Effective measure estimates for sets of real numbers related to Mahler's classification

Vasili Bernik
National Academy of Sciences of Belarus
Minsk, Belarus
bernik@im.bas-net.by

Friedrich Götze
Universität Bielefeld
Bielefeld, Germany
goetze@math.uni-bielefeld.de

Victoria Lebed
Belarussian State University
Minsk, Belarus
vvv_vikki@tut.by

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Abstract

Classical theorems of metric theory of Diophantine approximation state that some Diophantine inequalities have infinitely many solutions only on sets with Lebesgue measure equal to zero. In this paper we estimate the rate of convergence to zero of measures of sets with a given measure of transcendentality as the right sides of the inequalities tend to zero.

1 Introduction and results

Irrational (transcendental) numbers are defined as numbers which are not rational (resp. algebraic). Therefore it is natural to classify them on the basis of their approximation by rational (resp. algebraic) numbers. However, a slightly different approach is widely used. It is based on the rate of approximation of zero by values of integral polynomials in the given irrational (transcendental) point. Such approach was introduced by Mahler (see [1, 2]).

In studying the described classification the following sets are essential.

Let $I = [a, b)$ be a given interval and

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with integral coefficients of degree $\deg P = n$ and height $H = H(P) = \max_{1 \leqslant j \leqslant n} |a_j|$.

For a real $w \geqslant n$ we consider inequality

$$|P(x)| < H^{-w} \tag{1}$$

and denote $\mathcal{L}_n(w) = \{x \in I : (1)$ has infinitely many solutions in $P(x) \in \mathbb{Z}[x], \deg P \leqslant n\}$.

Similarly, for a monotonically decreasing function $\psi(x) : \mathbb{R}_+ \to \mathbb{R}_+$ we consider inequality

$$|P(x)| < H^{-n+1}\psi(H) \tag{2}$$

and denote $\mathcal{L}_n(\psi) = \{x \in I : (2)$ has infinitely many solutions in $P(x) \in \mathbb{Z}[x], \deg P \leqslant n\}$.

Of course, (1) is just a particular case of (2) (one may simply take $\psi(x) = x^{-w+n-1}$), but historically the set $\mathcal{L}_n(w)$ was introduced earlier and it is convenient to study.

By $\mu\, A$ we will denote the Lebesgue measure of a set $A \subset R$.

From Minkowski's theorem on linear forms (see, for example, [3]) one easily derives that $\mathcal{L}_n(n) = I$. In 1932 Mahler showed that $\mu\,\mathcal{L}_n(w) = 0$ as soon as $w > 4n$ and conjectured the same result for any $w > n$. This hypothesis was proved in 1964 by Sprindžuk ([4]). Soon a more general result was achieved by A. Baker ([5]). In that work he also conjectured that $\mu\,\mathcal{L}_n(\psi) = 0$ as soon as $\sum_{H=1}^{\infty} \psi(H)$ converges. In 1989 Bernik proved Baker's hypothesis ([6]), and in 1999 Beresnevich ([7]) showed that $\mu\,\mathcal{L}_n(\psi) = |I|$ for a divergent series $\sum_{H=1}^{\infty} \psi(H)$. The works [6] and [7] give a full analogue of Khinchin's metric theorem on the approximation of real numbers by rational numbers for polynomials of a given degree.

Bugeaud extended the above mentioned results to algebraic integers. He also remarked that from [6] and [7] one cannot estimate the rate of convergence of measures of sets like $\mathcal{L}_n(\psi)$ to zero as the upper bound for H tends to infinity. In the present paper we show how such sets can be effectively investigated.

The statement of the problem we study here was motivated by the following result of Götze ([8]). Let $d \in \mathbb{N}$, $D \in \mathbb{R}^{d\times d}$ be a nondegenerate matrix of a positively defined quadratic form and I be a given interval. For a vector $\overline{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ we consider its norm $|\overline{x}|_\infty = max_{1\leqslant j\leqslant d} |x_j|$. For a large enough real r and small enough positive real τ we denote by $\mathcal{L}_I(\tau)$ the set of $t \in I$ such that the system of inequalities

$$\begin{cases} |\overline{m} - tD\overline{n}|_\infty & < & \tau r^{-1}, \\ |\overline{n}|_\infty & < & \tau r \end{cases}$$

has a solution in vectors $\overline{m}, \overline{n} \in \mathbb{Z}^d$, $(\overline{m}, \overline{n}) \neq 0$. Then there is a constant $c(d)$ depending only on d such that we have the following estimation:

Theorem 1

$$\mu\,\mathcal{L}_I(\tau) < c(d)\tau^2|I|.$$

This theorem plays an important role in work [8] and provides a basis for some ultimate results on the number of integral points in ellipsoids.

Now we state our main result.

Fix $n \in \mathbb{N}$ and an interval $I \subset \mathbb{R}$. By $c_1, c_2, \dots$ we denote constants depending only on n or on I and n. We also use Vinogradov's symbols: $f \ll g$ denotes that $f \leqslant c_1 g$, and $f \asymp g$ denotes $g \ll f \ll g$. All constants we use can be calculated effectively.

Further, take an interval $J \subset I$ and real positive values Q and $\tau \leqslant 1$. Let $B'_{n,I}(Q,\tau,J)$ denote the set of all $x \in J$ such that the system of inequalities

$$\begin{cases} |P(x)| \leqslant \tau Q^{-n}, \\ H = H(P) \leqslant \tau Q \end{cases} \tag{3}$$

has a solution in nonzero polynomials $P(x) \in \mathbb{Z}[x], \deg P \leqslant n\}$.

(By $H(P)$ we will throughout denote the height of the polynomial $P(x) = a_n x^n + \dots + a_1 x + a_0$, $H(P) = \max_{1 \leqslant i \leqslant n} |a_i|$.)

Then we have the following

Theorem 2 *If $Q \geqslant Q_{00}(n,I)$, $Q^{-c_1(n)} \leqslant \tau \leqslant 1$ and $|J| \geqslant Q^{-c_2(n)}$, then*

$$\mu\, B'_{n,I}(Q,\tau,J) \leqslant c(n,I)\tau^{n+1}|J|. \tag{4}$$

Note that for $\tau > 1$ the theorem is obviously true.

2 Notations and lemmas

Let $\mathbb{Z}_n[x]$ denote the set of all integral polynomials $P(x)$ with $\deg(P) = n$. For a real x and a polynomial $P(x) \in \mathbb{Z}_n[x]$, $P(x) = a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$, we will write $x \rightarrowtail \alpha_i$ if the root α_i is the closest to x (or one of the closest if there are several).

In the following three lemmas we consider a polynomial $P(x) \in \mathbb{Z}_n[x]$, $P(x) = a_n(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$, and assume its roots to be ordered such that $|\alpha_1-\alpha_2| \leqslant |\alpha_1-\alpha_3| \leqslant \dots \leqslant |\alpha_1-\alpha_n|$. We also denote $A_j = \Pi_{i=j}^n|\alpha_1-\alpha_i|$ for $j = \overline{2,n}$ and $A_{n+1} = 1$.

Lemma 1 *If $x \rightarrowtail \alpha_1$ and $P'(\alpha_1) \neq 0$, then for all $j = \overline{1,n}$*

$$|x-\alpha_1| \leqslant (2^{n-j}\frac{|P(x)|\Pi_{i=2}^j|\alpha_1-\alpha_i|}{|P'(\alpha_1)|})^{1/j} = (2^{n-j}\frac{|P(x)|}{|a_n|A_{j+1}})^{1/j}.$$

Lemma 2 *For $j = \overline{1,n}$*

$$|P^{(j)}(\alpha_1)| \leqslant \frac{(n-1)!j}{(n-j)!}|a_n|A_{j+1}.$$

Moreover,

$$|P'(\alpha_1)| = |a_n|A_2.$$

These lemmas can be proved by obvious calculations.

