon \le u_\varepsilon(x, t) \le M,\ (x, t) \in \bar{Q}_{T_0}, \\ &0 \le k(x, y, t) \le M,\ (x, y, t) \in \partial\Omega \times \bar{\Omega} \times [0, T_0], \end{aligned} \tag{17}$$

where M is some positive constant. Then it is easy to see from (17) that $\theta_1^{m-1}(x, \tau)$, $\theta_2^{q-1}(x, \tau)$, $\theta_3^{p-1}(x, \tau)$ and $\theta_4^{l-1}(x, \tau)$ are positive and bounded functions in $\bar{Q}_{T_0}$ and, moreover, $\theta_1^{m-1}(x, \tau) \le \max\{\varepsilon^{m-1}, M^{m-1}\}$, $\theta_2^{q-1}(x, \tau) \le M^{q-1}$, $\theta_3^{p-1}(x, \tau) \le M^{p-1}$, $\theta_4^{l-1}(x, \tau) \le M^{l-1}$. Define a sequence $\{a_n\}$ in the following way: $a_n(x, t) \in C^\infty(\bar{Q}_{T_0})$, $a_n(x, t) \ge 0$ and $a_n(x, t) \to mb\theta_1^{m-1}(x, t)$ as $n \to \infty$ in $L^1(\bar{Q}_{T_0})$. Now, we consider a backward problem given by

$$\begin{cases} \varphi_\tau + \Delta\varphi - a_n\varphi = 0,\ (x, \tau) \in Q_t, \\ \dfrac{\partial\varphi(x, \tau)}{\partial\nu} = 0,\ (x, \tau) \in S_t, \\ \varphi(x, t) = \psi(x),\ x \in \Omega, \end{cases} \tag{18}$$

where $\psi(x) \in C_0^\infty(\Omega)$ and $0 \le \psi(x) \le 1$. Denote a solution of problem (18) as $\varphi_n(x, \tau)$. Then by the standard theory for linear parabolic equations (see, for example, [25]), we find that $\varphi_n(x, \tau) \in C^{2,1}(\bar{Q}_t)$, $0 \le \varphi_n(x, \tau) \le 1$ in $\bar{Q}_t$. Putting $\varphi = \varphi_n$ in inequality (16) and passing to the limit as $n \to \infty$, we infer

$$\begin{aligned} \int_\Omega w(x, t)\psi(x)dx \le \int_\Omega w_+(x, 0)dx + \varepsilon^m bT_0|\Omega| + \\ + \left\{a(p+q)M^{p+q-1}T_0 + l|\partial\Omega|M^l\right\}\int_0^t\int_\Omega w_+(x, \tau)dxd\tau, \end{aligned} \tag{19}$$

where $w_+ = \max(w, 0)$; $|\partial\Omega|$ and $|\Omega|$ are the Lebesgue measures of $\partial\Omega$ in $\mathbb{R}^{n-1}$ and Ω in $\mathbb{R}^n$ respectively. Since inequality (19) holds for every $\psi(x)$, we can choose a sequence $\{\psi_n(x)\}$ converging in $L^1(\Omega)$ to

$$\psi(x) = \begin{cases} 1, \text{ if } w(x, t) > 0, \\ 0, \text{ if } w(x, t) \le 0. \end{cases}$$

Passing to the limit as $n \to \infty$ in inequality (19), we obtain

$$\begin{aligned} \int_\Omega w_+(x, t)dx \le \int_\Omega w_+(x, 0)dx + \varepsilon^m bT_0|\Omega| + \\ + \left\{a(p+q)M^{p+q-1}T_0 + l|\partial\Omega|M^l\right\}\int_0^t\int_\Omega w_+(x, \tau)dxd\tau,\ t \in (0, T_0]. \end{aligned}$$

Applying now Gronwall's inequality, we have

$$\int_\Omega w_+(x, t)dx \le \left(\int_\Omega w_+(x, 0)dx + \varepsilon^m bT_0|\Omega|\right)\exp\left[\left\{a(p+q)M^{p+q-1}T_0 + l|\partial\Omega|M^l\right\}t\right]$$

for $t \in (0, T_0]$. Passing to the limit as $\varepsilon \to 0$, the conclusion of the theorem follows for $\min(p, q, l) \ge 1$. For the case $p = 0$, $\min(q, l) \ge 1$ we prove the theorem in the same way. If $\min(q, l) < 1$ or $0 < p < 1$ we can consider $w(x, t) = \underline{u}(x, t) - \bar{u}(x, t)$ and prove the theorem in a similar way using the positiveness of a subsolution or a supersolution. Theorem 3 is proved.

