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BAD–APPROXIMABLE POINTS AND DISTRIBUTION OF DISCRIMINANTS OF THE PRODUCT OF LINEAR INTEGER POLYNOMIALS[1]

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Abstract

The paper is devoted to the investigation of the distribution of the product of linear integer polynomials. The described method is based on the construction of the set of bad–approximable points, that has a big measure.

Аннотация

В работе изучено распределение дискриминантов произведения линейных многочленов. Основу метода составляет постоение множества плохо аппроксимируемых точек, имеющего большую меру.

1. Introduction

The discriminant of a polynomial is one of its the main characteristics both in algebra and in number theory. For example, if one consider the polynomial of second degree $P(x) = ax^2 + bx + c$, the value of the discriminant $D = b^2 - 4ac$ is necessary for calculating roots, determining whether they are real or not.

There are two ways to define the discriminant $D(P)$ of the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with height $H = \max_{0\leqslant j\leqslant n} |a_j|$ and roots $\alpha_1, \alpha_2, \dots, \alpha_n$.
$D(P)$ can be defined as a determinant

$$D(P) = (-1)^{n(n-1)/2} \begin{vmatrix} 1 & a_{n-1} & a_{n-2} & \dots & a_0 & 0 & \dots \\ 0 & a_n & a_{n-1} & \dots & a_1 & a_0 & \dots \\ & & & \dots & & & \\ 0 & \dots & 0 & a_n & \dots & a_1 & a_0 \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & a_1 & 0 & \dots \\ 0 & na_n & (n-1)a_{n-1} & \dots & 2a_2 & a_1 & \dots \\ & & & \dots & & & \\ 0 & \dots & 0 & na_n & \dots & 2a_2 & a_1 \end{vmatrix} \qquad (1)$$

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or as the transformed product of roots differences

$$D(P) = a_n^{2n-2} \prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j)^2. \tag{2}$$

In the following we consider polynomials with integer rational coefficients only. Hence from (1) we obtain, that if $D(P) \neq 0$ then

$$|D(P)| \geqslant 1. \tag{3}$$

From (2) it obviously follows that $D(P) \neq 0$ if and only if $P(x)$ has no multiple roots.

Now fix $n \in \mathbb{N}$. Let Q be a sufficiently large number, $Q > Q_0(n)$. By $\mathcal{P}_n(Q)$ we denote the class of polynomials $P(x)$ with $\deg P \leqslant n$ and $H(P) \leqslant Q$. By $c(n)$, $c_j(n)$ we denote constants depending on n only. We shall use Vinogradov's symbols: $A \ll B$ denotes that $A \leqslant c_1 B$, and $A \asymp B$ denotes $B \ll A \ll B$. From (1) we obtain that $|D(P)| < c_1(n) Q^{2n-2}$ and if $D(P) \neq 0$, by (3) we have

$$1 \leqslant |D(P)| < c_1(n) Q^{2n-2}. \tag{4}$$

From the restrictions on degree and height of polynomials we obtain that

$$\# \mathcal{P}_n(Q) = (2Q+1)^{n+1} < 3^{n+1} Q^{n+1}. \tag{5}$$

For $n \geqslant 4$ (4) and (5) ensure the existence of intervals in $[1, c_1(n) Q^{2n-2}]$, of length Q, such that we can't find a polynomial with discriminant from these intervals.

The paper is devoted to the study of the distribution of discriminants for special polynomials in $\mathcal{P}_n(Q)$ on the interval $[1, c_1(n) Q^{2n-2}]$.

2. Construction of bad–approximable points

In this paragraph we show a way of constructing for any interval $I \subset \mathbb{R}$ the set of points $B \subset I$ of Lebesgue measure $\mu B > c_2 \mu I = c_2 |I|$ with given properties of simultaneous approximation by rational numbers .