Lemma 3 *For an irreducible $P(x)$ holds*

$$1 \leqslant 2^{\frac{n(n-1)}{2}} |a_n|^{n-1} \Pi_{i=2}^{n} A_i.$$

The proof of this lemma is based on estimating the discriminant of the polynomial.

Lemma 4 *For any $n \in \mathbb{N}$, $n > 1$ and real $\delta > 0$ there is an effectively calculated bound $H_0(\delta, n)$ such that for any $H > H_0$ and positive real μ, τ, η the following holds. If $P_1(x), P_2(x) \in \mathbb{Z}[x]$, $max(\deg(P_1), \deg(P_2)) \leqslant n$ are coprime, $\max(H(P_1), H(P_2)) \leqslant H^{\mu}$ and there is an interval $I \subset \mathbb{R}$ with $|I| = H^{-\eta}$, such that for all $x \in I$ $\max(P_1(x), P_2(x)) < H^{-\tau}$, then*

$$\tau + \mu + 2\max\{\tau + \mu - \eta, 0\} < 2n\mu + \delta.$$

The proof can be found in [10].

This lemma is a quantitative expression of the natural fact that no two coprime integral polynomials of bounded degree and height can be too small on a large enough interval.

We will rewrite the system (3) in a slightly different form:

$$\begin{cases} |P(x)| \leqslant \tau^{n+1} Q_1^{-n} = Q_1^{-\lambda(n+1)-n}, \\ H(P) \leqslant Q_1, \end{cases} \tag{5}$$

where $Q_1 = \tau Q$ and $\tau = Q_1^{-\lambda}$, and consider the set $B_{n,I}(Q_1, \tau, J)$ or $B_{n,I}(Q_1, \lambda, J)$ instead of $B'_{n,I}(Q, \tau, J)$. Then the conditions of the theorem change to the following: $Q_1 \geqslant Q_{00}(n, I)$, $0 \leqslant \lambda \leqslant \frac{c_1}{1-c_1} = \lambda_0$ (we certainly have $c_1 < 1$ since $\tau Q > 1$) and $|J| \geqslant (\tau Q^{-1})^{c_2} = Q_1^{-c_2(1+\lambda)}$, and we are to prove that

$$\mu\, B_{n,I}(Q_1, \lambda, J) \leqslant c(n, I) Q_1^{-\lambda(n+1)} |J|. \tag{6}$$

To simplify the notation, we will write Q instead of Q_1.

3 From the general case to a particular

We will prove the theorem by induction. Case $n = 1$ is trivial, and case $n = 2$ can be found in [9]. So now we assume $n > 2$.

1. For the set $B_{n-1,I}(Q, \lambda\frac{n+1}{n}, J)$ induction hypothesis is applicable, provided that $Q_{00}(n, I) \geqslant Q_{00}(n-1, I)$, $\lambda_0(n) \leqslant \lambda_0(n-1)\frac{n}{n+1}$, $c_2(n) \leqslant c_2(n-1)$. So we get

$$\mu\, B_{n-1,I}(Q, \lambda\frac{n+1}{n}, J) \leqslant c(n-1, I) Q^{-\lambda\frac{n+1}{n}n} |J| = c(n-1, I) Q^{-\lambda(n+1)} |J|,$$

and we may consider in (5) only polynomials with $deg(P) = n$ and then simply add $c(n-1, I)$ to $c(n, I)$.

2. We first consider solutions of (5) with $H(P) \leqslant Q^{\frac{7/8}{n+1}}$. (One naturally assumes $Q_{00}(n, I)$ large enough, so that $Q^{\frac{7/8}{n+1}} \geqslant 1$.) Lemma 1 for $j = n$ gives

$$|x - \alpha_1| \leqslant (\frac{|P(x)|}{|a_n|})^{1/n} \leqslant |P(x)|^{1/n} \leqslant Q^{-\lambda(1+1/n)-1}.$$

For two good polynomials (of such small height) $P_1(x) = a_n x^n + \cdots + a_1 x + a_{01}$ and $P_2(x) = a_n x^n + \cdots + a_1 x + a_{02}$ at points x_1 and x_2 respectively we have

$$|a_{01} - a_{02}| \leqslant |P_1(x_1)| + |P_2(x_2)| + \Sigma_{j=1}^n |a_j||x_1^j - x_2^j| \ll$$

$$\ll Q^{-\lambda(n+1)-n} + Q^{\frac{7/8}{n+1}}|J| \ll Q^{\frac{7/8}{n+1}}|J|,$$

so there are at most $(2Q^{\frac{7/8}{n+1}} + 1)^n * \overline{c}(n, I) Q^{\frac{7/8}{n+1}}|J| \ll Q^{\frac{7/8n}{n+1}}|J|$ good polynomials. Thus in this case for the investigated measure we get the estimate

$$\leqslant c'(n, I) Q^{\frac{7/8n}{n+1}}|J| * Q^{-\lambda(1+1/n)-1} = c'(n, I) Q^{\frac{7/8n}{n+1} - \lambda(1+1/n)-1}|J| \leqslant$$

$$\leqslant c'(n, I) Q^{-\lambda(n+1)}|J|.$$

So we may consider only solutions with

$$H(P) > Q^{\frac{7/8}{n+1}}.$$

Now suppose we have proved the estimate for polynomials with

$$2^{-\frac{1}{n}} Q \leqslant H(P) \leqslant Q$$

and constants $Q_{00}^*(n, I), \lambda_0^*(n), c_2^*(n)$. Take $Q_{00}(n, I) = Q_{00}^*(n, I)^{\frac{n+1}{7/8}}$, $\lambda_0(n) = \frac{7/8}{2(n+1)} \lambda_0^*(n)$, $c_2(n) = \frac{7/8}{n+1} c_2^*(n)$ and apply the result for $B_{n,I}(Q_k = 2^{-\frac{k}{n}} Q,\ \tau_k = k^{\frac{-2}{n+1}} \tau,\ J)$ with $2^{-\frac{1}{n}} Q_k \leqslant H(P) \leqslant Q_k$, where of course $Q_k > Q^{\frac{7/8}{n+1}}$, i.e.

$$k < n(1 - \frac{7/8}{n+1}) \log_2 Q.$$

It can be done since we have $Q_k \geqslant Q^{\frac{7/8}{n+1}} \geqslant Q_{00}(n, I)^{\frac{7/8}{n+1}} = Q_{00}^*(n, I)$, $\tau_k = k^{\frac{-2}{n+1}} \tau \geqslant (n(1 - \frac{7/8}{n+1}) \log_2 Q)^{\frac{-2}{n+1}} Q^{-\lambda_0(n)} \geqslant$ [we take $Q_{00}(n, I)$ large enough, so that $n(1 - \frac{7/8}{n+1}) \log_2 Q \leqslant Q^{\frac{\lambda_0(n)(n+1)}{2}}$] $\geqslant Q^{-\lambda_0(n) - \lambda_0(n)} =$

$Q^{\frac{-\lambda_0^*(n)7/8}{n+1}} \geqslant Q_k^{-\lambda_0^*(n)}$, $|J| \geqslant Q^{-c_2(1+\lambda)} = Q^{\frac{-c_2^*(1+\lambda)7/8}{n+1}} \geqslant Q_k^{-c_2^*(1+\lambda)}$. We get

$$\mu\, B_{n,I}(Q_k, \tau_k, J) \leqslant c(n,I)\tau_k^{n+1}|J| = c(n,I)k^{-2}\tau^{n+1}|J|.$$

Now note that

$$\tau_k^{n+1}Q_k^{-n} = 2^k k^{-2}\tau^{n+1}Q^{-n} \geqslant \tau^{n+1}Q^{-n},$$

so $B_{n,I}(Q,\tau,J) \subseteq B_{n,I}(Q^{\frac{7/8}{n+1}},\tau,J) \cup (\cup_{k=1}^{[n(1-\frac{7/8}{n+1})\log_2 Q]} B_{n,I}(Q_k,\tau_k,J))$ and therefore $\mu\, B_{n,I}(Q,\tau,J) \leqslant \mu\, B_{n,I}(Q^{\frac{7/8}{n+1}},\tau,J) + \Sigma_{k=1}^{[n(1-\frac{7/8}{n+1})\log_2 Q]}$ $\mu\, B_{n,I}(Q_k,\tau_k,J) \leqslant c'(n,I)Q^{-\lambda(n+1)}|J| + \Sigma_{k=1}^{[n(1-\frac{7/8}{n+1})\log_2 Q]} c(n,I)k^{-2}\tau^{n+1}$ $|J| \leqslant c^*(n,I)\tau^{n+1}|J|$, since the sum $\Sigma_{k=1}^{\infty}k^{-2}$ converges.