Remark. For similar problem (1), (3), (4) with $p = 0$ the authors of work [22] suppose in the comparison principle that $\underline{u}(x, t) > 0$ or $\bar{u}(x, t) > 0$ in $Q_T \cup \Gamma_T$ if $\min(q, m, l) < 1$.

Lemma 1. *Let $u(x,\, t)$ be a solution of problem (1)–(3) in Q_T. Let $u_0(x) \not\equiv 0$ in Ω and $m \geq 1$. Then $u(x,\, t) > 0$ in $Q_T \cup S_T$. If $u_0(x) > 0$ in $\bar{\Omega}$ and $p < m < 1$ then $u(x,\, t) > 0$ in $Q_T \cup \Gamma_T$.*

P r o o f. Let $u_0(x) \not\equiv 0$ in Ω and $m \geq 1$. We denote

$$M = \sup_{Q_{T_0}} u(x,\, t),$$

where M is some positive constant; $T_0 \in (0,\, T)$. Now we put $h(x,\, t) = u(x,\, t)\exp(\lambda t)$ with $\lambda \geq bM^{m-1}$. Then in Q_{T_0} we have

$$h_t - \Delta h = \exp(\lambda t)(\lambda u + u_t - \Delta u) \geq u \exp(\lambda t)\left(\lambda - bu^{m-1}\right) \geq 0.$$

Since $h(x,\, 0) = u_0(x) \geq 0,\ x \in \Omega$, and $u_0(x) \not\equiv 0$ in Ω, by the strong maximum principle $h(x,\, t) > 0$ in Q_{T_0}. Hence, $u(x,\, t) > 0$ in Q_{T_0}. Let $h(x_0,\, t_0) = 0$ in some point $(x_0,\, t_0) \in S_T$. Then according to theorem 3.6 of work [26] it yields $\dfrac{\partial h(x_0,\, t_0)}{\partial \nu} < 0$, which contradicts boundary condition (2).

Let $u_0(x) > 0$ in $\bar{\Omega}$ and $p < m < 1$. Then there exist $\tau \in (0,\, T)$ and $\varepsilon > 0$ such that

$$u(x,\, t) \geq \varepsilon \text{ in } \bar{Q}_\tau$$

and, moreover, $u(x,\, t) \equiv \varepsilon_1 = \min\left(\varepsilon,\, \left[\dfrac{a\tau\varepsilon^q}{b}\right]^{\frac{1}{m-p}}\right)$ is the subsolution of problem (1)–(3) in $Q_{T_0} \backslash \bar{Q}_\tau$ with initial function $u(x,\, \tau)$ for $t = \tau$ instead of initial datum (3). Putting $\underline{u}(x,\, t) \equiv \varepsilon_1$ and $\bar{u}(x,\, t) \equiv u(x,\, t)$ and arguing as in the proof of theorem 3, we get

$$u(x,\, t) \geq \varepsilon_1 \text{ in } \bar{Q}_{T_0} \text{ for any } T_0 \in (0,\, T).$$

Lemma 1 is proved.

As a simple consequence of theorem 3 and lemma 1, we get the following uniqueness result for problem (1)–(3).

Theorem 4. *Let problem (1)–(3) have a positive in $Q_T \cup \Gamma_T$ solution or a solution in Q_T either with nonnegative initial data in Ω for $\min(p,\, q,\, l) \geq 1$ or with positive initial data in $\bar{\Omega}$ under the conditions $m \geq 1$ or $p < m < 1$. Then a solution of problem (1)–(3) is unique in Q_T.*

Now we will prove the nonuniqueness of solution of problem (1)–(3) with $u_0(x) \equiv 0$ for $l < \min(1,\, m)$ or $p + q < \min(1,\, m)$. We note that problem (1)–(3) with $u_0(x) \equiv 0$ has solution $u(x,\, t) \equiv 0$.