Lemma 1. *Let* I *be an interval,* $I \subset \mathbb{R}$, c_3 *and* c_4 *be positive constants such, that* $\max(c_3, c_4) \leqslant 1$. *For sufficiently large* Q *denote by* $\mathcal{L}_{1,Q}(c_3, c_4)$ *the set of* $x \in I$, *for which the system of inequalities*

$$\begin{cases} |qx - p| < c_3 Q^{-1}, \\ 1 \leqslant q \leqslant c_4 Q, \end{cases} \tag{6}$$

has solutions with coprime $(p, q) \in \mathbb{Z} \times \mathbb{N}$. *Then for* $c_3 c_4 < \lambda$, $0 < \lambda < \frac{1}{3}$, *we have*

$$\mu \mathcal{L}_{1,Q}(c_3, c_4) < 3\lambda |I|.$$

Proof. For fixed p and q the first inequality of (6) holds for

$$x \in I(p,q) = (\frac{p}{q} - c_3Q^{-1}q^{-1}, \frac{p}{q} + c_3Q^{-1}q^{-1}),$$

and $|I(p,q)| = 2c_3Q^{-1}q^{-1}$. For fixed q, provided that $x \in I$, the inequality does not hold hold for more than $q|I| + 2$ values of p. We estimate the measure of $\cup_p I(p,q)$ from above by

$$\mu\Big(\cup_p I(p,q)\Big) \leqslant 2c_3Q^{-1}q^{-1}(q|I|+2) < 3c_3Q^{-1}|I|.$$

Obviously

$$\mathcal{L}_{1,Q}(c_3,c_4) \in \cup_{1\leqslant q\leqslant c_4Q} \cup_p I(p,q),$$

therefore

$$\mu\mathcal{L}_{1,Q}(c_3,c_4) \leqslant 3c_3c_4|I| < 3\lambda|I|.$$

For $(p,q)=1$ the result can be improved.

Lemma 2. *Denote by* $M_n(Q)$ *the set of* $x \in I$*, for which the following* $2n$ *inequalities*

$$\begin{cases} \frac{Q^{-1}}{6^i(n+1)^i} < |q_ix - p_i| < \frac{Q^{-1}}{6^{i-1}(n+1)^{i-1}} \\ 6^{i-2}(n+1)^{i-2}Q \leqslant q_i \leqslant 6^{i-1}(n+1)^{i-1}Q, \ 1 \leqslant i \leqslant n \end{cases}$$

have solutions in $\{(p_i,q_i)\}_{i=1}^n$. *Then* $\mu M_n(Q) > \frac{|I|}{n+1}$.

proof. By Dirichlet's theorem the two inequalities

$$\begin{cases} |qx - p| < Q^{-1}, \\ 1 \leqslant q \leqslant Q \end{cases}$$

have solutions in integers p and q for any $Q > 1$ and $x \in I$. Denote by B_1 the set of $x \in I$, for which the following two inequalities hold for some integers p_1 and q_1

$$\begin{cases} \frac{Q^{-1}}{6(n+1)} \leqslant |q_1x - p_1| < Q^{-1} \\ \frac{Q}{6(n+1)} < q_1 \leqslant Q. \end{cases} \tag{7}$$

By Lemma 1 the measure of x, for which (7) is violated, doesn't exceed $\frac{|I|}{n+1}$. Indeed, the set A_1 of pointed x is the unit of two sets E_1 and F_1. The set E_1 denotes the set of solutions of the system

$$\begin{cases} |q_1x - p_1| < Q^{-1}, \\ q_1 < \frac{Q}{6(n+1)}. \end{cases}$$

The set F_1 denotes the set of solutions of the system

$$\begin{cases} |q_1x - p_1| < \frac{1}{6Q(n+1)}, \\ q_1 \leqslant Q. \end{cases}$$

