Journal of Number Theory 111, no.1 (2005) 33-56

ON APPROXIMATION OF P-ADIC NUMBERS BY P-ADIC ALGEBRAIC NUMBERS

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1. Introduction

Throughout $p \geqslant 2$ is a fixed prime number, $\mathbb{Q}_p$ is the field of p-adic numbers, $|\omega|_p$ is the p-adic valuation of $\omega \in \mathbb{Q}_p$, $\mu(S)$ is the Haar measure of a measurable set $S \subset \mathbb{Q}_p$, $\mathbb{A}_{p,n}$ is the set of algebraic numbers of degree n lying in $\mathbb{Q}_p$, $\mathbb{A}_p$ is the set of all algebraic numbers, $\mathbb{Q}_p^*$ is the extension of $\mathbb{Q}_p$ containing $\mathbb{A}_p$. There is a natural extension of p-adic valuation from $\mathbb{Q}_p$ to $\mathbb{Q}_p^*$ [Cas86, Lut55]. This valuation will also be denoted by $|\cdot|_p$. The disc in $\mathbb{Q}_p$ of radius r centered at α is the set of solutions of the inequality $|x-\alpha|_p < r$. Throughout, $\mathbb{R}_{>a} = \{x \in \mathbb{R} : x > a\}$, $\mathbb{R}_+ = \mathbb{R}_{>0}$ and $\Psi : \mathbb{N} \to \mathbb{R}_+$ is monotonic.

Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $a_n \neq 0$, $\deg P = n$ is the degree of P, $H(P) = \max_{0\leqslant i\leqslant n} |a_i|$ is the usual height of P. Also $H(\alpha)$ will stand for the usual height of $\alpha \in \mathbb{A}_p$, *i.e.* the height of the minimal polynomial for α. The notation $X \ll Y$ will mean $X = O(Y)$ and the one of $X \asymp Y$ will stand for $X \ll Y \ll X$.

In 1989 V. Bernik [Ber89] proved A. Baker's conjecture by showing that for almost all $x \in \mathbb{R}$ the inequality $|P(x)| < H(P)^{-n+1}\Psi(H(P))$ has only finitely many solutions in $P \in \mathbb{Z}[x]$ with $\deg P \leqslant n$ whenever and the sum

$$\sum_{h=1}^{\infty} \Psi(h) \tag{1}$$

converges. In 1999 V. Beresnevich [Ber99] showed that in the case of divergence of (1) this inequality has infinitely many solutions.

We refer the reader to [BBKM02, BD99, Ber02, BKM01, Spr79] for further development of the metric theory of Diophantine approximation. In this paper we establish a complete analogue of the aforementioned results for the p-adic case.

Theorem 1. *Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be monotonically decreasing and $M_n(\Psi)$ be the set of $\omega \in \mathbb{Q}_p$ such that the inequality*

$$|P(\omega)|_p < H(P)^{-n}\Psi(H(P)) \tag{2}$$

has infinitely many solutions in polynomials $P \in \mathbb{Z}[x]$, $\deg P \leqslant n$. Then $\mu(M_n(\Psi)) = 0$ whenever the sum (1) converges and $M_n(\Psi)$ has full Haar measure whenever the sum (1) diverges.

Date: February 13, 2008.

1991 *Mathematics Subject Classification.* Primary 11J83; Secondary 11K60.

Key words and phrases. Diophantine approximation, Khintchine type theorems, Metric theory of Diophantine approximation.

The following is a p-adic analogue of Theorem 2 in [Ber99].

Theorem 2. *Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be monotonically decreasing and $\mathbb{A}_{p,n}(\Psi)$ be the set of $\omega \in \mathbb{Q}_p$ such that the inequality*

$$|\omega - \alpha|_p < H(\alpha)^{-n}\Psi(H(\alpha)) \tag{3}$$

has infinitely many solutions in $\alpha \in \mathbb{A}_{p,n}$. Then $\mu(\mathbb{A}_{p,n}(\Psi)) = 0$ whenever the sum (1) converges and $\mathbb{A}_{p,n}(\Psi)$ has full Haar measure whenever the sum (1) diverges.

2. Reduction of Theorem 1

We are now going to show that the convergence part of Theorem 1 follows from the following two theorems. Also we show that the divergence part of Theorem 1 follows from Theorem 2.