Theorem 5. *Let $u_0(x) \equiv 0$ and either $l < \min(1,\, m)$ and*

$$k(x,\, y_0,\, t_0) > 0 \ \textit{for any } x \in \partial\Omega \textit{ and some } y_0 \in \partial\Omega \textit{ and } t_0 \in [0,\, T) \tag{20}$$

or $p + q < \min(1,\, m)$. Then a maximal solution of problem (1)–(3) $u_M(x,\, t) \not\equiv 0$ in Q_T.

P r o o f. As shown in theorem 2 a maximal solution $u_M(x,\, t) = \lim\limits_{\varepsilon \to 0} u_\varepsilon(x,\, t)$, where $u_\varepsilon(x,\, t)$ is some positive in $\bar{Q}_T$ supersolution of problem (1)–(3). To prove the theorem we construct a subsolution $\underline{u}(x,\, t) \not\equiv 0$ of problem (1)–(3) with $u_0(x) \equiv 0$. By theorem 3 we have $u_\varepsilon(x,\, t) \geq \underline{u}(x,\, t)$ and therefore maximal solution $u_M(x,\, t) \not\equiv 0$.

At first let $l < \min(1,\, m)$ and inequality (20) hold. To construct a subsolution we use the change of variables in a neighbourhood of $\partial\Omega$ as in work [27]. Let $\bar{x}$ be a point on $\partial\Omega$. We denote by $\hat{n}(\bar{x})$ the inner unit normal to $\partial\Omega$ at the point $\bar{x}$. Since $\partial\Omega$ is smooth it is well-known that there exists $\delta > 0$ such that the mapping $\psi : \partial\Omega \times [0,\, \delta] \to \mathbb{R}^n$ given by $\psi(\bar{x},\, s) = \bar{x} + s\hat{n}(\bar{x})$ defines new coordinates $(\bar{x},\, s)$ in a neighbourhood of $\partial\Omega$ in $\bar{\Omega}$.

Under the assumptions of the theorem, there exists $\bar{t}$ such that $k(x,\, y,\, t) > 0$ for $t_0 \leq t \leq t_0 + \bar{t}$, $x \in \partial\Omega$ and $y \in V(y_0)$, where $V(y_0)$ is some neighbourhood of y_0 in $\bar{\Omega}$.

Let $\dfrac{1}{1-l} < \alpha \leq \dfrac{1}{1-m}$ for $m < 1$ and $\alpha > \dfrac{1}{1-l}$ for $m \geq 1$, $2 < \beta < \dfrac{2}{1-m}$ for $m < 1$ and $\beta > 2$ for $m \geq 1$. Assume that $A > 0$, $0 < \xi_0 \leq 1$ and $0 < T_0 < \min\left(T - t_0,\, \bar{t},\, \delta^2\right)$. For points in $\partial\Omega \times [0,\, \delta] \times (t_0,\, t_0 + T_0]$ of coordinates $(\bar{x},\, s,\, t)$ define

$$\underline{u}(\bar{x}, s, t) = A(t - t_0)^{\alpha}\left(\xi_0 - \frac{s}{\sqrt{t - t_0}}\right)_+^{\beta}$$

and extend $\underline{u}$ as zero to the whole of $\bar{Q}_\tau$ with $\tau = t_0 + T_0$. Arguing as in work [18] we prove that $\underline{u}$ is the subsolution of problem (1)–(3) with $u_0(x) \equiv 0$ in Q_τ.

Now we suppose that $p + q < \min(1, m)$. Then it is easy to check that $\underline{u}(x, t) = t^{\gamma}$ is the subsolution of problem (1)–(3) with $u_0(x) \equiv 0$ in Q_τ for small values of τ if

$$\gamma > \max\left(\frac{2}{1 - (p + q)}, \frac{1}{m - (p + q)}\right).$$

Theorem 5 is proved.

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