By Lemma 1 we have $\mu E_1 \leqslant \frac{|I|}{2(n+1)}$, $\mu F_1 \leqslant \frac{|I|}{2(n+1)}$. Since $A_1 \subset E_1 \cap F_1$ we have $\mu A_1 \leqslant \mu E_1 + \mu F_1 < \frac{|I|}{n+1}$. Therefore $\mu B_1 > |I| - \frac{1}{n+1}|I| = \frac{n}{n+1}|I|$. Denote $Q_2 = 6(n+1)Q$. Similarly, by Dirichlet's theorem the two inequalities

$$\begin{cases} |qx - p| < Q_2^{-1}, \\ 1 \leqslant q \leqslant Q_2 \end{cases} \tag{8}$$

have solutions in (p, q). Denote by B_2 the subset of $I \setminus B_1$, for which the following two inequalities hold

$$\begin{cases} \frac{Q^{-1}}{6^2(n+1)^2} = \frac{Q_2^{-1}}{6(n+1)} \leqslant |q_2 x - p_2| < Q_2^{-1} = \frac{Q^{-1}}{6(n+1)} \\ Q = \frac{Q_2}{6(n+1)} < q_2 \leqslant Q_2 = 6(n+1)Q. \end{cases}$$

Again by Lemma 1 the measure of x for which one of these inequalities is violated does not exceed $\frac{|I|}{n+1}$. Therefore $\mu B_2 > \frac{n}{n+1}|I| - \frac{1}{n+1}|I| = \frac{n-1}{n+1}|I|$.

We repeat the described procedure $n-2$ times. Finally with $Q_n = 6^n(n+1)^n Q$ we shall construct the set B_n with measure $\mu B_n > \frac{1}{n+1}|I|$, for which n systems of inequalities hold. The last system being

$$\begin{cases} \frac{Q^{-1}}{6^n(n+1)^n} = \frac{Q_n^{-1}}{6(n+1)} \leqslant |q_i x - p_i| < Q_n^{-1} = \frac{Q^{-1}}{6^{n-1}(n+1)^{n-1}} \\ 6^{n-2}(n+1)^{n-2}Q = \frac{Q_n}{6(n+1)} < q_i \leqslant Q_n = 6^{n-1}(n+1)^{n-1}Q. \end{cases} \tag{9}$$

The systems of inequalities (7) – (9) proves Lemma 2.

By Lemma 2 one can show that there are a lot of polynomials $P(x) \in \mathcal{P}_n(Q)$ with absolute values of discriminants from the interval $[1, c(n)]$.

To prove it let us consider the $B_n \subset I$ of measure $\mu B_n > \frac{|I|}{n+1}$, constructed in Lemma 2. Hence for any point x of B_n we can construct n pairs (p_j, q_j), that satisfy the requirements of Lemma 2.

Consider the polynomial

$$T(x) = \prod_{j=1}^{n} (q_j x - p_j).$$

Denote $\rho = 6(n+1)$, $\alpha_j = \frac{p_j}{q_j}$. Then from the second inequality of (9) we obtain

$$\rho^{\frac{n(n-2)}{2}} Q^n < H(T) < \rho^{\frac{n(n-1)}{2}} Q^n.$$

For $i < j$ we obtain

$$|\alpha_i - \alpha_j| \leqslant |x - \alpha_i| + |x - \alpha_j| < \rho^{-(i-1)}Q^{-2} + \rho^{-(j-1)}Q^{-2} < 2\rho^{-(i-1)}Q^{-2} \tag{10}$$

and

$$\prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j)^2 < 2^{n(n-1)} \rho^{-(n-2)(n-1)} Q^{-2n(n-1)}. \tag{11}$$

From (10) and (11) we have

$$\begin{aligned} D(T) &= (\textstyle\prod_{j=1}^{n} q_j)^{2n-2} \prod_{1\leqslant i<j\leqslant n}(\alpha_i-\alpha_j)^2 < \\ &< Q^{2n(n-1)}\rho^{n(n-1)^2-(n-2)(n-1)}2^{n(n-1)}Q^{-2n(n-1)} = \\ &= \rho^{(n-1)(n^2-2)}2^{n(n-1)} = 6^{(n-1)(n^2-2)}2^{n(n-1)}(n+1)^{(n-1)(n^2-2)} < \\ &< 6^{(n-1)(n^2+n-2)}(n+1)^{(n-1)(n^2-2)} = K. \end{aligned} \tag{12}$$