Proposition 1. *Let $\delta, \xi \in \mathbb{R}_+$, $\xi < 1/2$, $Q \in \mathbb{R}_{>1}$ and K_0 be a finite disc in $\mathbb{Q}_p$. Given a disc $K \subset K_0$, let $E_1(\delta, Q, K, \xi)$ be the set of $\omega \in K$ such that there is a non-zero polynomial $P \in \mathbb{Z}[x]$, $\deg P \leqslant n$, $H(P) \leqslant Q$ satisfying the system of inequalities*

$$\begin{cases} |P(\omega)|_p < \delta Q^{-n-1}, \\ |P'(\alpha_{\omega,P})|_p \geqslant H(P)^{-\xi}, \end{cases} \tag{4}$$

where $\alpha_{\omega,P} \in \mathbb{A}_p$ is the root of P nearest to ω (if there are more than one root nearest to ω then we choose any of them). Then there is a positive constant c_1 such that for any finite disc $K \subset K_0$ there is a sufficiently large number Q_0 such that $\mu(E_1(\delta, Q, K, \xi)) \leqslant c_1\delta\mu(K)$ for all $Q \geqslant Q_0$ and all $\delta > 0$.

Proposition 2. *Let $\xi, C \in \mathbb{R}_+$, K_0 be a finite disc in $\mathbb{Q}_p$ and let $E_2(\xi, C, K_0)$ be the set of $\omega \in \mathbb{Q}_p$ such that there are infinitely many polynomials $P \in \mathbb{Z}[x]$, $\deg P \leqslant n$ satisfying the system of inequalities*

$$\begin{cases} |P(\omega)|_p < CH(P)^{-n-1}, \\ |P'(\alpha_{\omega,P})|_p < H(P)^{-\xi}. \end{cases} \tag{5}$$

Then $\mu(E_2(\xi, C, K_0)) = 0$.

Proof of the convergence part of Theorem 1 modulo Propositions 1 and 2. Let the sum (1) converges. Then it is readily verified that

$$\sum_{t=1}^{\infty} 2^t\Psi(2^t) < \infty \tag{6}$$

and

$$\Psi(h) = o(h^{-1}) \tag{7}$$

as $h \to \infty$. For the proofs of (6) see Lemma 5 in [Ber99]. The arguments for (7) can be found in the proof of Lemma 4 in [Ber99].

Fix any positive $\xi < 1/2$. By (7), $H(P)^{-n}\Psi(H(P)) < H(P)^{-n-1}$ for all but finitely many P. Then, by Proposition 2, to complete the proof of the convergence part of Theorem 1 it remains to show that for any finite disc K in $\mathbb{Q}_p$ the set $E_1(\xi, \Psi)$ consisting of $\omega \in \mathbb{Q}_p$ such that there are infinitely many polynomials $P \in \mathbb{Z}[x]$, $\deg P \leqslant n$ satisfying the system of inequalities

$$\begin{cases} |P(\omega)|_p < H(P)^{-n}\Psi(H(P)), \\ |P'(\alpha_{\omega,P})|_p \geqslant H(P)^{-\xi} \end{cases} \tag{8}$$

has zero measure.

The system (8) implies

$$\begin{cases} |P(\omega)|_p < (2^t)^{-n-1}2^t\Psi(2^t), \\ |P'(\alpha_{\omega,P})|_p \geqslant H(P)^{-\xi}, \end{cases} \tag{9}$$

where $t = t(P)$ with $2^t \leqslant H(P) < 2^{t+1}$, which means that $\omega \in E_1(2^{n+1}2^t\Psi(2^t), 2^{t+1}, K, \xi)$. The system (9) holds for infinitely many t whenever (8) holds for infinitely many P. Therefore,

$$E_1(\xi, \Psi) \subset \limsup_{t\to\infty} E_1(2^{n+1}2^t\Psi(2^t), 2^{t+1}, K, \xi).$$

By Proposition 1, $\mu(E_1(2^{n+1}2^t\Psi(2^t), 2^{t+1}, K, \xi)) \ll 2^t\Psi(2^t)$. Taking into account (6), the Borel-Cantelli lemma completes the proof. □

Next, we are going to show that the divergence part of Theorem 1 is a consequence of Theorem 2.

Proof of the divergence part of Theorem 1 modulo Theorem 2. Fix any finite disc K in $\mathbb{Q}_p$. Then there is a positive constant $C > 0$ such that $|\omega|_p \leqslant C$ for all $\omega \in K$. Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a given monotonic function such that the sum (1) diverges. Then the function $\tilde{\Psi}(h) = |n!|_p C^{1-n}\Psi(h)$ is also monotonic and the sum $\sum_{h=1}^{\infty} \tilde{\Psi}(h)$ diverges. By Theorem 2, for almost every $\omega \in K$ there are infinitely many $\alpha \in \mathbb{A}_{p,n}$ satisfying

$$|\omega - \alpha|_p < H(\alpha)^{-n}\tilde{\Psi}(H(\alpha)). \tag{10}$$

As Ψ decreases, the right hand side of (10) is bounded by a constant. Then we can assume that $|\omega-\alpha|_p \leqslant C$ for the solutions of (10). Then $|\alpha|_p = |\alpha-\omega+\omega|_p \leqslant \max\{|\alpha-\omega|_p, |\omega|_p\} \leqslant C$.

