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Simultaneous Approximation of Zero by Values of Integral Polynomials

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The direction of studies formulated in the title was indicated by the works of Sprindzhuk [6, 7] in which he posed the basic problem of this direction, solved particular cases of the problem, and pointed out some applications. The conjectures by V. G. Sprindzhuk were proved in 1980 [3]. Later various generalizations and applications were obtained [4, 9], but the method itself does not allow one in principle to pass from a power function on the right-hand side of the inequality to a logarithmic one; this does not allow one in principle to obtain many refined characteristics of classical sets appearing in the theory of transcendental numbers.

In this paper, we obtain a two-dimensional analog of the theorem in [3]; this analog can be considered as a proof of a two-dimensional generalization of the conjecture by Baker [8], which was proved in [1].

Suppose that $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is a polynomial with integer coefficients, $H = H(P) = \max_{0 \le i \le n} |a_i|$ is the height of $P(x)$, the function $\Psi(x)$ monotonically decreases for $x > 0$, and $\sum_{H=1}^{\infty} \Psi(H) < \infty$.

Theorem. *The system of inequalities*

$$\begin{cases} |P(\omega_1)| < H^{-w_1}\Psi^{v_1}(H) \\ |P(\omega_2)| < H^{-w_2}\Psi^{v_2}(H), \end{cases} \tag{1}$$

where $w_1 + w_2 = n - 2$, $v_1 + v_2 = 1$, *has only a finite number of solutions in polynomials* $P(x) \in \mathbb{Z}[x]$ *for almost all* $(w_1, w_2) \in \mathbb{R}^2$.

We introduce some notation and give several lemmas, which are necessary for further arguments. Denote by $c(n)$ positive functions depending only on n. We shall realize operations over $c(n)$ by the formal rules

$$c(n) + c(n) = c(n), \qquad c(n)c(n) = c(n),$$

the sense of these rules consists in the fact that the sum and the product are again a certain function depending on n. Since in the theorem one speaks on the finiteness or infiniteness of the number of solutions to the system of inequalities (1), we shall assume that $H > H_0$, where H_0 is a sufficiently large positive integer. Further, we shall assume that ω_1 and ω_2 both are transcendental numbers since the measure of $(\omega_1, \omega_2) \in \mathbb{R}^2$ for which at least one of the numbers ω_1 and ω_2 is algebraic is equal to zero.

Lemma 1. *Let $P_1(x), \dots, P_l(x) \in \mathbb{R}[x]$ be polynomials and let $P(x) = P_1(x)P_2(x)\cdots P_l(x)$. Then*

$$c_1(n)H(P_1)H(P_2)\cdots H(P_l) < H(P) < c_2(n)H(P_1)H(P_2)\cdots H(P_l).$$

Lemma 1 is proved, for example, in [7].

Lemma 2. *Let $G \subset \mathbb{R}^2$ be some bounded domain and let $B \subset G$ be a measurable set in the plane, $\mu B > c_1(n)\mu G$, where μB and μG are the Lebesgue measures of the sets B and G in $\mathbb{R}^2$. Let further the inequality*

$$|P(\omega_1)P(\omega_2)| < H^{-w}$$

with $\deg P(x) \le n$ *hold for $(\omega_1, \omega_2) \in B$. Then the inequality*

$$|P(\omega_1)P(\omega_2)| < c_2(n)H^{-w}$$

holds for all $(\omega_1, \omega_2) \in G$.

Lemma 2 is an analog of Lemma 10 in [2] and is proved with the help of the Lagrange interpolation formula.

Denote by $P_n(H)$ the class of irreducible polynomials with the condition $a_n(P) = H$. Suppose that $P_n = \bigcup_{H=1}^{\infty} P_n(H)$. Let $P(x) \in P_n(H)$ and let $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ be roots of this polynomial. We shall assume that the roots $\varkappa_1, \varkappa_2, \dots, \varkappa_n$ are ordered in such a way that $\operatorname{Re}\varkappa_1 \le \operatorname{Re}\varkappa_2 \le \dots \le \operatorname{Re}\varkappa_n$. In the case of the equality $\operatorname{Re}\varkappa_l = \operatorname{Re}\varkappa_t$, we shall previously write the root whose modulus of the imaginary part is less than that of the other ones, and in the case where the moduli of the imaginary parts are equal, we put previously the root whose imaginary part is positive. Choose any two roots $\varkappa_{11}$ and $\varkappa_{21}$. We order all other roots with respect to each of these roots in the following way:

$$|\varkappa_{11} - \varkappa_{12}| \le |\varkappa_{11} - \varkappa_{13}| \le \dots \le |\varkappa_{11} - \varkappa_{1n}|, \qquad |\varkappa_{21} - \varkappa_{22}| \le |\varkappa_{21} - \varkappa_{23}| \le \dots \le |\varkappa_{21} - \varkappa_{2n}|.$$

Introduce the following notation:

$$|\varkappa_{11} - \varkappa_{1i}| = H^{-\mu_i}, \quad i = 2, \dots, n, \qquad |\varkappa_{21} - \varkappa_{2s}| = H^{-\Theta_s}, \quad s = 2, \dots, n.$$

Fix ε. Set $\varepsilon_1 = \varepsilon d^{-1}$, where $d = d(n)$ is a sufficiently large value. Set $T = [\varepsilon_1^{-1}]$. Define integers l_i and s_i by the inequalities

$$\frac{l_i - 1}{T} \le \mu_i < \frac{l_i}{T}, \quad i = 2, \dots, n, \qquad \frac{s_i - 1}{T} \le \Theta_i < \frac{s_i}{T}, \quad i = 2, \dots, n.$$

We shall assign an integral vector $\overline{s}_{1,2} = \overline{s} = (l_2, \dots, l_n, s_2, \dots, s_n)$ to a fixed pair of roots $(\varkappa_{11}, \varkappa_{21})$ of the polynomial $P(x)$. All polynomials $P(x) \in P_n(H)$ having one and the same vector $\overline{s}$ are joined into the class $P_n(H, \overline{s})$. Let

$$p_i = \frac{l_{i+1} + \dots + l_n}{T}, \quad q_i = \frac{s_{i+1} + \dots + s_n}{T}, \qquad i = 1, \dots, n-1,$$

$$S(\varkappa_{11}) = \Big\{\omega_1 \in \mathbb{R} : \min_{1 \le l \le n} |\omega_1 - \varkappa_l| = |\omega_1 - \varkappa_{11}|\Big\},$$

$$S(\varkappa_{21}) = \Big\{\omega_2 \in \mathbb{R} : \min_{1 \le s \le n} |\omega_2 - \varkappa_s| = |\omega_2 - \varkappa_{21}|\Big\}.$$

Lemma 3. *The inequality*

$$|P(\omega_1)P(\omega_2)| < H^{-n+2-\delta} \tag{2}$$

has only a finite number of solutions in reducible integral polynomials of degree not exceeding n for any $\delta > 0$ and almost all $(\omega_1, \omega_2) \in \mathbb{R}^2$.