3. It is sufficient to prove the theorem only for the case

$$|a_n| \gg |H(P)|.$$

Let vice versa $|a_n| \ll |H(P)|$. Choose minimal $t = t(I) \in \mathbb{Z}$ such that $\forall x \in I\ t \geqslant x+1$. For an appropriately chosen constant $c_3 = c_3(n,I)$ holds

$$\max_{t\leqslant m\leqslant t+n} |P(m)| \geqslant c_3|H(P)|.$$

Let this maximum occur at $m = m_0$. Then for the polynomial

$$\widetilde{P}(x) = P(\frac{1}{x}+m_0)x^n = \widetilde{a}_n x^n + \cdots + \widetilde{a}_1 x + \widetilde{a}_0$$

we have: $\widetilde{P}(x) \in \mathbb{Z}_n[x]$, $H(\widetilde{P}) \ll Q$, $|\widetilde{a}_n| = |P(m_0)| \geqslant c_3|H(P)|$ and for $x \in I$ holds $|\widetilde{P}(\frac{1}{x-m_0})| \leqslant |P(x)|$. Now we can consider $\widetilde{P}(x)$ on $J' = [\frac{1}{b-m_0}, \frac{1}{a-m_0}]$ (where $J = [a,b]$) and, since for $[a',b'] \subseteq J$ holds

$$|[\frac{1}{a'-m_0} - \frac{1}{b'-m_0}]| = \frac{b'-a'}{(m_0-a')(m_0-b')} \asymp b'-a' = |[a',b']|,$$

apply the result for $|a_n| \gg |H(P)|$ and then change $c(n,I)$ appropriately.

4. We consider only polynomials $P(x)$ which are irreducible over $\mathbb{Z}[x]$. Here we show how to get measure estimation for reducible polynomials basing on the induction hypothesis.

   Let $P(x) = P_1(x)P_2(x)$, $P_1 \in \mathbb{Z}_{n_1}[x], P_2 \in \mathbb{Z}_{n_2}[x]$, $1 \leqslant n_1 \leqslant n_2 < n$, $n_1 + n_2 = n$. Then, taking into consideration the well-known property

$$H(P) \asymp H(P_1)H(P_2),$$

choose $k \in \{1, 2\}$ such that

$$Q^{\frac{k-1}{2}} \leqslant H(P_1) \leqslant Q^{\frac{k}{2}} = Q_1.$$

It follows that

$$H(P_2) \ll Q^{\frac{2-k+1}{2}} = Q_2.$$

If $Q_{00}(n, I) \geqslant Q_{00}(n_1, I)^2$, $\lambda_0(n) \leqslant \lambda_0(n_1)\frac{n_1+1}{2(n+1)}$, $c_2(n) \leqslant \frac{c_2(n_1)}{2(1+\lambda_0(n))}$, then induction hypothesis is applicable to the set $B_{n_1,I}(Q_1, \lambda\frac{2(n+1)}{k(n_1+1)}, J)$, and we get

$$\mu\, B_{n_1,I}(Q_1, \lambda\frac{2(n+1)}{k(n_1+1)}, J) \leqslant c(n_1, I) Q_1^{-\lambda\frac{2(n+1)}{k(n_1+1)}(n_1+1)}|J| =$$

$$= c(n_1, I) Q^{-\lambda(n+1)}|J|.$$

Similarly,

$$\mu\, B_{n_2,I}(Q_2, \lambda\frac{2(n+1)}{(2-k+1)(n_2+1)}, J) \leqslant c(n_2, I) Q^{-\lambda(n+1)}|J|.$$

So we may consider only x with

$$|P_1(x)P_2(x)| \leqslant Q^{-\lambda(n+1)-n},$$

$$|P_1(x)| > Q_1^{-\lambda\frac{2(n+1)}{k(n_1+1)}(n_1+1)-n_1} = Q^{-\lambda(n+1)-\frac{k}{2}n_1},$$

$$|P_2(x)| > Q_2^{-\lambda\frac{2(n+1)}{(2-k+1)(n_2+1)}(n_2+1)-n_2} = Q^{-\lambda(n+1)-\frac{(2-k+1)}{2}n_2}.$$

But then we have

$$Q^{-\lambda(n+1)-\frac{k}{2}n_1} < |P_1(x)| < Q^{\frac{(2-k+1)}{2}n_2-n},$$

hence $-\lambda(n+1) - \frac{k}{2}n_1 < \frac{(2-k+1)}{2}n_2 - n$, $n < \lambda(n+1) + \frac{k}{2}n_1 + \frac{(2-k+1)}{2}n_2 = \lambda(n+1) + \frac{k}{2}(n_1 - n_2) + \frac{(2+1)}{2}n_2 \leqslant$ [for $n_1 \leqslant n_2$] $\leqslant \lambda(n+1)+\frac{1}{2}(n_1-n_2)+\frac{(2+1)}{2}n_2 = \lambda(n+1)+\frac{1}{2}n_1+n_2 \leqslant \lambda_0(n)(n+1)+\frac{1}{2}+n-1$, which is contradictory for $\lambda_0(n)(n+1) < \frac{1}{2}$. So we should simply increase our $c(n, I)$ in the proper way.

Naturally, for $P(x) = a_n x^n + \cdots + a_1 x + a_0$ we will also demand

$$GCD(a_0, a_1, \ldots, a_n) = 1.$$

5. For each polynomial $P(x) \in \mathbb{Z}_n[x]$,

$$P(x) = a_n(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n),$$

we may choose one of its roots (say α_1) and consider only $x \in J$ such that $x \rightarrowtail \alpha_1$ (naturally, we should multiply the resulting constant $c(n, I)$ by n). Further, assume the roots to be ordered such that

$$|\alpha_1 - \alpha_2| \leqslant |\alpha_1 - \alpha_3| \leqslant \ldots \leqslant |\alpha_1 - \alpha_n|.$$

Denote

$$|\alpha_1 - \alpha_j| = H^{-\mu_j}, \ l_j = [\mu_j T], \ j = \overline{2, n},$$

where $T = \frac{n(n-1)}{2\theta}$ and θ is a small value (one can take for instance $\theta = (n+1)\lambda_0(n)$ and demand $(n+1)\lambda_0(n) < \frac{1}{16}$ and $nc_2(n)(1+\lambda_0(n))+(n+1)\lambda_0(n) < 2$ with some additional restrictments [they will appear further], showing that $\lambda_0(n)$ and $c_2(n)$ are small enough if compared with $\lambda_0(n_1)$ and $c_2(n_1)$ for all $n_1 < n$; changing θ slightly we change our requirements thus allowing τ or $|J|$ to be smaller, of course forcing the remaining parameter to be larger).

