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STOCHASTIC MODELING IN THE SCIENCES:
STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS
AND RANDOM MEDIA

1 April 2005 – 15 September 2005

Preprint No. 10/2005

METRIC THEORY FOR DIOPHANTINE APPROXIMATIONS OF DEPENDENT VALUES

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Let $\overline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Many of the important arithmetic properties of the vector $\overline{x}$ become apparent through solvability of the inequality

$$|a_n x_n + a_{n-1}x_{n-1} + \ldots + a_1 x + a_0| < H^{-\omega} \tag{1}$$

$$a_j \in \mathbb{Z},\ a \leqslant j \leqslant n,\ H = \max_{1\leqslant i\leqslant n} |a_i|.$$

For any $\overline{x}$ and $\omega \leqslant n$ the inequality (1) has infinitely many solutions in integer vectors $\overline{a} = (a_0, a_1, \ldots, a_n)$, that follows from Minkowski's linear forms theorem. The inequality (1) doesn't allow principal improvement. If the right part of (1) is multiplied on small enough constant, there are such $\overline{x}$ that this modified inequality doesn't hold. But according to Khinchine–Groshev theorem there aren't many such $\overline{x}$ [1], [2].

Let $\Psi(x)$ be monotonic decreasing function, μA is Lebesgue measure. $\mathcal{L}_n(\Psi)$ is the set of $\overline{x}$ from some parallelepiped $T = \prod_{i=1}^{n}(\alpha_i \beta_i)$ that the inequality

$$|a_n x_n + \ldots + a_1 x + a_0| < H^{-n+1}\Psi(H) \tag{2}$$

has infinitely many solutions in $\overline{a} \in \mathbb{Z}^{n+1}$.
Then

$$\mu\mathcal{L}_n(\Psi) = \begin{cases} 0, & \sum_{H=1}^{\infty} \Psi(H) < \infty \\ \prod_{i=1}^{n}(\beta_i - \alpha_i), & \sum_{H=1}^{\infty} \Psi(H) = \infty \end{cases} \tag{3}$$

In following papers there were received asymptotic for the number of solutions of the inequality (2) for the divergence case.

The problem become essentially more difficult, if $\overline{x}$ lies on some surface G in $\mathbb{R}^n$, $\dim G = m$, $1 \leqslant m < n$. If $G = (x, x^2, \ldots, x^n)$, the exponent $\omega = \omega(x)$ can be used for classification of transcendent. This was done by Mahler [3]. During last 70 years Koksma, Kubilius, Cassels, Folkman, Schmidt, Levquek, Sprindzuk, A. Baker, R. Baker, Dodson, Kovalevskaya study solvability of (1) for $\overline{x} \in G$. G was gradually considered under weaker sufficient conditions, when ω was very close to n and the inequality 1 had infinitely many solutions only for the set of zero Lebesgue m–measure. Finally in [4], [5], [6] Beresnevich, Bernik, Kleinbock and Margulis managed to receive full analogue of Khinchine–Groshev theorem for nondegenerate surfaces.

However in the number of problems the dependence of left part of inequalities (1) and (2) on coefficients is more complicated. If we consider the left part of inequality (1) $S(x) = P(x) + d = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 + d$, $d \in \mathbb{R}$ for $G = (x, x^2, \ldots, x^n)$, Minkowski's theorem can't be apply for convex body, described by (1), isn't centrosymmetric any more. First these nonhomogeneous problems were considered in [7], [8]. When we study the inequality $|P(x)| < H^{-\omega}$ we use facts that roots of $P(x) = 0$ are algebraic numbers, discriminant of $P(x)$ and resultant of $P_1(x)$ and $P_2(x)$ are integer numbers.

In addition two algebraic numbers can coincide or we can obtain the lower estimate for the absolute value of their difference. All listed properties of integer polynomials and algebraic numbers aren't right for the polynomials $S(x)$ for $d = \sqrt{2},\ e,\ \pi$. However in papers [9] countable number of integer polynomials were divided into finite number of classes. For big values $|P'(x)|$ the research of the inequality $|P(x) < H^{-\omega}|$ can be done directly, for small values of $P'(x)$ we can come to the difference of polynomials $S(x)$, that are with integer coefficients.

The divergence case of the problem $|S(x)| < H^{n-1}\Psi(H)$ demands constructing of polynomials $S(x)$ such, that $|S(x)| < c(n)H^{-n}$ holds. It is hardly possible for arbitrarily point x, however for the subset B of arbitrarily interval I, $\mu B > 0,5\mu I$ we can find $n+1$ linear independent polynomials $P(x)$, and using them construct $S(x)$. This construction was first proposed by Bougeaux [10], that was based on work of Davenport and Schmidt [11].