Proof. We shall consider $\omega_1 \in S(\varkappa_{11})$ and $\omega_2 \in S(\varkappa_{21})$. Denote by $S(n)$ the set of (ω_1, ω_2) for which inequality (2) holds infinitely many times. Let $M(H)$ be the set of polynomials $P(x)$ satisfying the condition $H(P) = H$. Set $M_t = \bigcup_{2^t \le H < 2^{t+1}} M(H)$, $M = \bigcup M_t$, $t > c(n) \ln H_0(\delta)$. If inequality (2) for $P(x) \in M_t$ holds for all (ω_1, ω_2) in a certain domain $S(P)$, then by Lemma 1, there exists a polynomial $P_1(x)$ which divides $P(x)$ and which satisfies the inequality

$$|P_1(\omega_1)P_1(\omega_2)| < c(n)H(P_1)^{-n+2-\delta}, \qquad \deg P_1 \le n-1, \tag{3}$$

on the set $S_1(P)$, $\mu S_1(P) > c(n)\mu S(P)$. By Lemma 2, inequality (3) holds for all $(\omega_1, \omega_2) \in S(P)$, maybe with another value of $c(n)$. Consider three cases appeared.

1. If $H(P_1) > 2^{t\varepsilon/(2(n+1)n^2)}$, then using the metric theorem in [3], we obtain that the set $S = \bigcup_{p \in M} S(P)$ can be covered by a system of domains whose total measure does not exceed an arbitrary given positive number.

2. If $H(P_1) \le 2^{t\varepsilon/(2(n+1)n^2)}$ and

$$|P_1(\omega_1)P_1(\omega_2)| < 2^{-t\varepsilon/n} \tag{4}$$

on the set $B_1(P)$, $\mu B_1(P) > c(n)\mu S(P)$, then again applying Lemma 2, we obtain that

$$|P_1(\omega_1)P_1(\omega_2)| < c(n)2^{-t\varepsilon/n}$$

for $(\omega_1, \omega_2) \in S(P)$. Further, we have $\mu S(P) < c(n)2^{-t\varepsilon/n}$. Since the number of polynomials $P_1(x)$ with the condition $H(P_1) \le 2^{t\varepsilon/(2(n+1)n^2)}$ does not exceed $c(n)2^{t\varepsilon/(2n^2)}$, we have

$$\sum_{P(x), H(P_1) \le 2^{t\varepsilon/(2(n+1)n^2)}} 1 < c(n)2^{-t\varepsilon/(2n^2)}.$$

Since the series $\sum_t 2^{-t\lambda}$ converges for $\lambda > 0$, we use the Borel–Cantelli lemma in order to complete the proof.

3. Assume that inequality (4) does not hold on the set $B_1(P)$ with the condition $\mu B_1(P) > c(n)\mu S(P)$. Then the inequality $|P_1(\omega_1)P_1(\omega_2)| > 2^{-t\varepsilon/n}$ holds on the set $B_2(P) = S(P) \setminus B_1(P)$, $\mu B_2(P) > c(n)\mu S(P)$. Set $P_2(x) = P(x)/P_1(x)$. Using Lemmas 1 and 2, we obtain from inequality (2) that

$$|P_2(\omega_1)P_2(\omega_2)| < c(n)H^{-n+2-\delta-\varepsilon/n}, \qquad c(n)2^{t(1-\varepsilon)/(2(n+1)n^2)} < H(P_2) < 2^{t+1}.$$

Again, we apply the metric theorem in [3].

Lemma 4. *We can consider only polynomials $P(x) \in P_n$ in the system of inequalities* (1).

The passing to polynomials whose leading coefficient is equal to the height is realized in the same way as in [3]. The passing to irreducible polynomials is realized by application of Lemma 3.

Lemma 5. *The number of the classes $P_n(H, \overline{s})$ is finite and depends only on n and ε.*

Lemma 5 is proved in [3].

Lemma 6. *Let $P(x) \in P_n(H)$ and let $\omega \in S(\varkappa_{11})$. Then*

$$|\omega - \varkappa_{11}| \le 2^n \frac{|P(\omega)|}{|P'(\varkappa_{11})|}, \tag{5}$$

$$|\omega - \varkappa_{11}| \le \min_{2\le j\le n} \left(2^n \frac{|P(\omega)|}{|P'(\varkappa_{11})|} |\varkappa_{11} - \varkappa_{12}| \cdots |\varkappa_{11} - \varkappa_{1j}| \right)^{1/j}. \tag{6}$$

Lemma 6 is proved in [3].

Lemma 7. *Let $P(x) \in P_n(H, \overline{s})$. Then*

$$|P^{(l)}(\varkappa_{11})| < c(n) H^{1-p_l+(n-l)\varepsilon_1}, \qquad l = 1, \dots, n-1.$$

Lemma 7 is proved in [3].

Lemma 8. *Let $\delta > 0$ be a certain real number and let $n \ge 2$ be a positive integer. Let $P(x)$ and $Q(x)$ be integral relatively prime polynomials of degree at most n and let $\max\bigl(H(P), H(Q)\bigr) \le H$. If for all (ω_1, ω_2) in a certain rectangle $K = I_1 \times I_2$, $|I_1| = H^{-\eta_1}$, $\eta_1 > 0$, $|I_2| = H^{-\eta_2}$, $\eta_2 > 0$, the inequalities*

$$\max\bigl(|P(\omega_1), Q(\omega_1)|\bigr) < H^{-\tau_1}, \qquad \max\bigl(|P(\omega_2), Q(\omega_2)|\bigr) < H^{-\tau_2}$$

hold, then

$$\tau_1 + \tau_2 + 2 + 2\bigl(\max(\tau_1 + 1 - \eta_1, 0) + \max(\tau_2 + 1 - \eta_2, 0)\bigr) \le 2n + \delta.$$

Lemma 8 is proved in [5].

Lemma 9. *Let $B(\delta, w_1, w_2, v_1, v_2)$ be the set of real vectors $\overline{\omega} = (\omega_1, \omega_2)$ for which the system of inequalities (1) has an infinite number of solutions in polynomials $P(x) \in P_n$ with the condition $|\varkappa_i - \varkappa_j| > \delta$ for any i and j and an arbitrary but fixed $\delta > 0$. Then we have $\mu B(\delta, w_1, w_2, v_1, v_2) = 0$ for any $\delta > 0$.*

The proof of Lemma 9 is slightly different from that of Lemma 10 in [3].

Lemma 10. *Let $B_2(w_1, w_2, v_1, v_2)$ be the set of $\overline{\omega}$ for which the system of inequalities (1) has an infinite number of solutions in polynomials $P(x) \in P_2$. Then $\mu B(\delta, w_1, w_2, v_1, v_2) = 0$.*

Lemma 10 is proved analogously to Lemma 11 in [3]. As in [3], it is proved that any set of $\overline{\omega}$ for which the system of inequalities (1) holds infinitely many times in polynomials $P(x) \in P_n$ with condition $|\varkappa_{11}(P) - \varkappa_{21}(P)| \le \delta$ for any $\delta > 0$ has zero measure. Therefore, in what follows, we can assume that $|\varkappa_{11}(P) - \varkappa_{21}(P)| > \delta$ for a certain arbitrary but fixed $\delta > 0$.