Take any $j \in \{2, \ldots, n\}$. If $|a_n| \geqslant c_3|H(P)|$, then $|\alpha_j| < 1 + \frac{1}{c_3}$, so $Q^{-\mu_j} < 2(1 + \frac{1}{c_3})$ and for sufficiently large $Q_{00}(n, I)$ we get

$$\mu_j > -\frac{1}{T}.$$

From lemma 3 follows

$$1 \leqslant 2^{\frac{n(n-1)}{2}} H^{n-1-\Sigma_{j=2}^{n}(j-1)\mu_j},$$

hence for sufficiently large $Q_{00}(n, I)$ holds

$$\mu_j < n - 1 + \theta.$$

So $l_j = [\mu_j T]$ takes only $c_4(n)$ values, and we can prove the theorem for a fixed set of l_j and then multiply the resulting constant by c_4^{n-1}.

We also denote $p_j = \frac{l_j + \cdots + l_n}{T}, \ j = \overline{2, n}, \ p_{n+1} = 0.$

Then for $j = \overline{2, n}$ we have

$$H^{-\frac{l_j}{T}-\theta} < H^{-\frac{l_j}{T}-\frac{1}{T}} < |\alpha_1 - \alpha_j| \leqslant H^{-\frac{l_j}{T}} \tag{7}$$

and for $j = \overline{2, n+1}$

$$H^{-p_j-\theta} < A_j \leqslant H^{-p_j}. \tag{8}$$

(A_j were introduced in section 2.)

We also readily see that l_j form a decreasing sequence and $p_{j+1} = p_j - \frac{l_j}{T} \leqslant p_j - \mu_j + \frac{1}{T} \leqslant p_j + 2\frac{1}{T}, \ j = \overline{2, n}$, so

$$p_{j+1} \leqslant p_j + 2\frac{1}{T}. \tag{9}$$

From lemma 3 for sufficiently large $Q_{00}(n, I)$ easily follows

$$\Sigma_{j=2}^{n} m_j \frac{l_j}{T} \leqslant n - 1 + \theta \tag{10}$$

for any set of $m_j \in \{0, 1, \ldots, j-1\}$, $j = \overline{2, n}$. Note also that for $j = \overline{2, n}$

$$p_j \geqslant -2(n+1-j)\frac{1}{T}.$$

4 The main case

In the previous section we showed that in order to achive the desired estimate we have to consider the set $B^*_{n,I}(Q, \lambda, J, l_2, \ldots, l_n)$ of all $x \in J$ such that the system of inequalities

$$\begin{cases} |P(x)| \leqslant Q^{-\lambda(n+1)-n}, \\ c_5 Q \leqslant H(P) \leqslant Q, \end{cases} \tag{11}$$

has a solution in irreducible integral polynomials $P(x) = a_n(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$ with $|a_n| \geqslant c_3|H(P)|$, $x \rightarrowtail \alpha_1$, $|\alpha_1 - \alpha_2| \leqslant |\alpha_1 - \alpha_3| \leqslant \ldots \leqslant |\alpha_1 - \alpha_n|$, $|\alpha_1 - \alpha_j| = H^{-\mu_j}$, $[\mu_j T] = l_j$, $j = \overline{2, n}$. We are to prove that

$$\mu\, B^*_{n,I}(Q, \lambda, J, l_2, \ldots, l_n) \leqslant c^*(n, I) Q^{-\lambda(n+1)}|J|. \tag{12}$$

For an integral polynomials $P(x)$ and an $x \in J$ we will write $x \rightarrowtail P$ and say 'x belongs to $P(x)$ ' if together they satisfy the above stated conditions; for an interval $J' \subseteq J$ we will write $J' \rightarrowtail P$ if there is an $x \in J'$ such that $x \rightarrowtail P$. We will call the polynomial good if $J \rightarrowtail P$.

We divide good polynomials into several groups according to their characteristics $l_2, \ldots, l_n$ and prove the statement separately for each group.

Case 1

$$p_2 + \frac{l_2}{T} \geqslant (n+1)(1+\lambda) - \theta. \tag{13}$$

In this case lemma 1 gives the sharpest estimate for $j = 2$:

$$|x - \alpha_1| \ll Q^{\frac{-(n+1)(1+\lambda)+\theta+p_3}{2}}. \tag{14}$$

Now we divide our interval J into $[|J|Q^{\frac{(n+1)(1-\lambda)-\theta-p_3}{2}}] \asymp |J|Q^{\frac{(n+1)(1-\lambda)-\theta-p_3}{2}}$ (since $\frac{-(n+1)(1-\lambda)+\theta+p_3}{2} \leqslant$ [see (10)] $\leqslant \frac{-(n+1)(1-\lambda)+\theta+\frac{n-1+\theta}{3}}{2} \leqslant -\frac{n}{3} - \frac{2(1-\theta)}{3} + \frac{\lambda_0(n)(n+1)}{2} \leqslant -c_2(n)(1+\lambda_0(n))$, and so $|J|Q^{\frac{(n+1)(1-\lambda)-\theta-p_3}{2}} \geqslant 1$) equal parts of length $\asymp Q^{\frac{-(n+1)(1-\lambda)+\theta+p_3}{2}}$. We prove that such a part J' can't belong to two different (and thus coprime) polynomials simultaneously.

Suppose the contrary: $x_i \in J',\ x_i \rightarrowtail P_i,\ x_i \rightarrowtail \alpha_{1_i},\ i = \overline{1,2},\ (P_1, P_2) = 1$. Then for any $x \in J'$ we have

$$|x - \alpha_{1_i}| \leqslant |x - x_i| + |x_i - \alpha_{1_i}| \ll$$
$$\ll |J'| + Q^{\frac{-(n+1)(1+\lambda)+\theta+p_3}{2}} \ll Q^{\frac{-(n+1)(1-\lambda)+\theta+p_3}{2}}$$

(see (14)) and, applying lemma 2, we have

$$|P_i^{(j)}(\alpha_{1_i})(x - \alpha_{1_i})^j| \ll Q^{1-p_{j+1}+\frac{j(-(n+1)(1-\lambda)+\theta+p_3)}{2}}.$$

(Here and throughout we take $j = \overline{1,n}$.) Denoting

$$f_j = 1 - p_{j+1} + \frac{j(-(n+1)(1-\lambda)+\theta+p_3)}{2},$$

we now show that

$$\max_{1 \leqslant j \leqslant n} f_j = f_2.$$

For $j \geqslant 3$ we estimate $f_j - f_{j-1} = \frac{l_j}{T} + \frac{-(n+1)(1-\lambda)+\theta+p_3}{2} \leqslant \frac{1}{2}(2\frac{l_3}{T} + p_3 - (n+1)(1-\lambda) + \theta) \leqslant \frac{1}{2}(\frac{l_2+l_3}{T} + p_3 - (n+1)(1-\lambda) + \theta) \leqslant \frac{1}{2}(n - 1 + \theta - n - 1 + (n+1)\lambda + \theta) = \theta + \frac{(n+1)\lambda}{2} - 1 < 0$ (we used (10)),while $f_2 - f_1 = \frac{l_2}{T} + \frac{-(n+1)(1-\lambda)+\theta+p_3}{2} = \frac{1}{2}(p_2 + \frac{l_2}{T} - (n+1)(1-\lambda) + \theta) \geqslant$ [see (13)] $\geqslant \frac{1}{2}((n+1)(1+\lambda) - \theta - (n+1)(1-\lambda) + \theta) = (n+1)\lambda > 0$. Therefore from the Taylor's expansion

$$P_i(x) = \Sigma_{j=1}^n \frac{1}{j!} P_i^{(j)}(\alpha_{1_i})(x - \alpha_{1_i})^j$$

we get

$$|P_i(x)| \ll Q^{f_2} = Q^{1-p_3-(n+1)(1-\lambda)+\theta+p_3} = Q^{-n+\theta+(n+1)\lambda}.$$

Now for $P_1(x)$ and $P_2(x)$ on J' we can apply lemma 4 with $\delta = 4\frac{1}{T},\ \mu = 1,\ \eta = \frac{(n+1)(1-\lambda)-\theta-p_3}{2},\ \tau = n - \theta - (n+1)\lambda : 3n - 3\theta - 3(n+1)\lambda + 3 - (n+1)(1-\lambda) + \theta + p_3 < 2n + \delta \Longrightarrow -2\theta - 2(n+1)\lambda + 2 + p_3 < \delta \Longrightarrow 1 < \theta + (n+1)\lambda_0(n) + (n-2)\frac{1}{T} + \frac{\delta}{2} = \theta + (n+1)\lambda_0(n) + n\frac{1}{T}$, which is a contradiction.