The problem become more difficult when we change Veronese curve $x, \ldots, x^n$ on arbitrary curve. Homogeneous case of this plane problem was first studied by Schmidt [12], analogue of the Khinchine–Groshev theorem for this curve was proved in [13], [14].

Now we show the scheme of proof of richness of content of the set $\mathcal{L}_n(\Psi)$ in terms of lower estimates of Hausdorff dimension of these sets.

Consider an interval $I \subset \mathbb{R}$ and a function $\lambda\ : \mathbb{R} \leftarrow \mathbb{R}$. Define an inequality

$$|a_n f_n(x) + a_{n-1} f_{n-1}(x) + \ldots + a_1 f_1(x) + a_0 + \lambda(x)| < H^{-w}. \qquad (4)$$

where $n \geqslant 2, f_1, f_2, \ldots, f_n \in C^{n+1}(I); \overline{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{Z}^{n+1}$ and $H = \max\{|a_0|, |a_1|, \ldots, |a_n|\}$. Denote by $X_n(w, \lambda) = X_n(w, I, f_1, \ldots, f_n, \lambda)$ the set of numbers $x \in I$ such that the set $(a_0, \ldots, a_n)$ of solutions 4 is infinite.

Denote by $\mathcal{F}_n$ the set of functions $a_n f_n(x) + \ldots + a_1 f_1(x) + a_0$ where $a_i \in \mathbb{Z},\ 0 \leqslant i \leqslant n$. Let $F(x) \in \mathcal{F}_n,\ F(x) = a_n f_n(x) + \ldots + a_1 f_1(x) + a_0$.

A value of expression $H(F) = \max\{|a_0|, \dots, |a_n|\}$ we'll call the height of the function F. We'll need another type of height $H^*(F) = \max\{|a_1|, \dots, |a_n|\}$.

Denote by $\mathfrak{F}_n(Q)$ and $\mathfrak{F}_n^*(Q)$ the sets of functions $F \in \mathfrak{F}_n$ such that $H(F) \leqslant Q$ and $H^*(F) \leqslant Q$ respectively.

Denote by $\Lambda_n(\epsilon)$ the set of $x \in I$ such that the system of inequalities

$$\begin{cases} |F(x)| < H^{-n} \\ |f'(x)| < H^{1-\epsilon} \end{cases} \tag{5}$$

has infinitely many solutions $f \in \mathfrak{F}_n$.

Now we'll formulate the following theorem.

Theorem 1 *Let $f_1(x) = x, f_2(x), \dots, f_n(x) \in C^n(I)$ and $\lambda(x) \in C^2(I)$ be functions. Let there exist $x_0 \in I$ such that $w(f_1'(x_0), \dots, f_n'(x_0)) \neq 0$. Moreover let there exist the constant K and the neighborhood J of the x_0 such that $\forall x \in J, \max\{\lambda(x), \lambda'(x), \lambda''(x)\} < K$. Then for any $w > n$*

$$\dim\{X_n(w, \lambda)\} \geqslant \frac{n+1}{w+1}.$$

REGULAR SYSTEMS OF POINTS

We will widely use the notion of the regular system.

Definition 1 *A countable set $A \subset I$ with a function $N\ A \leftarrow \mathbb{R}^+$ is called regular system if $\exists K = K(A, N, I) > 0$ such as for all finite interval $J \subset I$ $\exists\, T_0 = T_0(A, N, J) > 0 \mid \forall T > T_0$ there exists a set $(\alpha_1, \alpha_2, \dots, \alpha_t \in A \cap J$ with the following properties.*

1. *$N(\alpha_i) \leqslant T$ $1 \leqslant i \leqslant t$;*
2. *$|\alpha_i - \alpha_j| \geqslant T^{-1}$ $1 \leqslant i < j \leqslant t$;*
3. *$t \geqslant K|J|T$,*

where $|J|$ means Lebesgue measure of $|J|$.

Let $A_\lambda = \{\alpha \in I \mid \exists\ F \in \mathfrak{F}_n,\ F(\alpha) + \lambda(\alpha) = 0\}$. For all $\alpha \in A_\lambda$ we'll denote by the height of α he value $H(\alpha) = \min\{H(F) \mid F \in \mathfrak{F}_n,\ F(\alpha) + \lambda(\alpha) = 0\}$.