Denote by $L(w_1, w_2, v_1, v_2)$ the set of (ω_1, ω_2) for which the system of inequalities (1) has an infinite number of solutions in polynomials $P(x) \in \mathbb{Z}[x]$.

We divide the **proof of the theorem** into nine cases.

Proposition 1. *If*

$$l_2T^{-1} + p_1 \le w_1 + v_1 + 1, \tag{7}$$

$$s_2T^{-1} + q_1 \le w_2 + v_2 + 1, \tag{8}$$

$$n - 1 + 4n\varepsilon_1 < l_2T^{-1} + p_1 + s_2T^{-1} + q_1,$$

then $\mu L(w_1, w_2, v_1, v_2) = 0$.

Proof. We use the inequality $\Psi(H) < H^{-1}$ which can easily be obtained. Then system (1) takes the form

$$\begin{cases} |P(\omega_1)| < H^{-w_1-v_1} \\ |P(\omega_2)| < H^{-w_2-v_2}. \end{cases} \tag{9}$$

Define $P_t(\overline{s}) = \bigcup_{2^t \le H < 2^{t+1}} P_n(H, \overline{s})$. Since $a_n = H$ and the inequality $|\varkappa_i| \le 1 + H/|a_n|$ is known, we have $|\varkappa_i| \le 2$. Then it follows from inequality (2) for $j = n$ that all $(\omega_1, \omega_2) \in \mathbb{R}^2$ for which the system of inequalities (9) holds are in the interior of the square $[-3; 3] \times [-3; 3]$. Divide this square into equal rectangles K with sides $H^{-\eta_1}$ and $H^{-\eta_2}$, $\eta_1 = w_1 + v_1 + 1 - p_1 - \varepsilon_1/2$, $\eta_2 = w_2 + v_2 + 1 - q_1 - \varepsilon_1/2$. We shall say that the polynomial $P(x)$ belongs to the rectangle K if there exists $(\omega_1, \omega_2) \in K$ such that $|P(\omega_1)| < H^{-w_1-v_1}$, $|P(\omega_2)| < H^{-w_2-v_2}$. Assume that at most one polynomial $P(x) \in P_t(\overline{s})$ belongs to each rectangle K. Then we obtain from relation (5) that the measure of the set of $\omega_1 \in S(\varkappa_{11})$ for which the first inequality of system (9) holds does not exceed $c(n)2^{-t(w_1+v_1+1-p_1)}$, and the measure of the set of $\omega_2 \in S(\varkappa_{21})$ for which the second inequality of system (9) holds does not exceed $c(n)2^{-t(w_2+v_2+1-q_1)}$. Therefore, the measure of the set of (ω_1, ω_2), $\omega_1 \in S(\varkappa_{11})$, $\omega_2 \in S(\varkappa_{21})$ for which the system of inequalities (9) holds does not exceed $c(n)2^{-t(n+1-p_1-q_1)}$. The number of polynomials does not exceed the number of rectangles K. Hence the measure of the set of (ω_1, ω_2) for which the system of inequalities (9) holds for at least one polynomial $P(x) \in P_t(\overline{s})$ is estimated from above by

$$c(n)2^{-t(-n-1+p_1+q_1+w_1+v_1+1-p_1-\varepsilon_1/2+w_2+v_2+1-q_1-\varepsilon_1/2)} < c(n)2^{-t\varepsilon_1}.$$

Since the series $\sum_{t=1}^{\infty} 2^{-t\varepsilon_1}$ converges, the proof of the theorem in this case follows from the Borel–Cantelli lemma.

Assume now that there exist rectangles K such that they contain two or more polynomials $P(x) \in P_t(\overline{s})$. Let $P(x), Q(x) \in K$; then there exist points $(\omega_{11}, \omega_{21})$ and $(\omega_{12}, \omega_{22})$ belonging to K such that

$$\max\bigl(|P(\omega_{11})|, |Q(\omega_{12})|\bigr) < H^{-w_1-v_1}, \qquad \max\bigl(|P(\omega_{21}), |Q(\omega_{22})|\bigr) < H^{-w_2-v_2}. \tag{10}$$

Suppose that $\varkappa_{11}(P)$, $\varkappa_{21}(P)$, $\varkappa_{11}(Q)$, and $\varkappa_{21}(Q)$ are the nearest to ω_{11}, ω_{21}, ω_{12}, and ω_{22} roots of the polynomials $P(x)$ and $Q(x)$, respectively. We obtain from relations (6) and (10) that

$$\max\bigl(|\omega_{11} - \varkappa_{11}(P)|, |\omega_{12} - \varkappa_{11}(Q)|\bigr) < c(n)H^{-w_1-v_1-1+p_1},$$

$$\max\bigl(|\omega_{21} - \varkappa_{21}(P)|, |\omega_{22} - \varkappa_{21}(Q)|\bigr) < c(n)H^{-w_2-v_2-1+q_1}.$$

Therefore,

$$|\varkappa_{11}(P)-\varkappa_{11}(Q)| \le |\varkappa_{11}(P)-\omega_{11}|+|\omega_{11}-\omega_{12}|+|\omega_{12}-\varkappa_{11}(Q)| < c(n)(H^{-w_1-v_1-1+p_1}+H^{-\eta_1}) < c(n)H^{-\eta_1}, \tag{11}$$

$$|\varkappa_{21}(P)-\varkappa_{21}(Q)| \le |\varkappa_{21}(P)-\omega_{21}|+|\omega_{21}-\omega_{22}|+|\omega_{22}-\varkappa_{21}(Q)| < c(n)(H^{-w_2-v_2-1+q_1}+H^{-\eta_2}) < c(n)H^{-\eta_2}. \tag{12}$$

Let us estimate the difference $|\varkappa_{11}(P)-\varkappa_{1i}(Q)|$, $i=2,\dots,n$, taking into account that inequality (7) implies $l_2T^{-1}-\varepsilon_1<\eta_1$. We have

$$|\varkappa_{11}(P)-\varkappa_{1i}(Q)| \le |\varkappa_{11}(P)-\varkappa_{11}(Q)|+|\varkappa_{11}(Q)-\varkappa_{1i}(Q)| < c(n)(H^{-\eta_1}+H^{-l_iT^{-1}+\varepsilon_1}) < c(n)H^{-l_iT^{-1}+\varepsilon_1},$$

$$\prod_{i=1}^{n}|\varkappa_{11}(P)-\varkappa_{1i}(Q)| < c(n)H^{-\eta_1-p_1+(n-1)\varepsilon_1}. \tag{13}$$

Analogously, taking into account that inequality (8) implies $s_2T^{-1}-\varepsilon_1<\eta_2$, we obtain

$$|\varkappa_{21}(P)-\varkappa_{2i}(Q)| < c(n)H^{-s_iT^{-1}+\varepsilon_1}, \qquad i=2,\dots,n. \tag{14}$$

We have from relations (12) and (14) that

$$\prod_{i=1}^{n}|\varkappa_{21}(P)-\varkappa_{2i}(Q)| < c(n)H^{-\eta_2-q_1+(n-1)\varepsilon_1}. \tag{15}$$