So all x on J' with the required approximation properties lie near one and the same α_1 (more precisely, see (14)), so in this case we get the measure estimate

$$[|J|Q^{\frac{(n+1)(1-\lambda)-\theta-p_3}{2}}]*Q^{\frac{-(n+1)(1+\lambda)+\theta+p_3}{2}} \asymp |J|Q^{\frac{(n+1)(1-\lambda)-\theta-p_3-(n+1)(1+\lambda)+\theta+p_3}{2}} =$$
$$= Q^{-\lambda(n+1)}|J|.$$

In all other cases lemma 1 gives the sharpest estimate for $j = 1$:

$$|x - \alpha_1| \ll Q^{-(n+1)(1+\lambda)+\theta+p_2}. \tag{15}$$

Case 2

$$p_2 + \frac{l_2}{T} < (n+1)(1+\lambda) - \theta, \qquad (16)$$

$$2p_2 > n - 1 + 2\theta. \qquad (17)$$

We handle this case similarly to the previous one. We divide J into $[|J|Q^{n+1-\theta-p_2}]$ parts of length $\asymp Q^{-(n+1)+\theta+p_2}$, preliminary checking that $|J| > Q^{-(n+1)+\theta+p_2}$ (summing $p_2 + p_3 \leqslant n - 1 + \theta$ [see (10)] and (16), we get $3p_2 < 2n + (n+1)\lambda$, so $Q^{-(n+1)+\theta+p_2} < Q^{\frac{-n+(n+1)\lambda_0(n)}{3}-1+\theta} < Q^{-c_2(n)(1+\lambda_0(n))} \leqslant |J|$). Assuming like above the existence of an interval J' belonging to two different polynomials $P_1(x)$ and $P_2(x)$, for any $x \in J'$ we get

$$|x - \alpha_{1_i}| \leqslant |x - x_i| + |x_i - \alpha_{1_i}| \ll Q^{-(n+1)+\theta+p_2}$$

(see (15)), and

$$|P_i^{(j)}(\alpha_{1_i})(x - \alpha_{1_i})^j| \ll Q^{1-p_{j+1}+j(-(n+1)+\theta+p_2)}.$$

Let

$$f_j = 1 - p_{j+1} + j(-(n+1) + \theta + p_2).$$

For $j \geqslant 3$ we estimate $f_j - f_{j-1} = \frac{l_j}{T} - (n+1) + \theta + p_2 \leqslant n - 1 + \theta - (n+1) + \theta = 2(\theta - 1) < 0$. So f_j is maximal for $j = 2$ or $j = 1$. Accordingly, we distinguish two possibilities.

1. $f_2 > f_1$, i.e. $\frac{l_2}{T} + p_2 + \theta - (n+1) > 0$.

   From Taylor's expansion we get

   $$|P_i(x)| \ll Q^{f_2} = Q^{1-p_3+2(-(n+1)+\theta+p_2)} = Q^{\frac{l_2}{T}+p_2-2n-1+2\theta}.$$

   For $P_1(x)$ and $P_2(x)$ on J' we apply lemma 4 with $\delta = 4\frac{1}{T},\ \mu = 1,\ \eta = n+1-\theta-p_2,\ \tau = 2n+1-(\frac{l_2}{T}+p_2)-2\theta : 6n+3-3(\frac{l_2}{T}+p_2)-6\theta+3-2n-2+2\theta+2p_2 < 2n+\delta \Longrightarrow 2n+4 < 3\frac{l_2}{T}+p_2+4\theta+\delta = 2(\frac{l_2}{T}+p_2)-p_3+4\theta+\delta <$[from (16)]$< 2((n+1)(1+\lambda)-\theta)+2(n-2)\frac{1}{T}+4\theta+\delta \leqslant 2((n+1)(1+\lambda_0(n))+\theta)+2n\frac{1}{T} \Longrightarrow 1 < \theta+(n+1)\lambda_0(n)+n\frac{1}{T}$, which is a contradiction.

2. $f_2 \leqslant f_1$, i.e. $\frac{l_2}{T} + p_2 + \theta - (n+1) \leqslant 0$.

   From Taylor's expansion

   $$|P_i(x)| \ll Q^{f_1} = Q^{1-p_2-(n+1)+\theta+p_2} = Q^{-n+\theta}.$$

   Lemma 4 is applicable here with $\delta = \frac{\theta}{2},\ \mu = 1,\ \eta = n+1-\theta-p_2,\ \tau = n-\theta : 3n-3\theta+3-2n-2+2\theta+2p_2 < 2n+\delta \Longrightarrow 2p_2+1 < n+\theta+\delta = n+\frac{3\theta}{2}$, which contradicts (17).

So here we also get the measure estimate

$$[|J|Q^{n+1-\theta-p_2}] * Q^{-(n+1)(1+\lambda)+\theta+p_2} = Q^{-\lambda(n+1)}|J|.$$

Case 3

$$2p_2 \leqslant n-1+2\theta, \tag{18}$$

$$p_2 + \frac{l_2}{T} \geqslant 2-\theta. \tag{19}$$

This case covers the widest range of values of p_2 and $\frac{l_2}{T}$, and it is the most difficult to handle.

Like above, we divide J into $[|J|Q^{\frac{l_2}{T}}]$ parts of length $\asymp Q^{-\frac{l_2}{T}}$.
($\frac{l_2}{T} \geqslant \frac{p_2+\frac{l_2}{T}}{n} \geqslant \frac{2-\theta}{n} > c_2(n)(1+\lambda_0(n))$, so $|J| \gg Q^{-\frac{l_2}{T}}$.)

Denote

$$\mu = n+1-\theta-(\frac{l_2}{T}+p_2).$$

From (18) follows $p_2+\frac{l_2}{T} = 2p_2-p_3 \leqslant 2p_2+2(n-2)\frac{1}{T} \leqslant n-1+2\theta+2(n-2)\frac{1}{T}$, hence

$$\mu \geqslant n+1-\theta-n+1-2\theta-2(n-2)\frac{1}{T} = 2-3\theta-2(n-2)\frac{1}{T} > 1.$$

From (19) follows

$$\mu \leqslant n+1-\theta-2+\theta = n-1.$$

First consider only subintervals belonging to not more than Q^μ polynomials. Then the investigated measure is majorized using (15) by the value

$$[|J|Q^{\frac{l_2}{T}}] * Q^\mu * Q^{-(n+1)(1+\lambda)+\theta+p_2} \asymp Q^{\frac{l_2}{T}+\mu-(n+1)(1+\lambda)+\theta+p_2}|J| =$$

$$= Q^{-\lambda(n+1)}|J|.$$

Now we show that the measure of subintervals belonging to more than Q^μ polynomials is small.