Let P_α denote the minimal polynomial for α. Since $P_\alpha^{(i)}$ is a polynomial with integer coefficients of degree $n - i$, we have $|P_\alpha^{(i)}(\alpha)|_p \leqslant \max_{0\leqslant j\leqslant n-i} |\alpha|_p^j \leqslant C^{n-i}$. Then

$$\begin{aligned} |P_\alpha(\omega)|_p = |\omega - \alpha|_p \left| \sum_{i=1}^{n} i!^{-1} P_\alpha^{(i)}(\alpha)(\omega - \alpha)^{i-1} \right|_p \leqslant \\ \leqslant |\omega - \alpha|_p \cdot \max_{1\leqslant i\leqslant n} \left| i!^{-1} P_\alpha^{(i)}(\alpha)(\omega - \alpha)^{i-1} \right|_p \leqslant \\ \leqslant |\omega - \alpha|_p \cdot |n!|_p^{-1} C^{n-i} C^{i-1} = |n!|_p^{-1} C^{n-1} |\omega - \alpha|_p. \end{aligned}$$

Therefore (10) implies

$$|P_\alpha(\omega)|_p < H(\alpha)^{-n}\tilde{\Psi}(H(\alpha))|n!|_p^{-1}C^{n-1} = H(\alpha)^{-n}\Psi(H(\alpha)) = H(P_\alpha)^{-n}\Psi(H(P_\alpha)). \tag{11}$$

Inequality (10) has infinitely many solutions for almost all $\omega \in K$ and so has (11). As ω is almost every point of K, the proof is completed. □

3. Reduction of Theorem 2

Proof of the convergence part of Theorem 2. Given an $\alpha \in \mathbb{A}_{p,n}$, let $\chi(\alpha)$ be the set of $\omega \in \mathbb{Q}_p$ satisfying (3). The measure of $\chi(\alpha)$ is $\ll H(\alpha)^{-n}\Psi(H(\alpha))$. Then

$$\sum_{\alpha\in\mathbb{A}_{p,n}} \mu(\chi(\alpha)) = \sum_{h=1}^{\infty} \sum_{\alpha\in\mathbb{A}_{p,n},\ H(\alpha)=h} \mu(\chi(\alpha)) \ll$$

$$\ll \sum_{h=1}^{\infty} \sum_{\alpha\in\mathbb{A}_{p,n},\ H(\alpha)=h} h^{-n}\Psi(h) \ll \sum_{h=1}^{\infty} \Psi(h) < \infty.$$

Here we used the fact that the quantity of algebraic numbers of height h is $\ll h^n$. The Borel-Cantelli lemma completes the proof. □

The proof of the divergence part of Theorem 2 will rely on the regular systems method of [Ber99]. In this paper we give a generalization of the method for the p-adic case.

Definition 1. Let a disc K_0 in $\mathbb{Q}_p$, a countable set of p-adic numbers Γ and a function $N : \Gamma \to \mathbb{R}_+$ be given. The pare (Γ, N) is called *a regular system of points in* K_0 if there is a constant $C > 0$ such that for any disc $K \subset K_0$ for any sufficiently large number T there exists a collection

$$\gamma_1, \dots, \gamma_t \in \Gamma \cap K$$

satisfying the following conditions

$$N(\gamma_i) \leqslant T \quad (1 \leqslant i \leqslant t),$$
$$|\gamma_i - \gamma_j|_p \geqslant T^{-1} \quad (1 \leqslant i < j \leqslant t),$$
$$t \geqslant CT\mu(K).$$

Proposition 3. *Let (Γ, N) be a regular system of points in $K_0 \subset \mathbb{Q}_p$, $\tilde{\Psi} : \mathbb{R}_+ \to \mathbb{R}_+$ be monotonically decreasing function such that $\sum_{h=1}^{\infty} \tilde{\Psi}(h) = \infty$. Then $\Gamma_{\tilde{\Psi}}$ has full Haar measure in K_0, where $\Gamma_{\tilde{\Psi}}$ consists of $\omega \in K_0$ such that the inequality*

$$|x - \gamma|_p < \tilde{\Psi}(N(\gamma)) \tag{12}$$

has infinitely many solutions $\gamma \in \Gamma$.

This theorem is proved in [BK03]. The proof is also straitforward the ideas of the proof of Theorem 2 in [Ber99].

Proposition 4. *The pare (Γ, N) of $\Gamma = \mathbb{A}_{p,n}$ and $N(\alpha) = H(\alpha)^{n+1}$ is a regular system of points in any finite disc $K_0 \subset \mathbb{Q}_p$.*

Proof of the divergence part of Theorem 2 modulo Propositions 3 and 4. Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotonic function and the sum (1) diverges. Fix any finite disc $K_0 \subset \mathbb{Q}_p$.