Theorem 2 *Let the functions* $f_1, f_2, \ldots, f_n$ *have the same properties as in theorem 1. Then there exists a neighborhood of* x_0 *such that the set* A_λ *with the function* $N(\alpha) = H^{n+1}(\alpha)$ *is a regular system on that neighborhood.*

Select a neighborhood I^* of x_0 such that $\forall x \in I^*, |w(f_1'(x), \ldots, f_n'(x))| > |w(f_1'(x_0), \ldots, f_n'(x_0))|/2$ and $\max\{\lambda(x), \lambda'(x), \lambda''(x)\} < K$. Further we'll call this interval I. Such interval exists because the Vronscian is continuous function.

The functions $f_i^{(j)};\ 0 \leqslant j \leqslant n+1,\ 1 \leqslant i \leqslant n$ are continuous. Therefore without loss of generality we may consider that for all $x \in I$ we have

$$|f_i^{(j)}(x)| \leqslant C, \tag{6}$$

for some constant C. Also without loss of generality we can define $f_1(x) \equiv x$.

Lemma 1 *Let* $\Phi(Q, \delta)$ *be a set of* $x \in I$ *such that*

$$|F(x)| < \delta Q^{-n} \tag{7}$$

for some nonzero function $F \in \mathcal{F}_n^*(Q)$*. Then* $\exists Q_1 \mid \forall Q > Q_1, \forall \delta = n^{-1}3^{-n}2^{-5}, |\Phi(Q, \delta)| < |I|/2.$

The proof of this lemma can be found in [15][corollary of the proposition 2.]. The proof of that proposition needs another one property to be true. There should be $\epsilon \in (0,1)$ such that $|\Lambda_n(\epsilon)| = 0$. But it is followed from [5] that if the Vronscian is not equal to 0 for all $x \in I$ than this property is satisfied automatically.

Proof of the theorem 2 uses ideas of Y. Bugeaud [10].

BAKER–SCHMIDT'S LEMMA AND PROOF OF THE MAIN THEOREM

Let (A, N) be a regular system. Denote by (A, N, w) the set of real numbers ξ such that the inequality

$$|\xi - \alpha| < N(\alpha)^{-w}$$

has infinitely many solutions $\alpha \in A$.

Lemma 2 *Let* $0 < w^{-1} \leqslant 1$*. Then the Hausdorff dimension for the set* (A, N, w) *is at least* w^{-1}.

This lemma is a particular case of Baker-Schmidt lemma [16][chapter 1, Bakers lemma 5].

The proof of the main theorem.

Using theorem 2 we obtain that A_λ with a function $H(\alpha)^{n+1}$ is a regular system. Let $\alpha \in A_\lambda$ i.e. there exists $F \in \mathcal{F}_n$ such that $F(\alpha) + \lambda(\alpha) = 0$. Note that for such function $|F'(\alpha)| \leqslant nCH(F) = vH(F)$.

Consider an interval $J = (\alpha - (2v)^{-1}H(F)^{-n-1}, \alpha + (2v)^{-1}H(F)^{-n-1})$. For all $x \in J \cap I$ we have

$$F'(x) + \lambda'(x) = F'(\alpha) + \lambda'(\alpha) + (F''(x_1) + \lambda''(x_1))(x - \alpha).$$

Therefore we get $|F'(x) + \lambda(x)| \leqslant vH(F) + K + (C + K)(2v)^{-1}H^{-w-1}(F) \leqslant 2vH(F)$. Last inequality may not be true for a finite number of functions F when $H(F)$ is small enough.

By Lagrange's formulae we obtain

$$F(x) + \lambda(x) = F(\alpha) + \lambda(\alpha) + (F'(x_2) + \lambda'(x_2))(x - \alpha)$$

Using previously obtained inequality we get $|F(x) + \lambda(x)| \leqslant H(F)^{-w}$.

Denote by $\widetilde{X}_n(w)$ a set $\xi \in I$ such that th inequality

$$|\alpha - \xi| \leqslant (2v)^{-1}H(\alpha)^{-w-1} \tag{8}$$

has infinitely many solutions $\alpha \in A_\lambda$. Using proved proposition $\widetilde{X}_n(w) \subset X_n(w)$. Changing slightly (8) we get

$$|\alpha - \xi| < (2v)^{-1}N(\alpha)^{-\frac{w+1}{n+1}}.$$

By lemma 2 we finally obtain that $\dim\{\widetilde{X}_n(w)\} \geqslant \frac{n+1}{w+1}$. And thus $\dim\{X_n(w)\} \geqslant \dim\{\widetilde{X}_n(w)\} \geqslant \frac{n+1}{w+1}$. It finishes the proof.

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