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Vasilij I. Bernik; Ella I. Kovalevskaya

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The distribution of rational points close to a smooth manifold and Hausdorff dimension

V. I. Bernik
E. I. Kovalevskaya

Abstract: We find a connection between the distribution of rational points close to a smooth manifold and the lower bound for the Hausdorff dimension. We get also the new bound.

Key Words: diophantine approximation, Hausdorff dimension.

Mathematics Subject Classification: 11J13

Introduction

The method of trigonometric sums allows to get the asymptotic formulas for counting the number of rational points close to a smooth manifold only up to the certain level of closeness. We show how rational points are distributed outside this level by Khinchine's Transference Principle and by the construction of a regular system according to A.Baker and W.M.Schmidt [1].

Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a point in $\mathbb{R}^k$ and let $\| y \|$ be the distance from $y \in \mathbb{R}$ to the nearest integer. We consider the system of inequalities

$$\max(\| \alpha_1 q \|, \dots, \| \alpha_k q \|) < q^{-v} \tag{1}$$

where $q \in \mathbb{N}$, v is a real fixed number, $v > 0$. Let $v(\alpha)$ be the least upper bound of those $v > 0$ for which the system of inequalities (1) has infinitely many solutions in integers $q > 0$. In this case, Dirichlet's "pigeonhole principle" gives the inequality $v(\alpha) \geq k^{-1}$.

Now we replace the point α in (1) by the point belonging to the manifold $M = (x_1, \dots, x_m, f_1, \dots, f_n) \subset \mathbb{R}^{m+n}$ where $1 \leq n < m$, $f_j = f_j(x) = f_j(x_1, \dots, x_m)$ $(1 \leq j \leq n)$ are real three times continuously differentiable functions defined on the domain $X \subset \mathbb{R}^m$. Then the system of inequalities (1) can be written in the form

$$\max_{1 \leq i \leq m, 1 \leq j \leq n} (\| x_i q \|, \| f_j(x) q \|) < q^{-v}. \tag{2}$$

It is clear that the system (2) has infinitely many solutions in integers $q > 0$ for all $x \in X$ if $v \leq (m+n)^{-1}$. In 1972, V.Sprindžuk [2 ,pp. 82-92, 95-99] showed that the system of inequalities (2) has only a finite number of solutions q for almost all $x \in X$ ("almost all" in the sense of Lebesgue measure on M) if $1 \leq n < m$, $v > (m+n)^{-1}$ and the functions $f_j(x)$ $(1 \leq j \leq n)$ satisfy the following conditions:

1) the determinant

$$\det(\partial^2 f_j/\partial x_1 \partial x_k)_{j,k=1,\dots,n}$$

is different from zero almost everywhere in X,

2) every linear combination

$$\varphi(x_k) = c_1 \partial^2 f_1/\partial x_1 \partial x_k + \dots + c_n \partial^2 f_n/\partial x_1 \partial x_k$$

with integer coefficients, regarded as a function of one variable x_k $(1 \leq k \leq n)$ with the other variables fixed, is such that every interval on which it is defined can be divided into a bounded number (independent of $c_1, \dots, c_n$) of subintervals on which $\varphi(x_k)$ is monotonic.

This is one of the general result in Diophantine approximation on manifolds. For any $v > (m+n)^{-1}$ let $M(v)$ be a set of those $x \in X$ for which the system (2) has infinitely many solutions q. Then Sprindžuk's theorem asserts for Lebesgue measure: mes $M(v) = 0$. If we wish to know more precise metric characteristic of $M(v)$ then we have to deal with the Hausdorff dimension of $M(v)$. The lower and the upper estimates for the Hausdorff dimension $\dim M(v)$ were found in [3]:

$$\frac{1}{(m+n)(1+v)} \leq \dim M(v) \leq \frac{m^2 + m(n+1)}{(m+n)(1+v)}, \tag{3}$$

if $m > n^2 - n + 1$ and the functions $f_1, \dots, f_n$ satisfy the conditions 1)–2) by the method of trigonomertic sums.

The upper estimate in (3) reflects the essence of the phenomenon since the set $M(v)$ can be empty if the index v is large. For example, it is the case under $f_j = \sqrt[3]{1 - x_j{}^3}$ $(1 \leq j \leq n)$. It takes place when the rational approximations in (2) are realized only by the points belonging to the manifold M. Meanwhile it is clear that there are no rational points with the same denominator $q \geq 2$ on the curves $y_j = \sqrt[3]{1 - x_j{}^3}$ $(1 \leq j \leq n)$.