Let (Γ, N) be a regular system defined in Proposition 3 and let Ψ be a monotonic function such that the sum (1) diverges. Define a function $\tilde{\Psi}$ by setting $\tilde{\Psi}(x) = x^{-n/(n+1)}\Psi(x^{1/(n+1)})$. Using the monotonicy of Ψ, we obtain

$$\sum_{h=1}^{\infty} \tilde{\Psi}(h) = \sum_{t=1}^{\infty} \sum_{(t-1)^{n+1}<h\leqslant t^{n+1}} \tilde{\Psi}(h) \geqslant \sum_{t=1}^{\infty} \sum_{(t-1)^{n+1}<h\leqslant t^{n+1}} t^{-n}\Psi(t) =$$

$$= \sum_{t=1}^{\infty} \left(t^{n+1} - (t-1)^{n+1}\right) t^{-n}\Psi(t) \asymp \sum_{h=1}^{\infty} \Psi(h) = \infty.$$

In is obvious that $\tilde{\Psi}$ is monotonic. Then, by Proposition 2, for almost all $\omega \in K_0$ the inequality

$$|x - \alpha|_p < \tilde{\Psi}(N(\alpha)) = H(\alpha)^{-n}\Psi(H(\alpha)) \tag{13}$$

has infinitely many solutions in $\alpha \in \mathbb{A}_{p,n}$. The proof is completed. □

4. Proof of Proposition 1

Fix any finite $K \subset K_0$ in $\mathbb{Q}_p$. Let $\chi(P)$ be the set of $\omega \in K$ satisfying (4) and let $\mathcal{P}_n(Q, K)$ be the set of non-zero polynomials P with integer coefficients, $\deg P \leqslant n$, $H(P) \leqslant Q$ and with $\chi(P) \neq \emptyset$. We will use the following

Lemma 1. *Let $\alpha_{\omega,P}$ is the nearest root of a polynomial P to $\omega \in \mathbb{Q}_p$. Then*

$$|\omega - \alpha_{\omega,P}|_p \leqslant |P(\omega)|_p |P'(\alpha_{\omega,P})|_p^{-1}.$$

For the proof see [Spr69, p. 78].

Given a polynomial $P \in \mathcal{P}_n(Q, K)$, let $\mathcal{Z}_P$ be the set of roots of P. It is clear that $\#\mathcal{Z}_P \leqslant n$. Given an $\alpha \in \mathcal{Z}_P$, let $\chi(P, \alpha)$ be the subset of $\chi(P)$ consisting of ω with $|\alpha - \omega|_p = \min\{|\alpha' - \omega|_p : \alpha' \in \mathcal{Z}_P\}$.

By Lemma 1, for any $P \in \mathcal{P}_n(Q, K)$ and any $\alpha \in \mathcal{Z}_P$ one has

$$\mu(\chi(P, \alpha)) \ll \delta Q^{-n-1} |P'(\alpha)|_p^{-1}. \tag{14}$$

Given a $P \in \mathcal{P}_n(Q, K)$ and an $\alpha \in \mathcal{Z}_P$, define the disc

$$\overline{\chi}(P, \alpha) = \left\{\omega \in K : |\omega - \alpha|_p \leqslant \left(4Q|P'(\alpha)|_p\right)^{-1}\right\}. \tag{15}$$

It is readily verified that if $\overline{\chi}(P, \alpha) \neq \emptyset$ then $\mu(\overline{\chi}(P, \alpha)) \gg \left(4Q|P'(\alpha)|_p\right)^{-1}$. Using (14) we get

$$\mu(\chi(P, \alpha)) \ll \delta Q^{-n-1} \mu(\overline{\chi}(P, \alpha)) \tag{16}$$

with the implicit constant depending on p only.

Fix any $P \in \mathcal{P}_n(Q, K)$ and an $\alpha \in \mathcal{Z}_P$ such that $\chi(P, \alpha) \neq \emptyset$. Let $\omega \in \overline{\chi}(P, \alpha)$. Then

$$P(\omega) = P'(\alpha)(\omega - \alpha) + (\omega - \alpha)^2 \left(\sum_{i=2}^{n} P^{(i)}(\alpha)(\omega - \alpha)^{i-2}\right). \tag{17}$$