If such a subinterval $J' \rightarrowtail P$, i.e. there is an $x_0 \in J'$ with $x_0 \rightarrowtail P$, then for any $x \in J'$ we have

$$|x-\alpha_1| \leqslant |x-x_0| + |x_0-\alpha_1| \ll$$

$$\ll Q^{-\frac{l_2}{T}} + Q^{-(n+1)(1+\lambda)+\theta+p_2} \ll Q^{-\frac{l_2}{T}},$$

since $p_2+\frac{l_2}{T} \leqslant n-1+2\theta+2(n-2)\frac{1}{T} \leqslant n+1-\theta \leqslant (n+1)(1+\lambda)-\theta$. Further,

$$|P^{(j)}(\alpha_1)(x-\alpha_1)^j| \ll Q^{1-p_{j+1}-j\frac{l_2}{T}} \ll Q^{1-p_2-\frac{l_2}{T}},\ j=\overline{1,n},$$

so

$$|P(x)| \ll Q^{1-p_2-\frac{l_2}{T}}. \tag{20}$$

Denote $dg = n - [\mu - \theta]$. From the estimations for μ one easily derives

$$2 \leqslant dg \leqslant n - 1. \tag{21}$$

Divide the segment $[-Q, Q]$ into $[2Q^{\frac{\{\mu-\theta\}}{dg}}] \asymp 2Q^{\frac{\{\mu-\theta\}}{dg}}$ equal segments T_i of length $\asymp Q^{1-\frac{\{\mu-\theta\}}{dg}}$. Note that

$$Q^{1-\frac{\{\mu-\theta\}}{dg}} > Q^{1-\frac{1}{2}} = Q^{\frac{1}{2}},$$

so for sufficiently large $Q_{00}(n, I)$ the number of integral points on T_i is $\asymp Q^{1-\frac{\{\mu-\theta\}}{dg}}$. We say that polynomials $P_1(x)$ and $P_2(x)$ belong to the same class if their $[\mu-\theta]$ highest coefficients are equal, $n-[\mu-\theta]$ next coefficients belong to the same set of segments T_i, and the lowest coefficient is arbitrary. Each of the Q^μ polynomials our J' belongs to gets into one of the

$$(2Q+1)^{[\mu-\theta]}(2Q^{\frac{\{\mu-\theta\}}{dg}})^{n-[\mu-\theta]} \asymp Q^{\mu-\theta}$$

classes, hence, according to Dirichlet principle, there are at least $Q^\mu/Q^{\mu-\theta} = Q^\theta$ polynomials $P_i(x)$ in one of the classes. Consider polynomials

$$S_i(x) = P_i(x) - P_1(x),\ i = \overline{2, [Q^\theta]}.$$

For such polynomials we have

$$\deg(S_i) \leqslant dg,$$

$$H(S_i) \ll Q^{1-\frac{\{\mu-\theta\}}{dg}} = Q_0 \tag{22}$$

and, from (20), throughout on J'

$$|S_i(x)| \ll Q^{1-p_2-\frac{l_2}{T}}. \tag{23}$$

Now we consider three possibilities.

1. If at least two polynomials $S_i(x)$ (say, $S_2(x)$ and $S_3(x)$) are relatively prime, we can apply lemma 4 with $n = dg,\ \delta = 4\frac{1}{T},\ \mu = 1-\frac{\{\mu-\theta\}}{dg},\ \eta = \frac{l_2}{T},\ \tau = p_2+\frac{l_2}{T}-1 : 3p_2+3\frac{l_2}{T}-3+3-3\frac{\{\mu-\theta\}}{dg}-2\frac{l_2}{T} < 2dg(1-\frac{\{\mu-\theta\}}{dg})+\delta \Longrightarrow 3p_2+\frac{l_2}{T}-3\frac{\{\mu-\theta\}}{dg} < 2dg-2\{\mu-\theta\}+\delta = 2(n-[\mu-\theta]-\{\mu-\theta\})+\delta = 2(n-\mu+\theta)+\delta = 2(n-(n+1-\theta-(\frac{l_2}{T}+p_2))+\theta)+\delta = 2(-1+\theta+\frac{l_2}{T}+p_2+\theta)+\delta \Longrightarrow p_3 = p_2-\frac{l_2}{T} < -2+2\theta+2\theta+3\frac{\{\mu-\theta\}}{dg}+\delta < -\frac{1}{2}+2\theta+2\theta+\delta$, but $p_3 \geqslant -2(n-2)\frac{1}{T}$, which is a contradiction.

2. Now let one of the polynomials $S_i(x)$ (say, $S_2(x)$) be reducible over $\mathbb{Z}[x]$, i.e. $S_2(x) = P_1(x)P_2(x)$, $P_1 \in \mathbb{Z}_{n_1}[x], P_2 \in \mathbb{Z}_{n_2}[x]$, $1 \leqslant n_1 \leqslant n_2 < dg$, $n_1 + n_2 \leqslant dg$. Choose $k \in \{1, 2\}$ such that

$$Q_0^{\frac{k-1}{2}} \leqslant H(P_1) \leqslant Q_0^{\frac{k}{2}} = Q_1.$$

It follows that

$$H(P_2) \ll Q_0^{\frac{3-k}{2}} = Q_2.$$

If $Q_{00}(n, I) \geqslant Q_{00}(n_1, I)^4$, $\lambda_0(n) \leqslant \lambda_0(n_1)\frac{n_1+1}{4(n+1)}$, $c_2(n) \leqslant \frac{c_2(n_1)}{4(1+\lambda_0(n))}$, then induction hypothesis is applicable to the set

$$B_{n_1,I}(Q_1, \lambda\frac{2(n+1)}{k(n_1+1)(1-\{\mu-\theta\}/dg)}, J),$$

and we get

$$\mu\, B_{n_1,I}(Q_1, \lambda\frac{2(n+1)}{k(n_1+1)(1-\{\mu-\theta\}/dg)}, J) \leqslant$$

$$\leqslant c(n_1, I)Q_1^{-\lambda\frac{2(n+1)}{k(n_1+1)(1-\{\mu-\theta\}/dg)}(n_1+1)}|J| =$$

$$= c(n_1, I)Q^{-\lambda(n+1)}|J|.$$

Similarly,

$$\mu\, B_{n_2,I}(Q_2, \lambda\frac{2(n+1)}{(3-k)(n_2+1)(1-\{\mu-\theta\}/dg)}, J) \leqslant$$

$$\leqslant c(n_2, I)Q^{-\lambda(n+1)}|J|.$$

So we may consider only x with

$$|P_1(x)P_2(x)| \leqslant Q^{1-p_2-\frac{l_2}{T}},$$

$$|P_1(x)| > Q_1^{-\lambda\frac{2(n+1)}{k(n_1+1)(1-\{\mu-\theta\}/dg)}(n_1+1)-n_1} =$$

$$= Q^{-\lambda(n+1)-\frac{k}{2}n_1(1-\{\mu-\theta\}/dg)},$$

$$|P_2(x)| > Q^{-\lambda(n+1)-\frac{(3-k)}{2}n_2(1-\{\mu-\theta\}/dg)}.$$

But then we have

$$Q^{-\lambda(n+1)-\frac{k}{2}n_1(1-\{\mu-\theta\}/dg)} <$$

$$< |P_1(x)| <$$

$$< Q^{1-p_2-\frac{l_2}{T}+\lambda(n+1)+\frac{(3-k)}{2}n_2(1-\{\mu-\theta\}/dg)},$$

hence $-\lambda(n+1)-\frac{k}{2}n_1(1-\{\mu-\theta\}/dg) < 1-p_2-\frac{l_2}{T}+\lambda(n+1)+\frac{(3-k)}{2}n_2(1-\{\mu-\theta\}/dg) \Longrightarrow p_2+\frac{l_2}{T} < 1+2\lambda(n+1)+\frac{(3-k)}{2}n_2(1-\{\mu-\theta\}/dg)+\frac{k}{2}n_1(1-\{\mu-\theta\}/dg) \leqslant$[see similar arguments for reducible polynomials in section 3]$\leqslant 1+2\lambda(n+1)+(1-\{\mu-\theta\}/dg)(dg-1+\frac{1}{2}) = 1+2\lambda(n+1)+dg-\{\mu-\theta\}-\frac{1}{2}+\frac{\{\mu-\theta\}}{2dg} < \frac{1}{2}+2\lambda_0(n)(n+1)+n-\mu+\theta+\frac{1}{4} = -\frac{1}{4}+\theta+\frac{l_2}{T}+p_2+\theta+2\lambda_0(n)(n+1) \Longrightarrow \frac{1}{4} < \theta+\theta+2\lambda_0(n)(n+1)$, which is contradictory.