All rational points α_i belong to an interval, containing x of length not more than $12(n+1)Q^{-2}$. Thus we may choose a point x from the interval

$$A_1 = [\inf\{x : x \in B_n\}, \inf\{x : x \in B_n + 12(n+1)Q^{-2}\}]$$

and then exclude the interval A_1 from the set B_n. Denote by $B_{n+1} = B_n \setminus A_1$ the resulting set, such that $\mu B_{n+1} > \frac{|I|}{n+1} - 12(n+1)Q^{-2}$. Then we choose a new point x from the interval

$$A_2 = [\inf\{x : x \in B_{n+1}\}, \inf\{x : x \in B_{n+1} + 12(n+1)Q^{-2}\}].$$

Similarly we construct a new polynomial $T_1(x)$ for the corresponding x. The absolute value of the discriminant $D(T_1)$ will be smaller than the right-hand side (12). In such a way we may construct more than $[\frac{|I|Q^2}{12(n+1)^2}]$ different polynomials with absolute values of discriminants less then K. This shows

Theorem 1. *For any sufficiently large* M *there at least* $\left[\frac{|I|Q^2}{12(n+1)^2}\right]$ *polynomials* $P(x) \in \mathcal{P}_n(M)$ *with absolute values of discriminants from the interval*

$$[1, 6^{(n-1)(n^2+n-2)}(n+1)^{(n-1)(n^2-2)}].$$

3. The existence of the polynomials with discriminants from an arbitrary interval

In this section we will show how an improvement of the proposed method allows to construct polynomials with discriminants from any part of the possible range. We construct $n-1$ approximations $\frac{p_i}{q_i}$ of the point x as before, but the pair (p_n, q_n), $q_n > q_{n-1}$ will be chosen in a special way to provide the necessary value for the discriminant. For some $0 \leqslant \alpha < \frac{1}{n}$, the values q_i, $1 \leqslant i < n$, will be chosen to be of the same order Q^{α}, but q_n will be of order $Q^{1-(n-1)\alpha}$. Then the approximations of the point x by fractions α_j, $1 \leqslant j \leqslant n-1$ will be best possible, i.e. of order q_j^{-2}, but the approximation by α_n will be of an arbitrarily order $Q^{u(1-(n-1)\alpha)}$, where $-2 \leqslant u \leqslant 0$. We shall obtain polynomials with discriminants from any chosen intervals by appropriate choices of the parameters α and u.

Consider a point x_0 from the $B_n \subset I$. For $Q > Q_0$ and $0 < \alpha \leqslant \frac{1}{n}$ assume $T_1 = [Q^{\alpha}]$. As in Lemma 2 we construct $n-1$ approximating functions $q_j x - p_j$, $1 \leqslant j \leqslant n-1$, such that

$$\begin{cases} \frac{T_1^{-1}}{6^j(n+1)^j} = \frac{T_j^{-1}}{6(n+1)} < |q_j x_0 - p_j| < T_j^{-1} = \frac{T_1^{-1}}{6^{j-1}(n+1)^{j-1}}, \\ 6^{j-2}(n+1)^{j-2}T_1 = \frac{T_j}{6(n+1)} \leqslant q_j \leqslant T_j = 6^{j-1}(n+1)^{j-1}T_1. \end{cases} \tag{13}$$

The last approximating function $q_n x - p_n$ we will construct using: $T_n = Q^{1-\alpha(n-1)}$

$$\begin{cases} \frac{T_n^{-1}}{6(n+1)} < |q_n x_0 - p_n| < T_n^{-1}, \\ \frac{T_n}{6(n+1)} \leqslant q_n \leqslant T_n. \end{cases} \tag{14}$$