By the inequalities $|P'(\alpha)|_p \geqslant H(P)^{-\xi}$ and $H(P) \leqslant Q$, we have $|P'(\alpha)|_p^{-1} \leqslant Q^{\xi}$. Then by (15), $|\omega - \alpha|_p \leqslant Q^{-1+\xi}$. Next, as $\omega \in K$ and K is finite, it is readily verified that $|P^{(i)}(\alpha)|_p \ll 1$, where the constant in this inequality depends on K. Then

$$\left|(\omega - \alpha)^2 \left(\sum_{i=2}^{n} P^{(i)}(\alpha)(\omega - \alpha)^{i-2}\right)\right|_p \ll Q^{-2+2\xi}. \tag{18}$$

By (15), we have $|P'(\alpha)(\omega-\alpha)|_p \leqslant (4Q)^{-1}$. Using this inequality, (18) and $\xi < 1/2$, we conclude that

$$|P(\omega)|_p \leqslant (4Q)^{-1}, \qquad \omega \in \chi(P,\alpha) \tag{19}$$

if Q is sufficiently large.

Assume that $P_1, P_2 \in \mathcal{P}_n(Q,K)$ satisfy $P_1 - P_2 \in \mathbb{Z}_{\neq 0}$ and assume that there is an $\omega \in \overline{\chi}(P_1) \cap \overline{\chi}(P_2)$. Then $\omega \in \overline{\chi}(P_1,\alpha) \cap \overline{\chi}(P_2,\beta)$ for some $\alpha \in \mathcal{Z}_{P_1}$ and $\beta \in \mathcal{Z}_{P_2}$. Then, (19), $|P_1(\omega)-P_2(\omega)|_p < (4Q)^{-1}$. On the other $P_1(\omega)-P_2(\omega)$ is an integer not greater than $2Q$ in absolute value. Therefore, $|P_1(\omega)-P_2(\omega)|_p \geqslant (2Q)^{-1}$ that leads to a contradiction. Hence there is no such an ω and $\overline{\chi}(P_1) \cap \overline{\chi}(P_2) = \emptyset$. Therefore

$$\sum_{P \in \mathcal{P}_n(Q,K,a_n,\dots,a_1)} \mu(\overline{\chi}(P)) \leqslant \mu(K), \tag{20}$$

where $\mathcal{P}_n(Q,K,a_n,\dots,a_1)$ is the subset of $\mathcal{P}_n(Q,K)$ consisting of P with fixed coefficients $a_n,\dots,a_1$.

By (16) and (20), $\sum_{P \in \mathcal{P}_n(Q,K,a_n,\dots,a_1)} \mu(\chi(P)) \ll \delta Q^{-n}\mu(K)$. Summing this over all $(a_n,\dots,a_1) \in \mathbb{Z}^n$ with coordinates at most Q in absolute value gives

$$\sum_{P \in \mathcal{P}_n(Q,K)} \mu(\chi(P)) \ll \delta\mu(K). \tag{21}$$

It is obvious that

$$E_1(\delta,Q,K,\xi) = \bigcup_{P \in \mathcal{P}_n(Q,K)} \chi(P). \tag{22}$$

As the Haar measure is subadditive (21) and (22) imply the statement of Proposition 1.

5. Reduction to irreducible primitive leading polynomials in Proposition 2

The following lemma shows us that there is no loss of generality in neglecting reducible polynomials while proving Proposition 2.

Lemma 2 (Lemma 7 in [BDY99]). *Let $\delta \in \mathbb{R}_+$ and $E(\delta)$ be the set of $\omega \in \mathbb{Q}_p$ such that the inequality*

$$|P(\omega)|_p < H(P)^{-n-\delta}$$

has infinitely many solutions in reducible polynomials $P \in \mathbb{Z}[x]$, $\deg P \leqslant n$. Then $\mu(E(\delta)) = 0$.

Also, by Sprindžuk's theorem [Spr69] there is no loss of generality in assuming that $\deg P = n$. From now on, $\mathcal{P}$ will denote the set of irreducible polynomials $P \in \mathbb{Z}[x]$ with $\deg P = n$.

Next, a polynomial $P \in \mathbb{Z}[x]$ is called *primitive* if the gcd (greatest common divisor) of its coefficients is 1. To perform the reduction to primitive polynomials we fix an ω such that the system (5) has infinitely many solutions in polynomials $P \in \mathcal{P}$ and show that either ω belongs to a set of measure zero or (5) holds for infinitely many primitive $P \in \mathcal{P}$.