Similarly, we estimate

$$\prod_{i=1}^{n}|\varkappa_{12}(P)-\varkappa_{1i}(Q)| < c(n)\prod_{i=1}^{n}\big(|\varkappa_{12}(P)-\varkappa_{11}(P)|+|\varkappa_{11}(P)-\varkappa_{11}(Q)|+|\varkappa_{11}(Q)-\varkappa_{1i}(Q)|\big) < c(n)H^{-l_2T^{-1}-p_1+n\varepsilon_1}, \tag{16}$$

$$\prod_{i=1}^{n}|\varkappa_{22}(P)-\varkappa_{2i}(Q)| < c(n)H^{-s_2T^{-1}-q_1+n\varepsilon_1}. \tag{17}$$

Since the polynomials $P(x)$ and $Q(x)$ are from $P_t(\overline{s})$, they have no common roots and the modulus of their resultant is $|R(P,Q)|\ge 1$. We have from relations (13), (15)–(17) that

$$1 \le |R(P,Q)| < c(n)2^{2tn}\prod_{1\le i,j\le n}|\varkappa_i(P)-\varkappa_j(Q)| < c(n)2^{t(2n-\eta_1-\eta_2-2p_1-2q_1-l_2T^{-1}-s_2T^{-1}+(4n-2)\varepsilon_1)} < c(n)2^{-t\varepsilon_1}.$$

The obtained inequality for large t leads one to a contradiction, and this indicates that the rectangles K to which at most one polynomial belongs do not exist.

Proposition 2. *If inequalities* (7) *and* (8) *hold and if*

$$3-\frac{\varepsilon}{2}\le l_2T^{-1}+p_1+s_2T^{-1}+q_1\le n-1+4n\varepsilon_1, \tag{18}$$

then

$$\mu L(w_1,w_2,v_1,v_2)=0.$$

Proof. Again, we pass from the system of inequalities (1) to the system of inequalities (9). We set

$$k=n+1-l_2T^{-1}-p_1-s_2T^{-1}-q_1. \tag{19}$$

Assume that

$$\{k\}>\varepsilon. \tag{20}$$

Then we obtain from relations (18)–(20) that $n-[k]\ge 3$. Taking into account relations (8), (9), and (18), we can assume that at least one of the following two inequalities holds:

$$l_2T^{-1}+p_1\le w_1+v_1+1-2(n+1)\varepsilon_1, \tag{21}$$

$$s_2T^{-1}+q_1\le w_2+v_2+1-2(n+1)\varepsilon_1. \tag{22}$$

Divide the square $[-3,3]\times[-3,3]$ into equal rectangles with sides $H^{-\sigma_1}$ and $H^{-\sigma_2}$. If inequality (21) holds, and this is not the case for inequality (22), then we set $\sigma_1=l_2T^{-1}+2(n+1)\varepsilon_1$, $\sigma_2=s_2T^{-1}$; otherwise we set $\sigma_1=l_2T^{-1}$, $\sigma_2=s_2T^{-1}+2(n+1)\varepsilon_1$. If both inequalities (21) and (22) hold, then we set $\sigma_1=l_2T^{-1}+(n+1)\varepsilon_1$, $\sigma_2=s_2T^{-1}+(n+1)\varepsilon_1$. For example, suppose that both inequalities (21) and (22) hold. Other cases are considered analogously. Denote by $N(K)$ the number of polynomials belonging to K. If $N(K)<c(n)H^{\nu}$, $\nu=k-1-0,1\varepsilon$, then the total measure of the set of $(\omega_1,\omega_2)\in[-3,3]\times[-3,3]$ for which the system of inequalities (9) holds for at least one $P(x)\in P_n(H,\overline{s})$ is estimated from above by

$$c(n)H^{-w_1-v_1-1+p_1-w_2-v_2-1+q_1+l_2T^{-1}+s_2T^{-1}+(2n+2)\varepsilon_1+k-1-0.1\varepsilon}<c(n)H^{-1-0.1\varepsilon+(2n+2)\varepsilon_1}$$
$$<c(n)H^{-1-\varepsilon_1}.$$

Since the series $\sum_{H=1}^{\infty}H^{-1-\varepsilon_1}$ converges, in order to complete the proof of Proposition 2 in this case, it is sufficient to apply the Borel–Cantelli lemma.

It remains to assume that there exist rectangles K such that $N(K)>c(n)H^{[k]-1}H^{\{k\}-0.1\varepsilon}$. We fix one of such rectangles K and enumerate $P_1(x),\dots,P_m(x)$ belonging to K. Two such polynomials

$$\begin{aligned}P_i(x)&=Hx^n+a_{n-1}^{(i)}x^{n-1}+\dots+a_1^{(i)}x+a_0^{(i)},\\ P_j(x)&=Hx^n+a_{n-1}^{(j)}x^{n-1}+\dots+a_1^{(j)}x+a_0^{(j)},\end{aligned}\qquad 1\le i<j\le m,$$

are united in one class if

$$a_{n-1}^{(i)}=a_{n-1}^{(j)},\quad\dots,\quad a_{n-[k]+1}^{(i)}=a_{n-[k]+1}^{(j)}.$$

Let us apply the Dirichlet box principle. Since the number of different classes does not exceed $c(n)H^{[k]-1}$, among $c(n)H^{\nu}$ polynomials, there are at least $c(n)H^{\{k\}-0.1\varepsilon}$ polynomials belonging to one and the same class. We enumerate these polynomials $P_0(x), \dots, P_l(x)$, $l = H^{\{k\}-0.1\varepsilon}$ and form new polynomials

$$t_1(x) = P_1(x) - P_0(x), \quad \dots, \quad t_l(x) = P_l(x) - P_0(x).$$

All polynomials $t_i(x)$ are different from each other, have the degree at most $n - [k]$ and height not more than $2H$. Expand any polynomial $P(x)$ belonging to K into the Taylor series in the neighborhood of the root $\varkappa_{11}(P)$:

$$P(\omega_1) = P'(\varkappa_{11})(\omega_1 - \varkappa_{11}) + \frac{1}{2!}P''(\varkappa_{11})(\omega_1 - \varkappa_{11})^2 + \dots + \frac{1}{n!}P^{(n)}(\varkappa_{11})(\omega_1 - \varkappa_{11})^n. \tag{23}$$

Since $P(x) \in K$, there exists a point $(\omega_{01}, \omega_{02}) \in K$ such that $|P(\omega_{01})| < H^{-w_1-v_1}$ and $|P(\omega_{02})| < H^{-w_2-v_2}$. We obtain from inequality (5)

$$|\omega_{01} - \varkappa_{11}| < c(n)H^{-w_1-v_1-1+p_1}.$$

If $(\omega_1, \omega_2) \in K$, then $|\omega_1 - \omega_{01}| < H^{-\sigma_1}$. Hence we have

$$|\omega_1 - \varkappa_{11}| < c(n)H^{-l_2T^{-1}-(n+1)\varepsilon_1}. \tag{24}$$