Furthermore we pick a linear expression $sx_0 - t$, $q_n \leqslant s < 2q_n$ such, that $|sx_0 - t| \ll q_n$. The set of values $|sx_0 - t|$ contains the interval $S = [6^{-1}(n+1)^{-1}T_n^{-1}, 6^{-1}(n+1)^{-1}T_n]$, that is the set of possible values for the function $6^{-1}(n+1)^{-1}Q^{u(1-\alpha(n-1))}$ for $-1 \leqslant u \leqslant 1$. Form now $\alpha_n = \frac{t}{s}$, so $|x_0 - \alpha_n| \asymp Q^{u(1-\alpha(n-1))}$. Let us estimate the discriminant $D(A)$ of the polynomial

$$A(x) = \prod_{i=1}^{n-1}(q_i x - p_i)(sx - t).$$

The height of the polynomial satisfies the inequalities

$$\prod_{i=1}^{n} q_i \leqslant H(A) \leqslant 2\prod_{i=1}^{n} q_i.$$

From the inequalities (13), (14) we obtain

$$\left(\prod_{j=1}^{n-1} 6^{j-2}(n+1)^{j-2}\right)T_1^{n-1}T_n \leqslant \prod_{i=1}^{n} q_i \leqslant 2\left(\prod_{j=1}^{n-1} 6^{j-1}(n+1)^{j-1}\right)T_1^{n-1}T_n,$$

which gives us the estimate of the height of the polynomials $A(x)$ as

$$H(A) \asymp Q. \tag{15}$$

First of all by (13) let us estimate the absolute values of the differences between roots $\alpha_j = \frac{p_j}{q_j}$ of the polynomial $A(x)$ for $1 \leqslant i < j \leqslant n-1$:

$$\begin{aligned} |\alpha_i - \alpha_j| &\leqslant |x_0 - \tfrac{p_i}{q_i}| + |x_0 - \tfrac{p_j}{q_j}| \leqslant \\ &\leqslant T_1^{-1}q_i^{-1}6^{-(i-1)}(n+1)^{-(i-1)} + T_1^{-1}q_j^{-1}6^{-(j-1)}(n+1)^{-(j-1)} \leqslant \\ &\leqslant 2 \cdot 6^{-(i-1)}(n+1)^{-(i-1)}T_1^{-2} \ll Q^{-2\alpha} \end{aligned} \tag{16}$$

and

$$|\alpha_i - \alpha_j| \geqslant \left|\frac{p_i q_j - p_j q_i}{q_i q_j}\right| \geqslant \frac{1}{q_i q_j} \gg Q^{-2\alpha}. \tag{17}$$

The inequalities (16) and (17) yield the estimate $|\alpha_i - \alpha_j| \asymp Q^{-2\alpha}$. In addition we bound the differences $|\alpha_i - \alpha_n|$ and obtain $|\alpha_i - \alpha_n| \asymp Q^{u(1-\alpha)}$. Thus one can estimate

$$\prod_{1 \leqslant i < j \leqslant n-1} (\alpha_i - \alpha_j)^2 \asymp Q^{-2(n-2)(n-1)}. \tag{18}$$

and the second factor of the discriminant can be bounded via

$$\prod_{k=1}^{n-1}(\alpha_k - \alpha_n)^2 \asymp Q^{2u(1-\alpha(n-1))(n-1)}. \tag{19}$$

Let us replace the expression $Q^{2u(1-\alpha(n-1))}$ by

$$Q^{\nu(1-\alpha(n-1))}, \ -2 \leqslant \nu \leqslant 0. \tag{20}$$

The following upper estimate for the discriminant follows from (15) – (20)

$$|D(A)| \asymp Q^{2n-2-2\alpha(n-1)(n-2)+2\nu(1-\alpha(n-1))(n-1)} = Q^{(2n-2)(1-\alpha(n-2)+\nu(1-\alpha(n-1)))}.$$