The lower estimate in (3) is obtained on a basis of the uniform distribution of the sequence $\{qf\} = \{qf_1(a_1/q, \dots, a_m/q), \dots, qf_n(a_1/q, \dots, a_m/q)\}$, $(q = 1, 2, \dots)$ in $[0,1)^n$. It is proved in [3, Lemma 1]. Namely, let q be a fixed natural number and let K be an n-dimensional cube in $[0,1)^n$, $K = \prod_{j=1}^n [\beta_j, \gamma_j)$ where $\gamma_j - \beta_j = \xi$ $(1 \leq j \leq n)$ and

$$q^{-1/(n^2+1)+\varepsilon} \leq \xi \leq q^{-1/(m+n)}$$

with $\varepsilon > 0$ (ε is arbitrary). Suppose

$$N(q) = \#\{a = (a_1, \dots, a_m) \in \mathbb{Z}^m, a \in qK = \prod_{j=1}^n [q\beta_j, q\gamma_j) :$$

$$\{qf\} \in K \bmod 1, \text{when } 0 \leq a_i \leq q (1 \leq i \leq m)\}\}$$

and $m > n^2 - n + 1$. Then

$$N(q) = 2^n \xi^n q^m + c\xi^{-1/n} q^{m-1/n+\varepsilon_1} \tag{4}$$

where c is some positive absolute constant and $\varepsilon_1 = \varepsilon(n + 1/n)$. It follows from (4) that $N(q) > 1$ if $\xi \geq c_1 q^{-1/(n^2+1)+\varepsilon}$ with the suitable constant $c_1 > 0$. Hence, there exists a rational points $a/q = (a_1/q, \ldots, a_m/q)$ in K. Given integer $q > 0$ we define integer t as $2^t \leq q < 2^{t+1}$. Just as in [3] we can prove that there exists a rational point a/q_1 with the following properties: (i) $a/q_1 \in K$ where $\gamma_j - \beta_j \geq 2\xi$ $(1 \leq j \leq n)$ and the denominator q_1 satisfies the inequality $2^t \leq q_1 < 2^{t+1}$; (ii) $\{qf(a/q_1)\} \in K \bmod 1$.

The better regular system than the one in [3] can be constructed if every cube K has the large number of those rational points. In that case, we can make more precise lower bound than in (3):

$$\dim M(v) \geq (m - nv)/(v + 1). \tag{5}$$

Thus, in this paper we find a connection between the distribution of rational points close to M and the lower bound for the Hausdorff dimension of the set $M(v)$. We prove the following

Theorem. *For $m > n^2 - n + 1$ and the functions $f_1(x), \ldots, f_n(x)$ satisfying the above formulated conditions 1)-2) we have (5).*

Regular systems.

The concept of the Hausdorff dimension was used in Diophantine Approximation by V.Jarnik and A.Besicovitch in 1929. But the intensive application of it in the theory began after 1970 when A.Baker and W.M.Schmidt suggested the method of obtaining lower bounds for the dimension by the construction of regular systems [1 and 4].

Definition. We shall call a countable set Γ of real numbers together with a positive-valued function N defined on Γ *a regular system* (Γ, N) if for every interval J there is a positive number $L(J)$ such that, for all $T \geq L(J)$, there are elements $\gamma_1, \ldots, \gamma_t$ of Γ such that, for each j, k with $1 \leq j, k \leq t$ $(j \neq k)$, we have

$$\gamma_j \in J, \quad N(\gamma_j) \leq T, \quad |\gamma_j - \gamma_k| \geq T^{-1}, \quad t \geq c_2 \, |J| \, T,$$

where $c_2 = c_2(\Gamma, N) > 0$; here $|J|$ denotes the length of J.

For any regular system (Γ, N) and any positive function $h(x)$ defined for $x > 0$ we denote by $(\Gamma, N; h)$ the set of all real numbers y for which there exist infinitely many γ of Γ such that $|y - \gamma| < h(N(\gamma))$.

Further, for any real set S and any positive function $g(x)$ defined for $x > 0$ we shall write $S \prec g$ if, for every $\lambda > 0$, $\delta > 0$, S is covered by some countable set $I_\delta(\lambda, g)$ of intervals $I_1, I_2, \ldots$ with $|I_j| \leq \lambda$ and

$$\sum_{j=1}^{\infty} g(|I_j|) < \delta.$$

The following lemma allows to obtain the lower bound for the Hausdorff dimension of the set of real numbers having the given type of approximation by elements of regular system.