Using Lemma 7 and inequality (24), we obtain

$$\begin{aligned} |P'(\varkappa_{11})(\omega_1 - \varkappa_{11})| &< c(n)H^{1-p_1-l_2T^{-1}-2\varepsilon_1}, \\ |P^{(i)}(\varkappa_{11})(\omega_1 - \varkappa_{11})^i| &< c(n)H^{1-p_i+(n-i)\varepsilon_1-il_2T^{-1}-i(n+1)\varepsilon_1} \\ &< c(n)H^{1-p_1-l_2T^{-1}-2\varepsilon_1}, \qquad i = 2, \dots, n-1, \\ |P^{(n)}(\varkappa_{11})(\omega_1 - \varkappa_{11})^n| &< c(n)H^{1-nl_2T^{-1}-n(n+1)\varepsilon_1} < c(n)H^{1-p_1-l_2T^{-1}-2\varepsilon_1}. \end{aligned} \tag{25}$$

It follows from relations (23) and (25) that

$$|P(\omega_1)| < c(n)H^{1-p_1-l_2T^{-1}-2\varepsilon_1}. \tag{26}$$

Expanding the polynomial $P(x)$, which belongs to K, into the Taylor series in a neighborhood of the root $\varkappa_{21}(P)$, we obtain in a similar way that

$$|P(\omega_2)| < c(n)H^{1-q_1-s_2T^{-1}-2\varepsilon_1}. \tag{27}$$

Since inequalities (26) and (27) hold for any $P(x) \in P_n(H, \overline{s})$ belonging to K, for any polynomial $t_i(x)$, $1 \le i \le l$, the following inequalities hold:

$$|t_i(\omega_1)| < c(n)H^{1-p_1-l_2T^{-1}-2\varepsilon_1}, \qquad |t_i(\omega_2)| < c(n)H^{1-q_1-s_2T^{-1}-2\varepsilon_1}. \tag{28}$$

Consider the following three possibilities that appear:

(a) All $t_i(x) = a_i t(x)$, $a_i \in \mathbb{Z}$. Then there is a polynomial $R(x)$ among all polynomials $t_i(x)$ whose height does not exceed $c(n)H^{1-\{k\}+0.1\varepsilon}$. We obtain from relations (28) that

$$|R(\omega_1)| < c(n)H(R)^{(1-p_1-l_2T^{-1}-2\varepsilon_1)/(1-\{k\}+0.1\varepsilon)},$$

$$|R(\omega_2)| < c(n)H(R)^{(1-q_1-s_2T^{-1}-2\varepsilon_1)/(1-\{k\}+0.1\varepsilon)}.$$

Since, under condition (20), we have

$$\frac{p_1 + l_2T^{-1} + q_1 + s_2T^{-1} - 2 + 4\varepsilon_1}{1 - \{k\} + 0.1\varepsilon} > n - [k] - 1,$$

in this case, in order to complete the proof, it is sufficient to apply the metric theorem in [3].

(b) There are reducible polynomials among $t_i(x)$. We apply here Lemma 3 since

$$p_1 + l_2T^{-1} + q_1 + s_2T^{-1} - 2 + 4\varepsilon_1 > n - [k] - 2.$$

(c) There are at least two polynomials among the polynomials $t_i(x)$ which have no common roots. Then we apply Lemma 8. Here, under condition (20), we have

$$\tau_1 = p_1 + l_2T^{-1} - 1 + 2\varepsilon_1, \qquad \tau_2 = q_1 + s_2T^{-1} - 1 + 2\varepsilon_1,$$

$$\eta_1 = l_2T^{-1} + (n+1)\varepsilon_1, \qquad \eta_2 = s_2T^{-1} + (n+1)\varepsilon_1.$$

We obtain

$$p_1 + q_1 - (4n-8)\varepsilon_1 \le -2 + 2\{k\} + l_2T^{-1} + s_2T^{-1} + \delta. \tag{29}$$

Since $p_1 \ge l_2T^{-1}$ and $q_1 \ge s_2T^{-1}$, for $\delta < 2 - 2\{k\} - (4n-8)\varepsilon_1$, inequality (29) leads one to a contradiction. The case $0 \le \{k\} \le \varepsilon$ requires some modifications, which are connected with the choice of the parameters. It is clear that under the validity of relations (7), (8), and (18), at least one of the inequalities

$$l_2T^{-1} + p_1 \le w_1 + v_1 + 0.2, \qquad s_2T^{-1} + q_1 \le w_2 + v_2 + 0.2. \tag{30}$$

holds. For example, suppose that the first of inequalities (30) holds, then we set $\delta_1 = l_2T^{-1} + 0.8$, $\delta_2 = s_2T^{-1}$, $\nu = k - 1.8 - \varepsilon_1$ and realize similar arguments.

Proposition 3. *If relations* (7) *and* (8) *hold and if*

$$\varepsilon < l_2T^{-1} + p_1 + s_2T^{-1} + q_1 < 3 - \frac{\varepsilon}{2}, \tag{31}$$

then $\mu L(w_1, w_2, v_1, v_2) = 0$.

Proof. Let us unite the polynomials $P(x) = Hx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ from the set $P_n(H, \overline{s})$ into one class $P_n(H, \overline{s}, \overline{a})$ if they have one and the same vector $\overline{a} = (a_{n-1}, \dots, a_3)$. Let $\sigma_1(P)$ be the set of real numbers ω_1 satisfying the inequality

$$|\omega_1 - \varkappa_{11}| < 2^n H^{-w_1} \Psi^{v_1}(H) |P'(\varkappa_{11})|^{-1},$$

$\sigma_2(P)$ be the set of real numbers ω_2 satisfying the inequality

$$|\omega_2 - \varkappa_{21}| < 2^n H^{-w_2}\Psi^{v_2}(H)|P'(\varkappa_{21})|^{-1},$$

$\tau_1(P)$ be the set of real numbers ω_1 satisfying the inequality

$$|\omega_1 - \varkappa_{11}| < 2^n H^{-\alpha_1}|P'(\varkappa_{11})|^{-1},$$

$\tau_2(P)$ be the set of real numbers ω_2 satisfying the inequality

$$|\omega_2 - \varkappa_{21}| < 2^n H^{-\alpha_2}|P'(\varkappa_{21})|^{-1},$$

where α_1 and α_2 are chosen in the following way:

$$l_2T^{-1}+p_1 < 1+\alpha_1-\frac{\varepsilon_1}{5}, \quad s_2T^{-1}+q_1 < 1+\alpha_2-\frac{\varepsilon_1}{5}, \qquad \alpha_1+\alpha_2 = 1, \quad \alpha_1 \le w_1+v_1, \quad \alpha_2 \le w_2+v_2.$$