Define $a_P = \gcd(a_n, \dots, a_1, a_0) \in \mathbb{N}$. Given a $P \in \mathcal{P}$, there is a uniquely defined primitive polynomial P_1 (*i.e.* $a_{P_1} = 1$) with $P = a_P P_1$. Then $H(P) = a_P H(P_1)$. Let $P \in \mathcal{P}$ be a solution of (5). By (5), P_1 satisfies the inequalities

$$\begin{cases} |a_P|_p |P_1(\omega)|_p = |P(\omega)|_p \ll H(P)^{-n-1} = (a_P H(P_1))^{-n-1}, \\ |a_P|_p |P_1'(\alpha_{\omega,P})|_p = |P'(\alpha_{\omega,P})|_p < H(P)^{-\xi} = (a_P H(P_1))^{-\xi}. \end{cases} \tag{23}$$

As $|a_P|_p^{-1} \leqslant a_P$, (23) implies

$$|P_1(\omega)|_p \ll H(P_1)^{-n-1} a_P^{-n}, \qquad |P_1'(\alpha_{\omega,P})|_p < H(P_1)^{-\xi} a_P^{1-\xi}. \tag{24}$$

If (24) takes place only for a finite number of different polynomials $P_1 \in \mathcal{P}$, then there exists one of them such that (5) has infinitely many solutions in polynomials P with the same P_1. It follows that ω is a root of P_1 and thus belongs to a set of measure zero. Further we assume that there are infinitely many P_1 satisfying (24).

If $\xi \geqslant 1$ then the reduction to primitive polynomials is obvious as $a_P \in \mathbb{N}$. Let $\xi < 1$. Then, if (5) holds for infinitely many polynomials $P \in \mathcal{P}$ such that $a_P \geqslant H(P_1)^{\xi'}$, where $\xi' = \xi/(2-2\xi)$, then the first inequality in (24) implies that $|P_1(\omega)|_p \ll H(P_1)^{-n-1} a_P^{-n} \leqslant H(P_1)^{-n-1-n\xi'}$ holds for infinitely many polynomials $P_1 \in \mathcal{P}$. By Sprindžuk's theorem [Spr69], the set of those ω has zero measure.

If (5) holds for infinitely many polynomials $P \in \mathcal{P}$ such that $a_P < H(P_1)^{\xi'}$ then (24) implies that the system of inequalities

$$|P_1(\omega)|_p \ll H(P_1)^{-n-1}, \qquad |P'(\alpha_{\omega,P})|_p < H(P_1)^{-\xi+(1-\xi)\xi'} < H(P_1)^{-\xi/2}$$

holds for infinitely many polynomials P_1. Thus, we get the required statement with a smaller ξ.

A polynomial $P \in \mathbb{Z}[x]$ with the leading coefficient a_n will be called *leading* if

$$a_n = H(P) \qquad \text{and} \qquad |a_n|_p > p^{-n}. \tag{25}$$

Let $\mathcal{P}_n(H)$ be the set of irreducible primitive leading polynomials $P \in \mathbb{Z}[x]$ of degree n with the height $H(P) = H$. Also define

$$\mathcal{P}_n = \bigcup_{H=1}^{\infty} \mathcal{P}_n(H). \tag{26}$$

Reduction to leading polynomials is completed with the help of

Lemma 3. *Let Ω be the set of points $\omega \in \mathbb{Q}_p$ for which* (5) *has infinitely many solutions in irreducible primitive polynomials $P \in \mathbb{Z}[x]$, $\deg P = n$. Let Ω_0 be the set of points $\omega \in \mathbb{Q}_p$ for which* (5) *has infinitely many solutions in polynomials $P \in \mathcal{P}_n$, where $\mathcal{P}_n$ is defined in* (26)*. If Ω has positive measure then so has Ω_0 with probably a different constant C in* (5)*.*

Proof of this lemma is very much the same as the one of Lemma 10 in [Spr69] and we leave it as an exercise.

Every polynomial $P \in \mathcal{P}_n$ has exactly n roots, which can be ordered in any way: $\alpha_{P,1}, \dots, \alpha_{P,n}$. The set $E_2(\xi, C, K_0)$ can be expressed as a union of subsets $E_{2,k}(\xi, C, K_0)$ with $1 \leqslant k \leqslant n$, where $E_{2,k}(\xi, C, K_0)$ is defined to consist of $\omega \in K_0$ such that (5) holds infinitely often with $\alpha_{\omega,P} = \alpha_{P,k}$. To prove Proposition 2 it suffices to show that

$E_{2,k}(\xi, C, K_0)$ has zero measure for every k. The consideration of these sets will not depend on k. Therefore we can assume that $k=1$ and omit this index in the notation of $E_{2,k}(\xi, C, K_0)$. Also whenever there is no risk of confusion we will write $\alpha_1, \dots, \alpha_n$ for $\alpha_{P,1}, \dots, \alpha_{P,n}$.

6. Auxiliary statements and classes of polynomials

Lemma 4. *Let $\alpha_1, \dots, \alpha_n$ be the roots of $P \in \mathcal{P}_n$. Then $\max_{1 \leqslant i \leqslant n} |\alpha_i|_p < p^n$.*

For the proof see [Spr69, p. 85].