Lemma. *Let $h(x), g(x)$ be positive functions defined for $x > 0$ such that $h(x)$ decreases and $h(x) \leq 1/(2x)$ for large x, $g(x)$ and $x/g(x)$both increase and tend to zero with x, and $xg(\frac{1}{2}h(x)) \to \infty$ as $x \to \infty$. Then, for any regular system (Γ, N) we have $(\Gamma, N; h) \not\prec g$. In fact, for any regular systems (Γ_i, N_i) $(i = 1, 2, \dots)$ we have*

$$\bigcap_{i=1}^{\infty} (\Gamma_i, N_i; h) \not\prec g.$$

Proof. It is Lemma 1 in [1].

Take $h(x) = x^{-\sigma}$, $g(x) = x^{\rho}$ where $0 < \rho < \sigma^{-1} < 1$. In view of the Lemma we have

Proposition. *Let (Γ, N) be a regular system and (Γ, N, σ) be the set of all real numbers x for which there exist infinitely many $\gamma \in \Gamma$ with $| x - \gamma | < N(\gamma)^{-\sigma}$. Then* $\dim(\Gamma, N, \sigma) \geq \sigma^{-1}$.

Proof of the estimate (5). Given a fixed integer t, suppose that rational points $a_r/q_r = (a_{1r}/q_r, \dots, a_{mr}/q_r)$ $(r = 1, 2, \dots, 2^t)$ lie in one of the cube with volume $(2\xi)^n$ as much close as the following inequalities permit

$$| a_{ir}/q_r - a_{is}/q_s | \geq (q_r q_s)^{-1} > 2^{-2t} \qquad (1 \leq i \leq m).$$

For every i $(1 \leq i \leq m)$ the number of those points is equal to 2^t. Hence, they belong to a cube K_1 with volume $| K_1 |$: $2^{-mt} \leq | K_1 | < c_2 2^{-mt}$. In this case, a series consisting of the ρ-covering of the set $M(v)$ is majorized by the following series

$$\sum_{i=1}^{\infty} c_3 2^{-t\rho(v+1)+t(m-nv)}.$$

Therefore we obtain

$$\dim M(v) \leq (m - nv)/(v + 1). \tag{6}$$

Now let $w(f)$ be the least upper bound of those $w' > 0$ for which the inequality

$$| a_1 x_1 + \dots + a_m x_m + a_{m+1} f_1(x) + \dots + a_{m+n} f_{m+n}(x) + a_0 | < A^{-(m+n)-w'}, \tag{7}$$

where $A = \max(| a_0 |, | a_1 |, \dots, | a_{m+n} |) \neq 0$ has infinitely many solutions in integer vectors $(a_0, a_1, \dots, a_{m+n})$. And let $v(f)$ be the least upper bound of those $v' > 0$ for which the system of inequalities

$$\max_{1 \leq i \leq m, 1 \leq j \leq n} (\| x_i q \|, \| f_j(x) q \|) < q^{-(1+v')/(m+n)}$$

has infinitely many integer solutions $q > 0$. According to Khintchine's Transference Principle [5] we have

$$\frac{w(f)}{(m+n)^2 + (m+n-1)w(f)} \leq v(f) \leq w(f). \tag{8}$$

Now if for some $v : (m+n)^{-1} < v < (m+n-1)^{-1}$ the system (2) has infinitely many integer solutions q then in view of (8) the inequality (7) with the index $(-(m+n) - w'')$, where

$$w'' = \frac{(m+n)^2[(m+n)v - 1]}{(m+n) - (m+n-1)[(m+n)v-1]}, \tag{9}$$

has also infinitely many solutions in integer vectors $(a_0, a_1, \ldots, a_{m+n})$. By virtue of the above mentioned Sprindžuk's result for the manifold M and by the Lemma we can construct a regular system consisting of zeros of the functions which form the left part of the inequality (7). Then in view of Proposition, the lower bound for the dimension is obtained:

$$\dim M(v) \geq m - 1 + (m+n+1)/(m+n+1+w''), \tag{10}$$

where w'' is defined by (9).

A comparison of estimates (10) and (6) shows that we have a contradiction. Hence, our hypothesis about the behaviour of rational points close to a smooth manifold is not true. The points have a uniform distribution in some sense.

Remark. For the first time the similar investigation was given in [6] for the topological product of m plane curves i.e. $M = (x_1, \ldots, x_m, f_1(x_1), \ldots, f_m(x_m))$ when $1 \leq n < m$.

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Author's address: Institute of mathematics, 220072, ul. Surganova 11, Minsk, Belarus

E-mail: `imanb%imanb.belpak.minsk.by@demos.su`

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