Clearly, $\sigma_1(P) \subset \tau_1(P)$ and $\sigma_2(P) \subset \tau_2(P)$. We obtain from relation (5) that all $\omega_1 \in S(\varkappa_{11})$ and $\omega_2 \in S(\varkappa_{11})$ satisfying the inequalities

$$|P(\omega_1)| < H^{-w_2}\Psi^{v_2}(H), \qquad |P(\omega_2)| < H^{-w_2}\Psi^{v_2}(H), \qquad |P(\omega_1)| < H^{-\alpha_1}, \qquad |P(\omega_2)| < H^{-\alpha_2}$$

belong to the sets $\sigma_1(P)$, $\sigma_2(P)$, $\tau_1(P)$, and $\tau_2(P)$, respectively. Let $(\omega_1, \omega_2) \in \tau_1(P) \times \tau_2(P)$. Then we have

$$\begin{aligned} |P'(\varkappa_{11})(\omega_1 - \varkappa_{11})| &< 2^n H^{-\alpha_1}, \\ \left|\frac{1}{i!}P^{(i)}(\varkappa_{11})(\omega_1 - \varkappa_{11})^i\right| &< c(n)H^{1-p_i+(n-i)\varepsilon_1-i\alpha_1-i+ip_1} \\ &< c(n)H^{-\alpha_1}, \qquad i = 2,\dots,n-1, \\ \left|\frac{1}{n!}P^{(n)}(\varkappa_{11})(\omega_1 - \varkappa_{11})^n\right| &< c(n)H^{1-n\alpha_1-n+np_1} < c(n)H^{-\alpha_1}. \end{aligned} \tag{32}$$

Using expansion (23) of the polynomial $P(x)$, for $\omega_1 \in \tau_1(P)$, we obtain from inequalities (32)

$$|P(\omega_1)| < H^{-\alpha_1}.$$

Analogously, for $\omega_2 \in \tau_2(P)$, we obtain

$$|P(\omega_2)| < H^{-\alpha_2}.$$

The domain $\Delta(P) = \tau_1(P) \times \tau_2(P)$ is called *nonessential* if there is a polynomial $Q(x)$ in the class $P(H, \overline{s}, \overline{a})$ such that

$$\mu\big(\Delta(P) \cap \Delta(Q)\big) \ge \frac{1}{2}\mu\Delta(P).$$

Otherwise, the domain $\Delta(P)$ is called *essential.*

If the domain $\Delta(P)$ is essential, then each point $(\omega_1, \omega_2) \in [-3,3] \times [-3,3]$ belongs to not more than four essential domains, and therefore,

$$\sum_{P(x)\in P_n(H,\overline{s},\overline{a})} \mu\Delta(P) < \text{const}\,.$$

We obtain from the inequality $\mu\big(\sigma_1(P)\times\sigma_2(P)\big) < \mu\Delta H^{-n+3}\Psi(H)$ that

$$\sum_{P(x)\in P_n(H,\overline{s},\overline{a})} \mu\big(\sigma_1(P)\times\sigma_2(P)\big) < c(n)H^{-n+3}\Psi(H).$$

Since the series $\sum_{H=1}^{\infty}\Psi(H)$ converges, by the Borel–Cantelli lemma, the set of (ω_1,ω_2) which occur in an infinite number of essential domains $\Delta(P)$ has zero measure.

If the domain $\Delta(P)$ is nonessential, then

$$|\tau_1(P)\cap\tau_1(Q)| \ge \frac{1}{2}|\tau_1(P)|, \qquad |\tau_2(P)\cap\tau_2(Q)| \ge \frac{1}{2}|\tau_2(P)|,$$

and on the intervals $J_1=\tau_1(P)\cap\tau_1(Q)$ and $J_2=\tau_2(P)\cap\tau_2(Q)$, the polynomial $R(x)=P(x)-Q(x)$ satisfies the conditions

$$|R(\omega_1)| < c(n)H^{-\alpha_1}, \qquad |R(\omega_2)| < H^{-\alpha_2}, \qquad \deg R(x)\le 2. \tag{33}$$

If $|\varkappa_1(R)-\varkappa_2(R)| > \delta$, where δ is an arbitrary, but fixed number, then the height of the polynomial $R(x)$ satisfies the inequalities

$$H(R) < c(n)|P'(\varkappa_{11})|, \qquad H(R) < c(n)|P'(\varkappa_{21})|. \tag{34}$$

We obtain from relations (31), (33), and (34) that for $(\omega_1,\omega_2)\in J_1\times J_2$,

$$|R(\omega_1)R(\omega_2)| < c(n)H^{-1/(1-\varepsilon)} < c(n)H^{-1-\varepsilon}.$$

Further, using Lemma 2 and the metric theorem in [3], we conclude that the set of (ω_1,ω_2) which belong to an infinite number of nonessential domains $\Delta(P)$ has zero measure. In the case $|\varkappa_1(R)-\varkappa_2(R)| < \delta$, we argue similarly as we did when proving Lemma 10.

Proposition 4. *If relations* (7) *and* (8) *hold and if*

$$l_2T^{-1}+p_1+s_2T^{-1}+q_1 < \varepsilon,$$

then $\mu L(w_1,w_2,v_1,v_2)=0$.

Proof. We unite polynomials $P(x)=Hx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ from the set $P_n(H,\overline{s})$ into one class $P_n(H,\overline{s},\overline{\beta})$ if they have one and the same vector $\overline{\beta}=(a_{n-1},\dots,a_2)$. Define $\sigma_1(P)$, $\sigma_2(P)$ as in Proposition 3, while $\tau_1(P)$ and $\tau_2(P)$ are defined as the sets of real numbers ω_1 and ω_2 satisfying the inequalities

$$|\omega_1-\varkappa_{11}| < 2^{-n-1}(n+1)^{-1}|P'(\varkappa_{11})|^{-1},$$

and

$$|\omega_2-\varkappa_{21}| < 2^{-n-1}(n+1)^{-1}|P'(\varkappa_{21})|^{-1}$$

respectively. Using the Taylor series expansion (23) of the polynomial $P(x)$ and the inequality $|P^{(i)}(\omega)| < i!\,3^n(n+1)H$ for $\omega_1\in[-3,3]$, we obtain

$$|P'(\varkappa_{11})(\omega_1-\varkappa_{11})| < 2^{-n-1}(n+1)^{-1},$$

$$\frac{1}{i!}|P^{(i)}(\varkappa_{11})(\omega_1-\varkappa_{11})^i| < 3^n(n+1)H\times 2^{-i(n-1)}(n+1)^{-i}\times H^{-i+\varepsilon i}$$

$$< (n+1)^{-1}2^{-n-1}, \qquad i=2,\dots,n,$$

$$|P(\omega_1)| < 2^{-n-1}.$$

Similarly, for $\omega_2 \in [-3, 3]$, we obtain

$$|P(\omega_2)| < 2^{-n-1}.$$

If the domain $\Delta(P)$ is essential, then we obtain

$$\sum_{P(x)\in P_n(H,\overline{s})} \mu\big(\sigma_1(P) \times \sigma_2(P)\big) < c(n) \sum_{\overline{\beta}} \sum_{P(x)\in P_n(H,\overline{s},\overline{\beta})} \mu\Delta(P)H^{-n+2}\Psi(H) < c(n)\Psi(H)$$

since

$$\mu\big(\sigma_1(P), \sigma_2(P)\big) \le c(n)\mu\Delta(P)H^{-n+2}\Psi(H).$$

Then we apply the Borel–Cantelli lemma. If the domain $\Delta(P)$ is nonessential, then we have

$$|R(\omega_1)| < 2^{-n}, \qquad |R(\omega_2)| < 2^{-n}, \qquad \deg R(x) \le 1 \tag{35}$$

for the polynomial $R(x) = P(x) - Q(x)$. Hence, the distance from (ω_1, ω_2) for which inequalities (35) hold and the line $\omega_1 = \omega_2$ does not exceed a given positive number. In order to complete the proof, we apply Lemma 2.