3. The only possibility left is that all polynomials $S_i(x)$ are of the form $k_iS_1(x)$, where $S_1(x) \in \mathbb{Z}_{dg}[x]$ and $k_i \in \mathbb{Z}$. Among these Q^θ pairwise distinct integers there must be one with $|k_i| \gg Q^\theta$. Then $S_1(x)$ possesses very strong approximating properties on J' (see (22),(23)):

$$H(S_1) \ll Q^{1-\frac{\{\mu-\theta\}}{dg}-\theta} = Q_0,$$

$$|S_1(x)| \ll Q^{1-p_2-\frac{l_2}{T}-\theta} = Q_0^{\frac{1-p_2-\frac{l_2}{T}-\theta}{1-\{\mu-\theta\}/dg-\theta}}.$$

If $Q_{00}(n,I) \geqslant Q_{00}(dg,I)^{\frac{1}{1/2+\theta}}, \theta \leqslant \lambda_0(dg)\frac{3/2-\theta dg}{(dg-1)}, c_2(n) \leqslant \frac{c_2(dg)(1/2-\theta)}{1+\lambda_0(n)}$, then our induction hypothesis is applicable to the set

$$B_{dg,I}(Q_0, \frac{\theta(dg-1)}{(dg+1)(1-\{\mu-\theta\}/dg-\theta)}, J),$$

and we get

$$\mu\, B_{dg,I}(Q_0, \frac{\theta(dg-1)}{(dg+1)(1-\{\mu-\theta\}/dg-\theta)}, J) \leqslant$$

$$\leqslant c(dg,I)Q_0^{-\frac{\theta(dg-1)}{1-\{\mu-\theta\}/dg-\theta}}|J| = c(dg,I)Q^{-\theta(dg-1)}|J| \leqslant$$

$$\leqslant c(dg,I)Q^{-\lambda(n+1)}|J|$$

when $\lambda_0(n) \leqslant \frac{\theta}{1+n}$.

So we see that all such J' can be covered by a set of measure $\ll Q^{-\lambda(n+1)}|J|$.

Case 4

$$p_2 + \frac{l_2}{T} < 2-\theta, \tag{24}$$

$$p_2 > 1-\theta-c_2(n)(1+\lambda). \tag{25}$$

Let $\mathcal{P}$ be the set of all good polynomials satisfying (24) and (25). For a $P \in \mathcal{P}$ denote

$$\sigma(P) = \left\{x \in \mathbb{R} : |x-\alpha_1| < c_6Q^{-(n+1)(1+\lambda)+\theta+p_2}\right\}, \tag{26}$$

$$\sigma'(P) = \left\{ x \in \mathbb{R} : |x - \alpha_1| < c_6 Q^{-2+\theta+p_2} \right\}, \qquad (27)$$

where c_6 is the constant implied in (15). We are interested (due to (15)) in the measure of $\cup_{P\in\mathcal{P}}\sigma(P)$.

For a vector $\overline{v} = (a_n, \dots, a_2) \in (\mathbb{Z} \cap [-Q, Q])^{n-1}$ let $\mathcal{P}(\overline{v})$ be the set of all polynomials $P \in \mathcal{P}$ with the first $n-1$ coefficients equal to $a_n, \dots, a_2$. We call a polynomial $P \in \mathcal{P}(\overline{v})$ *nonessential* if there is a $\widetilde{P} \in \mathcal{P}(\overline{v})$ such that $\mu\, \sigma'(P) \cap \sigma'(\widetilde{P}) \geqslant \frac{1}{2}\, \mu\, \sigma'(P)$ and *essential* otherwise. Let $\mathcal{P}_1(\overline{v})$ ($\mathcal{P}_2(\overline{v})$) be the set of all essential (resp. nonessential) polynomials $P \in \mathcal{P}(\overline{v})$; further, $\mathcal{P}_1 = \cup_{\overline{v}\in(\mathbb{Z}\cap[-Q,Q])^{n-1}}\mathcal{P}_1(\overline{v})$ and $\mathcal{P}_2 = \cup_{\overline{v}\in(\mathbb{Z}\cap[-Q,Q])^{n-1}}\mathcal{P}_2(\overline{v})$. We will consider $\cup_{P\in\mathcal{P}_1}\sigma(P)$ and $\cup_{P\in\mathcal{P}_2}\sigma(P)$ separately.

1. Firstly, we consider essential polynomials. For an interval $J = [a, b]$ we 'stretch' it to the interval

   $$J' = [a - 2c_6(b - a), b + 2c_6(b - a)].$$

   From (24) follows $p_2 \leqslant \frac{p_2 + \frac{l_2}{T}}{1+1/(n-1)} < \frac{(2-\theta)(n-1)}{n} < 2 - \theta - c_2(n)(1 + \lambda_0(n))$, hence $Q^{-2+\theta+p_2} < |J|$. Since for good $P(x)$ $\sigma(P)$ and therefore $\sigma'(P)$ intersect J, we conclude that $\sigma'(P) \subseteq J'$. One easily sees that no point of J' belongs to more than two intervals $\sigma'(P_i)$ for essential P_i-s. So

   $$\Sigma_{P\in\mathcal{P}_1(\overline{v})}\, \mu\, \sigma'(P) \leqslant 2|J'| = 2(4c_6 + 1)|J|.$$

   Now

   $$\mu \cup_{P\in\mathcal{P}_1}\sigma(P) \leqslant \Sigma_{P\in\mathcal{P}_1}\, \mu\, \sigma(P) = \Sigma_{\overline{v}\in(\mathbb{Z}\cap[-Q,Q])^{n-1}}\Sigma_{P\in\mathcal{P}_1(\overline{v})}\, \mu\, \sigma(P) =$$

   $$= Q^{1-n-\lambda(n+1)}\Sigma_{\overline{v}\in(\mathbb{Z}\cap[-Q,Q])^{n-1}}\Sigma_{P\in\mathcal{P}_1(\overline{v})}\, \mu\, \sigma'(P) \leqslant$$

   $$\leqslant Q^{1-n-\lambda(n+1)}(2Q + 1)^{n-1}2(4c_6 + 1)|J| \ll Q^{-\lambda(n+1)}|J|,$$

   which is exactly what we wanted.