Now we have to show that the function

$$f(\alpha,\nu) = 1 - \alpha(n-2) + \nu(1-\alpha(n-1)),$$

$$0 \leqslant \alpha \leqslant \frac{2}{n}, \ -2 \leqslant \nu \leqslant 0 \tag{21}$$

takes any value from the interval $[0,1]$. We will prove that for any $\theta \in [0,1]$ there are α and ν, satisfying (21), such that $f(\alpha,\nu) = \theta$. From

$$\theta = 1 - \alpha(n-2) + \nu(1-\alpha(n-1))$$

we obtain

$$\alpha = \frac{\nu + 1 - \theta}{(n-2) + \nu(n-1)}. \tag{22}$$

Its derivative

$$\alpha'_\nu = \frac{\theta(n-1) - 1}{((n-2) + \nu(n-1))^2}$$

exists on the whole interval $[-2,0]$ except for the point $\nu = -\frac{n-2}{n-1}$. If $\frac{1}{n-1} < \theta \leqslant 1$ the function $\alpha(\nu)$ increases on the intervals $[-2, -\frac{n-2}{n-1})$ and $(-\frac{n-2}{n-1}, 0]$. The values $\alpha(-2) = \frac{1+\theta}{n}$ and $\alpha(0) = \frac{1-\theta}{n-2}$ demonstrate that for $0 < \theta \leqslant 1$ the equation (22) has solutions for $0 \leqslant \alpha \leqslant \frac{1}{n}$ only when $\theta - 1 \leqslant \nu$. For $\frac{1-\theta}{n-2} \geqslant \frac{1}{n}$, i.e. for $\theta \leqslant \frac{2}{n}$, the maximum is attained. In the other case it equals $\frac{1-\theta}{n-2}$.

If $0 \leqslant \theta < \frac{1}{n-1}$ the function $\alpha(\nu)$ decreases on the intervals $[-2, -\frac{n-2}{n-1}) \cup (-\frac{n-2}{n-1}, 0]$ and can take any value θ for $n\theta - 2 \leqslant \nu \leqslant \theta - 1$ and $\max \alpha(\nu) = \frac{1}{n}$ for $\nu = n\theta - 2$.

Thus we have proved that if the discriminant involves the continuous function $f(\alpha,\nu)$ there always exist α and ν that $D(A) \in (c_5(n)Q^\theta, c_6(n)Q^\theta)$ for any $0 \leqslant \theta \leqslant 2n-2$. But the values $sx - t$ are discrete. However under the restrictions on x we can prove by using the Erdös–Turan inequality [4], that in any interval $I \subset [Q^{-1}, Q]$ of the length $|I| > Q^{-\frac{1}{2}}$ there exists a number $sx - t$ and therefore for any θ we can find s and t, such that the discriminant will satisfy the inequality

$$|D(a) - Q^\theta| < c(n)Q^\theta.$$

Thus we have proved

Theorem 2. *For any* θ, $0 \leqslant \theta \leqslant 2n-2$, *there exist polynomials* $P(x) \in \mathcal{P}_n(Q)$ *with discriminants satisfying the inequalities*

$$c_7(n)Q^{\theta} < |D(P)| < c_8(n)Q^{\theta}.$$

REFERENCES

[1] Sprindzuk V. G. Mahler's problem in metric Number Theory // Nauka i Tehnika. Minsk, 1967.

[2] Bernik V. I., Dodson M. M. Metric Diophantine approximation on manifolds // Cambridge Univ. Press: Cambridge Tracts in Math. 1999. Vol. 137.

[3] Győry K., Polynomials and binary forms with given discriminant // Publ. Math. Debrecen. 2006. 69/4. P. 473 – 499.

[4] Keipers L., Niederreiter G. Uniform distributin of sequences. Pure and Applied Mathematics // Wiley-Interscience [John Wiley & Sons], New York-London-Sydney. 1974. 390 pp.

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