For the roots $\alpha_1, \dots, \alpha_n$ of P we define the sets

$$S(\alpha_i) = \{\omega \in \mathbb{Q}_p : |\omega - \alpha_i|_p = \min_{1 \leqslant j \leqslant n} |\omega - \alpha_j|_p\} \quad (1 \leq i \leq n).$$

Let $P \in P_n$. As α_1 is fixed, we reorder the other roots of P so that $|\alpha_1 - \alpha_2|_p \leqslant |\alpha_1 - \alpha_3|_p \leqslant \dots \leqslant |\alpha_1 - \alpha_n|_p$. We can assume that there exists a root α_m of P for which $|\alpha_1 - \alpha_m|_p \leqslant 1$ (see [Spr69, p. 99]). Then we have

$$|\alpha_1 - \alpha_2|_p \leqslant |\alpha_1 - \alpha_3|_p \leqslant \dots \leqslant |\alpha_1 - \alpha_m|_p \leqslant 1 \leqslant \dots \leqslant |\alpha_1 - \alpha_n|_p. \tag{27}$$

Let $\varepsilon > 0$ be sufficiently small, $d > 0$ be a large fixed number and let $\varepsilon_1 = \varepsilon/d$, $T = [\varepsilon_1^{-1}] + 1$. We define real numbers ρ_j and integers l_j by the relations

$$|\alpha_1 - \alpha_j|_p = H^{-\rho_j}, \quad (l_j - 1)/T \leqslant \rho_j < l_j/T \quad (2 \leqslant j \leqslant m). \tag{28}$$

It follows from (27) and (28) that $\rho_2 \geqslant \rho_3 \geqslant \dots \geqslant \rho_m \geqslant 0$ and $l_2 \geqslant l_3 \geqslant \dots \geqslant l_m \geqslant 1$. We assume that $\rho_j = 0$ and $l_j = 0$ if $m < j \leqslant n$.

Now for every polynomial $P \in \mathcal{P}_n(H)$ we define a vector $\bar{l} = (l_2, \dots, l_n)$ having non-negative components. In [Spr69, p. 99–100] it is shown that the number of such vectors is finite and depends on n, p and T only. All polynomials $P \in \mathcal{P}_n(H)$ corresponding to the same vector $\bar{l}$ are grouped together into a class $\mathcal{P}_n(H, \bar{l})$. We define

$$\mathcal{P}_n(\bar{l}) = \bigcup_{H=1}^{\infty} \mathcal{P}_n(H, \bar{l}). \tag{29}$$

Let $K_0 = \{\omega \in \mathbb{Q}_p : |\omega|_p < p^n\}$ be the disc of radius p^n centered at 0. Define

$$r_j = r_j(P) = (l_{j+1} + \dots + l_n)/T \quad (1 \leqslant j \leqslant n-1).$$

Lemma 5. *Let $\omega \in S(\alpha_1)$ and $P \in \mathcal{P}_n(H)$. Then*

$$H^{-r_1} \ll |P'(\alpha_1)|_p \ll H^{-r_1 + (m-1)\varepsilon_1},$$

$$|P^{(j)}(\alpha_1)|_p \ll H^{-r_j + (m-j)\varepsilon_1} \quad \textit{for} \quad 2 \leqslant j \leqslant m,$$

$$|P^{(j)}(\alpha_1)|_p \ll 1 \quad \textit{for} \quad m < j \leqslant n.$$

Proof. From (25) we have $p^{-n} < |H|_p \leqslant 1$. Then, on differentiating the identity $P(\omega) = H(\omega - \alpha_1) \cdots (\omega - \alpha_n)$ j times $(1 \leqslant j \leqslant n)$ and using (27), (28) we get the statement of the lemma. □

Lemma 6. *Let $\delta \in \mathbb{R}_+$, $\sigma \in \mathbb{R}_+$, $n \geqslant 2$ be a natural number and $H = H(\delta, n)$ be a sufficiently large real number. Further let P, Q in $\mathbb{Z}[x]$ be two relatively prime polynomials of degree at most n with $\max(H(P), H(Q)) \leqslant H$. Let $K(\alpha, p^{-t})$ be a disc of radius p^{-t} centered at α where t is defined by the inequalities $p^{-t} \leqslant H^{-\sigma} < p^{-t+1}$. If there exists a number $\tau > 0$ such that for all $\omega \in K(\alpha, p^{-t})$ one has*

$$\max(|P(\omega)|_p, |Q(\omega)|_p) < H^{-\tau}$$

then $\tau + 2\max(\tau - \sigma, 0) < 2n + \delta$.

For the proof see Lemma 5 in [BDY99].

7. Proof of Proposition 2

As in the previous section $K_0 = \{\omega \in \mathbb{Q}_p : |\omega|_p < p^n\}$.