2. For a polynomial $P \in \mathcal{P}_2(\overline{v})$ there must be a $\widetilde{P} \in \mathcal{P}(\overline{v})$ such that

   $$\mu\, \sigma'(P) \cap \sigma'(\widetilde{P}) \geqslant \frac{1}{2}\, \mu\, \sigma'(P).$$

   From (27) and lemma 2, for any $x \in \sigma'(P)$ we have

   $$|P^{(j)}(\alpha_1)(x - \alpha_i)^1| \ll Q^{1-p_{j+1}+j(-2+\theta+p_2)}.$$

   Denoting

   $$f_j = 1 - p_{j+1} + j(-2 + \theta + p_2),$$

   we now show that

   $$\max_{1\leqslant j\leqslant n} f_j = f_1.$$

For $j \geqslant 2$ we estimate $f_j - f_{j-1} = \frac{l_j}{T} - 2 + \theta + p_2 \leqslant \frac{l_2}{T} + p_2 - 2 + \theta < 0$ (we used (24)). Therefore

$$|P(x)| \ll Q^{-1+\theta}.$$

Similarly from the Taylor's expansion

$$P'(x) = \Sigma_{j=0}^{n} \frac{1}{j!} P^{(j+1)}(\alpha_1)(x - \alpha_1)^j$$

one gets the estimate for derivative

$$|P'(x| \ll Q^{1-p_2}.$$

But the same inequalities hold for $\widetilde{P}$ on $\sigma'(\widetilde{P})$. So for

$$S(x) = P(x) - \widetilde{P}(x) = ax + b$$

on $\sigma'(P) \cap \sigma'(\widetilde{P})$ we get:

$$\begin{cases} |S(x)| = |ax + b| & \leqslant & c_7 Q^{-1+\theta}, \\ |S'(x)| = |a| & \leqslant & c_7 Q^{1-p_2}. \end{cases} \tag{28}$$

We suppose Q large enough, so that $c_7 Q^{-1+\theta} < 1$ and thus $a \neq 0$. We also suppose $c_7 Q^{1-p_2} \geqslant 1$: otherwise we get $\mathcal{P}_2 = \emptyset$.
Denote $\mathcal{L} = \{x \in J : (28) \text{ has a solution in } (a, b) \in \mathbb{Z}^2 / \{(0, 0)\}\}$. Then

$$\mu\,\mathcal{L} \leqslant \Sigma_{a=1}^{[c_7 Q^{1-p_2}]} \lceil |J| a \rceil \frac{2 c_7 Q^{-1+\theta}}{a} \leqslant \Sigma_{a=1}^{[c_7 Q^{1-p_2}]} (|J| a + 1) \frac{2 c_7 Q^{-1+\theta}}{a} \ll$$

$$\ll Q^{1-p_2} * Q^{-1+\theta} * |J| + Q^{-1+\theta} * (1 + (1 - p_2) \ln Q) \ll$$

[for sufficiently large Q]

$$\ll Q^{-p_2+\theta} |J| + Q^{-1+\theta+\theta} \ll$$

[see (25)]

$$\ll Q^{-1+2\theta+c_2(n)(1+\lambda)} |J| + Q^{-1+2\theta} \ll$$

$$\ll Q^{-\lambda(n+1)} |J|,$$

since $2\theta + \lambda(n+1) + c_2(n)(1+\lambda) < 1$.
It is obvious that $\mathcal{L}$ can be represented as $\cup_{i=1}^{C} |a_i, b_i|$. Now 'stretch' it to the set

$$\mathcal{L}' = \cup_{i=1}^{C} |a_i - (b_i - a_i), b_i + (b_i - a_i)|.$$

For each $P \in \mathcal{P}_2$ we have $\sigma'(P) \subseteq \mathcal{L}'$, and

$$\cup_{P \in \mathcal{P}_2} \sigma(P) \subseteq \cup_{P \in \mathcal{P}_2} \sigma'(P) \subseteq \mathcal{L}',$$

so

$$\mu \cup_{P \in \mathcal{P}_2} \sigma(P) \leqslant \mu\,\mathcal{L}' \leqslant 3\,\mu\,\mathcal{L} \ll Q^{-\lambda(n+1)} |J|,$$

which is what we wanted.

Case 5

$$p_2 + \frac{l_2}{T} < 2 - \theta, \tag{29}$$

$$p_2 \leqslant 1 - \theta - c_2(n)(1+\lambda). \tag{30}$$

We first show that in this case $\alpha_1 \in \mathbb{R}$. Suppose the contrary. Then $P(x)$ must have a root $\alpha_i = \overline{\alpha}_1$. For $x_0 \rightarrowtail P$ we have according to (7): $H^{-\frac{l_j}{T}-\theta} \leqslant |\alpha_1 - \alpha_i| \leqslant |\alpha_1 - x_0| + |\alpha_i - x_0| = 2|\alpha_1 - x_0| \ll Q^{-(n+1)(1+\lambda)+\theta+p_2} \Longrightarrow$ [for sufficiently large $Q_{00}(n,I)$] $\Longrightarrow (n+1)(1+\lambda) - 3\theta \leqslant p_2 + \frac{l_j}{T} \leqslant p_2 + \frac{l_2}{T} \leqslant 2 - \theta$ (see (29)), which is a contradiction.

Now note that due to (30)

$$|\alpha_1 - x_0| \ll Q^{-(n+1)(1+\lambda)+\theta+p_2} \ll Q^{-n-(n+1)\lambda - c_2(n)(1+\lambda)},$$

hence

$$|\alpha_1 - x_0| \leqslant |J|.$$

So we get $\alpha_1 \in J'$, where J' is a 'stretched' interval J :

$$J' = [a - (b-a), b + (b-a)].$$

Like in case 4, we define $\mathcal{P}$ as the set of all good polynomials satisfying (29) and (30). For a vector $\overline{v} = (a_n, \ldots, a_1) \in (\mathbb{Z} \cap [-Q, Q])^n$ let $\mathcal{P}(\overline{v})$ be the set of all polynomials $P \in \mathcal{P}$ with the first n coefficients equal to $a_n, \ldots, a_1$ and define

$$P_{\overline{v}} = a_n x^n + \ldots + a_1 x.$$

We also denote

$$\sigma(P) = \left\{ x \in \mathbb{R} : |x - \alpha_1| \leqslant 2^{n-1} \frac{|P(x)|}{|P'(\alpha_1)|} \right\}. \tag{31}$$

Due to lemma 1 weare interested in the measure of $\cup_{P \in \mathcal{P}} \sigma(P)$.

We fix $\overline{v} = (a_n, \ldots, a_1) \in (\mathbb{Z} \cap [-Q, Q])^n$ and estimate (noting of course that is $\mathcal{P}(\overline{v})$ finite)

$$\mu \cup_{P \in \mathcal{P}(\overline{v})} \sigma(P) \leqslant \Sigma_{P \in \mathcal{P}(\overline{v})} \, \mu \, \sigma(P) \leqslant$$

$$\leqslant 2^n Q^{-\lambda(n+1)-n} \Sigma_{P \in \mathcal{P}(\overline{v})} \, \mu \, \frac{1}{|P'(\alpha_1)|} =$$

$$= 2^n Q^{-\lambda(n+1)-n} \Sigma_{P \in \mathcal{P}(\overline{v})} \, \mu \, \frac{1}{|P'_{\overline{v}}(\alpha_1)|}.$$

Now, since we have

$$|P'(\alpha_1)| \geqslant c_8 Q^{1-p_2-\theta} \geqslant c_8 Q^{c_2(n)(1+\lambda)} \geqslant c_8 |J|^{-1},$$

using arguments similar to those of Proposition 1 in [7], one shows that

$$\Sigma_{P\in\mathcal{P}(\overline{v})}\, \mu \frac{1}{|P'_{\overline{v}}(\alpha_1)|} \leqslant |J'| + (2n-2)c_8^{-1}|J| \leqslant (3+(2n-2)c_8^{-1})|J|.$$

So we get

$$\mu \cup_{P\in\mathcal{P}} \sigma(P) = \mu \cup_{\overline{v}\in(\mathbb{Z}\cap[-Q,Q])^n} (\cup_{P\in\mathcal{P}(\overline{v})}\sigma(P)) \leqslant$$

$$\leqslant \Sigma_{\overline{v}\in(\mathbb{Z}\cap[-Q,Q])^n}\, \mu \cup_{P\in\mathcal{P}(\overline{v})}\sigma(P) \leqslant$$

$$\leqslant \Sigma_{\overline{v}\in(\mathbb{Z}\cap[-Q,Q])^n} 2^n Q^{-\lambda(n+1)-n}(3+(2n-2)c_8^{-1})|J| \leqslant$$

$$\leqslant (2Q+1)^n 2^n (3+(2n-2)c_8^{-1}) Q^{-\lambda(n+1)-n}|J| \ll$$

$$\ll Q^{-\lambda(n+1)}|J|,$$

and that finishes the proof.

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