Proposition 5. *If*

$$l_2T^{-1} + p_1 > w_1 + v_1 + 1, \tag{36}$$

$$s_2T^{-1} + q_1 \le w_2 + v_2 - 1 + 2n\varepsilon_1, \tag{37}$$

$$s_2T^{-1} + q_1 > 2 - w_1 - v_1 - (n-1)\varepsilon_1,$$

then $\mu L(w_1, w_2, v_1, v_2) = 0$.

Proof. We pass from the system of inequalities (1) to the system of inequalities (9). We set

$$k = w_2 + v_2 + 1 - s_2T^{-1} - q_1.$$

If $\{k\} > \varepsilon$, then we set $\sigma_1 = w_1 + v_1 + 1 - p_1$, $\sigma_2 = s_2T^{-1}$, and $\nu = k - 1 - 0.1\varepsilon$; then we proceed as in the proof of Proposition 2. If $0 \le \{k\} \le \varepsilon$, then we set $\sigma_1 = w_1 + v_1 + 1 - p_1$, $\sigma_2 = s_2T^{-1} + 0.8$, $\nu = k - 1.8 - \varepsilon_1$ and proceed just as in the proof of Proposition 2.

Proposition 6. *If relations* (36) *and* (37) *hold and if*

$$\varepsilon < s_2T^{-1} + q_1 < 2 - w_1 - v_1 - (n-1)\varepsilon_1,$$

then $\mu L(w_1, w_2, v_1, v_2) = 0$.

Proof. We unite the polynomials $P(x) \in P_n(H, \overline{s})$ into one class $P_n(H, \overline{s}, \overline{a})$ if they have equal coefficients of the powers $x^{n-1}, \dots, x^3$. Define the domains $\sigma_1(P)$ and $\sigma_2(P)$ as in Proposition 3. Let $\tau_1(P) = \sigma_1(P)$ and let $\tau_2(P)$ be the set of real ω_2 satisfying the inequality

$$|\omega_2 - \varkappa_{21}| < 2^n H^{-1-w_1}\Psi^{v_1}(H)|P'(\varkappa_{21})|^{-1}.$$

Further, we proceed as in the proof of Proposition 3.

Proposition 7. *If relations* (36) *and* (37) *hold and if*

$$s_2 T^{-1} + q_1 < \varepsilon,$$

then $\mu L(w_1, w_2, v_1, v_2) = 0$.

Proof. We unite the polynomials $P(x)$ from the set $P_n(H, \overline{s})$ into one class $P_n(H, \overline{s}, \overline{\beta})$ if they have one and the same vector $\overline{\beta} = (a_{n-1}, \dots, a_2)$. Define $\sigma_1(P)$, $\sigma_2(P)$, and $\tau_1(P)$ as in Proposition 6. Let $\tau_2(P)$ be the set of real ω_2 satisfying the inequality

$$|\omega_2 - \varkappa_{21}| < 2^{-n-1}(n+1)^{-1} H^{-w_1} \Psi^{v_1}(H) |P'(\varkappa_{21})|^{-1}.$$

Then we proceed as in the proof of Proposition 4.

Proposition 8. *If inequality* (36) *holds and if*

$$s_2 T^{-1} + q_1 > w_2 + v_2 + 1, \tag{38}$$

then $\mu L(w_1, w_2, v_1, v_2) = 0$.

Proof. We pass from the system of inequalities (1) to the system of inequalities (9). We can obtain from inequalities (36) and (38) that

$$l_2 T^{-1} > \frac{w_1 + v_1 + 1 - p_2 - 1/T}{2}, \tag{39}$$

$$s_2 T^{-1} > \frac{w_2 + v_2 + 1 - q_2 - 1/T}{2}. \tag{40}$$

Let us show that there exist two integers m and r ($2 \le m \le n-1$, $2 \le r \le n-1$) for which the following system of inequalities holds:

$$\frac{l_m}{T} \frac{w_1 + v_1 + 1 - p_m - 1/T}{m} \ge \frac{l_{m+1}}{T}, \tag{41}$$

$$\frac{s_r}{T} \frac{w_2 + v_2 + 1 - q_r - 1/T}{r} \ge \frac{s_{r+1}}{T}. \tag{42}$$

The left-hand side of inequality (41) follows from inequality (39) for $m = 2$. If

$$\frac{w_1 + v_1 + 1 - p_2 - 1/T}{2} \ge \frac{l_3}{T},$$

then inequality (41) holds for $m = 2$. If

$$\frac{l_3}{T} > \frac{w_1 + v_1 + 1 - p_2 - 1/T}{2},$$

then, taking into account the fact that $p_2 = l_3 T^{-1} + p_3$, we arrive at the inequality

$$3 l_3 T^{-1} > w_1 + v_1 + 1 - p_3 - \frac{1}{T}$$

and therefore,

$$\frac{l_3}{T} > \frac{w_1 + v_1 + 1 - p_3 - 1/T}{3}.$$

If now

$$\frac{w_1 + v_1 + 1 - p_3 - 1/T}{3} \geq \frac{l_4}{T},$$

then we obtain that inequality (41) holds for $m = 3$. Otherwise, we pass to the inequality

$$\frac{l_4}{T} > \frac{w_1 + v_1 + 1 - p_4 - 1/T}{4}$$

and so on. Clearly, we arrive at m for which relation (41) holds, since $|\varkappa_{11} - \varkappa_{21}| > \delta$ and therefore, for a certain $m \leq n-1$ we have $l_s T^{-1} < 2/T$, $p_{s-1} < 2(s-2)/T$ for $s \geq m$. Similarly, we can prove the existence of r for which inequality (42) holds. Assume that

$$|\varkappa_{11}(P) - \varkappa_{11}(Q)| < H^{-(w_1+v_1+1-p_m-1/T)/m}, \tag{43}$$

$$|\varkappa_{21}(P) - \varkappa_{21}(Q)| < H^{-(w_2+v_2+1-q_r-1/T)/r}, \tag{44}$$

where $P(x)$, $Q(x) \in P_t(\overline{s})$. Then it follows from the inequalities

$$\frac{l_2}{T} \geq \frac{l_3}{T} \geq \cdots \geq \frac{l_m}{T} > \frac{w_1 + v_1 + 1 - p_m - 1/T}{m}$$

and from inequality (43) that

$$\begin{aligned} |\varkappa_{1i}(P) - \varkappa_{1j}(Q)| &\leq |\varkappa_{1i}(P) - \varkappa_{11}(P)| + |\varkappa_{1j}(Q) - \varkappa_{11}(Q)| \\ &\leq |\varkappa_{1i}(P) - \varkappa_{11}(P)| + |\varkappa_{1j}(Q) - \varkappa_{11}(Q)| + |\varkappa_{11}(P) - \varkappa_{11}(Q)| \\ &\leq 3H^{-(w_1+v_1+1-p_m-1/T)/m} \end{aligned} \tag{45}$$

for any i, j, $2 \leq i \leq m$, $2 \leq j \leq m$. Therefore we obtain from relation (45) that

$$\prod_{1 \leq i,j \leq m} |\varkappa_{1i}(P) - \varkappa_{1j}(Q)| < c(n) H^{-m(w_1+v_1+1-p_m-1/T)}. \tag{46}$$

For $i \leq m$ and $j > m$ it follows from relation (41) that

$$|\varkappa_{1i}(P) - \varkappa_{1j}(Q)| \leq |\varkappa_{1i}(P) - \varkappa_{11}(P)| + |\varkappa_{11}(P) - \varkappa_{11}(Q)| + |\varkappa_{11}(Q) - \varkappa_{1j}(Q)| < c(n) H^{-l_j/T}. \tag{47}$$

Also, we obtain the following inequality for $i > m$, $j \leq m$, which is analogous to inequality (47):

$$|\varkappa_{1i}(P) - \varkappa_{1j}(Q)| < c(n) H^{-l_j/T}. \tag{48}$$

It follows from relations (47) and (48) that

$$\prod_{\max(i,j) > m} |\varkappa_{1i}(P) - \varkappa_{1j}(Q)| < c(n) H^{-2mp_m}. \tag{49}$$

If now we use the inequalities

$$\frac{s_2}{T} \geq \frac{s_3}{T} \geq \cdots \geq \frac{s_r}{T} > \frac{w_2 + v_2 + 1 - q_r - 1/T}{r}$$

and inequalities (42) and (44), then we obtain the following results which are analogous to inequalities (46) and (49):

$$\prod_{1 \leq i,j \leq m} |\varkappa_{2i}(P) - \varkappa_{2j}(Q)| < c(n) H^{-r(w_2+v_2+1-q_r-1/T)}, \qquad \prod_{\max(i,j)>m} |\varkappa_{2i}(P) - \varkappa_{2j}(Q)| < c(n) H^{-2rq_r}. \tag{50}$$

The polynomials $P(x)$ and $Q(x)$ belonging to $P_t(\overline{s})$ have no common roots, and thus, $|R(P,Q)| \geq 1$. We have from inequalities (46), (49), and (50) that

$$\begin{aligned} 1 \leq |R(P,Q)| &\leq c(n) 2^{t(2n-m(w_1+v_1+1-p_m-1/T)-2mp_m)} 2^{t(-r(w_2+v_2+1-q_r-1/T)-2rq_r)} \\ &< c(n) 2^{t(2n-m(w_1+v_1+1-1/T))} 2^{t(-r(w_2+v_2+1-1/T))} < c(n) 2^{-t(2-(m+r)/T)}. \end{aligned} \tag{51}$$

For large t, inequality (51) is contradictory. Therefore there are no roots $\varkappa_{11}(Q)$ and $\varkappa_{21}(Q)$ of any other polynomial $Q(x) \in P_t(\overline{s})$ in the rectangle

$$|\omega_1 - \varkappa_{11}(P)| < H^{-(w_1+v_1+1-p_m-1/T)/m}, \qquad |\omega_2 - \varkappa_{21}(P)| < H^{-(w_2+v_2+1-q_r-1/T)/r}.$$

This means that the number of polynomials $P(x) \in P_t(\overline{s})$ does not exceed

$$c(n) 2^{t((w_1+v_1+1-p_m-1/T)/m + (w_2+v_2+1-q_r-1/T)/r)}. \tag{52}$$

It follows from relation (5) that the measure of the set of (ω_1, ω_2), $\omega_1 \in S(\varkappa_{11})$, $\omega_2 \in S(\varkappa_{21})$ for which the system of inequalities (9) holds does not exceed

$$c(n) 2^{-t((w_1+v_1+1-p_m)/m + (w_2+v_2+1-q_r)/r)}. \tag{53}$$

We conclude from expressions (52) and (53) that the measure of the set of (ω_1, ω_2) for which the system of inequalities (9) holds for at least one polynomial $P(x) \in P_t(\overline{s})$ does not exceed

$$c(n) 2^{-t(1/(mT)+1/(rT))} < c(n) 2^{-2t/(Tn)}. \tag{54}$$

Since the series $\sum_{t=1}^{\infty} 2^{-t\lambda}$ converges for any $\lambda > 0$, Proposition 8 follows from inequality (54) and the Borel–Cantelli lemma.

Proposition 9. *If inequalities* (36) *hold and if*

$$w_2 + v_2 - 1 + 2n\varepsilon_1 < s_2 T^{-1} + q_1 \leq w_2 + v_2 + 1,$$

then $\mu L(w_1, w_2, v_1, v_2) = 0$.

Proof. Again, we pass to the system of inequalities (9). Divide the square $[-3,3] \times [-3,3]$ into equal rectangles K with sides $H^{-\eta_1}$, $H^{-\eta_2}$, $\eta_1 = (w_1 + v_1 + 1 - p_m - 1/T)/m$, $\eta_2 = w_2 + v_2 + 1 - q_1 - \varepsilon_1/2$, where m is chosen in such a way that inequality (41) holds. Suppose that not more than one polynomial $P(x) \in P_t(\overline{s})$ belongs to each rectangle K. Then the measure of the set of (ω_1, ω_2), $\omega_1 \in S(\varkappa_{11})$, $\omega_2 \in S(\varkappa_{21})$ for which the system of inequalities (9) holds for at least one polynomial $P(x) \in P_t(\overline{s})$ is estimated from above by

$$c(n)2^{t(-(w_1+v_1+1-p_m)/m-w_2-v_2-1+q_1+(w_1+v_1+1-p_m-1/T)/m+w_2+v_1-q_1-\varepsilon_1/2)} < c(n)2^{-t(1/(mT)+\varepsilon_1/2)}.$$

Since the series $\sum_{t=1}^{\infty} 2^{-t(1/(mT)+\varepsilon_1/2)}$ converges, we apply the Borel–Cantelli lemma in order to complete the proof. Assume now that there exist rectangles K such that they contain two or more polynomials $P(x) \in P_t(\overline{s})$. Let $P(x)$ and $Q(x)$ belong to K. Since $P(x), Q(x) \in P_t(\overline{s})$, they have no common roots. Then, using inequalities (15), (17), (46), and (49), we obtain

$$1 \le |R(P,Q)| \le c(n)2^{t(2n-m(w_1+v_1+1-p_m-1/T)-2mp_m)}2^{t(-\eta_2-2q_1-s_2T^{-1}+(2n-1)\varepsilon_1)}$$
$$< c(n)2^{t(2n-2w_1-2v_1-2-2/T-w_2-s_2T^{-1}-q_1+(2n-0.5)\varepsilon_1-v_2)} < c(n)2^{-2t/T}.$$

This inequality is contradictory for large t. Consequently, there are no rectangles K to which more than one polynomial belong.

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