Let $A(\bar{l}, \xi)$ be the set of points $\omega \in K_0$ for which

$$\begin{cases} |P(\omega)|_p < CH(P)^{-n-1}, \\ |P'(\alpha_1)|_p < H(P)^{-\xi} \end{cases} \tag{30}$$

has infinitely many solutions in polynomials $P \in \mathcal{P}_n(\bar{l})$, where $\mathcal{P}_n(\bar{l})$ is defined in (29). It follows from the previous discussion that to prove Proposition 2 it suffices to show that $A(\bar{l}, \xi)$ has zero measure for all possible vectors $\bar{l}$.

The following investigation essentially depends on the value of $r_1 + l_2/T$. According to Lemma 5 we have $|P'(\alpha_1)|_p \gg H^{-r_1}$. It follows from this and the second inequality of (30) that $H^{-r_1} \leqslant cH^{-\xi}$, *i.e.*

$$r_1 \geqslant \xi - \ln c/\ln H > \xi/2 \quad \text{for} \quad H \geqslant H_0. \tag{31}$$

Further we assume that r_1 satisfies (31). Further we set ε to be $\xi/2$.

Lemma 7. *If $r_1 + l_2/T > n$ then the set of points $\omega \in K_0$ for which the inequality*

$$|P(\omega)|_p < H(P)^{-n-\varepsilon}$$

holds for infinitely many polynomials $P \in \mathcal{P}_n(\bar{l})$ has zero measure.

For the proof see Proposition 3 in [Spr69, p. 111].

The proof of Proposition 2 is divided into 3 cases, each corresponding to one of the propositions of this section (see below).

Let $\chi(P) = \{\omega \in K_0 \cap S(\alpha_{P,1}) : |P(\omega)|_p < H^{-n-1}\}$. Thus, we investigate the set of ω that belong to infinitely many $\chi(P)$.

Proposition 5. *If $n - 1 + 2n\varepsilon_1 < r_1 + l_2/T$ then $\mu(A(\bar{l}, \xi)) = 0$.*

Proof. Let $r_1 + l_2/T > n$. Using Lemma 7 with $\varepsilon < 1$ we get $\mu(A(\bar{l}, \xi)) = 0$.

Let $n - 1 + 2n\varepsilon_1 < r_1 + l_2/T \leqslant n$ and t be a sufficiently large fixed natural number. We define the set

$$\mathcal{M}_t(\bar{l}) = \bigcup_{2^t \leqslant H < 2^{t+1}} \mathcal{P}_n(H, \bar{l}).$$

We divide the set K_0 into the discs of radius $2^{-t\sigma}$, where $\sigma = n + 1 - r_1 - \varepsilon_1$.

First, we consider the polynomials $P \in \mathcal{M}_t(\bar{l})$ such that there is one of the introduced discs, say K, such that $\chi(P) \cap K \neq \emptyset$ and $\chi(Q) \cap K = \emptyset$ for $Q \in \mathcal{M}_t(\bar{l}) \setminus \{P\}$. The number of the discs and respectively the number of the polynomials is at most $p^n 2^{t\sigma}$. From Lemmas 1 and 5 we get

$$\mu(\chi(P)) \ll |P(\omega)|_p |P'(\alpha_1)|_p^{-1} \ll 2^{-t(n+1-r_1)}$$

and thus summing the measures of $\chi(P)$ for the polynomials P of this class leads to

$$\sum_P \mu(\chi(P)) \ll 2^{t(n+1-r_1-\varepsilon_1-n-1+r_1)} = 2^{-t\varepsilon_1}.$$

The latest gives the convergent series and, by the Borel-Cantelli lemma, completes the proof in this case.

Now we consider the other type of polynomials. Let P and Q be different polynomials of $\mathcal{M}_t(\bar{l})$ such that $\chi(P)$ and $\chi(Q)$ intersect the same disc D introduced above. Then there exist the points ω_1 and ω_2 belonging to D such that

$$\max(|P(\omega_1)|_p, |Q(\omega_2)|_p) \ll 2^{-t(n+1)}. \tag{32}$$

Let $\alpha_{P,1}$ and $\alpha_{Q,1}$ be the nearest roots of P and Q to ω_1 and ω_2 respectively. By (32), Lemmas 1 and 5 we get

$$\max(|\omega_1 - \alpha_{P,1}|_p, |\omega_2 - \alpha_{Q,1}|_p) \ll 2^{-t(n+1-r_1)}.$$

Hence, according to the definition of the σ we have

$$|\alpha_{P,1} - \alpha_{Q,1}|_p \leqslant \max(|\alpha_{P,1} - \omega_1|_p, |\omega_1 - \omega_2|_p, |\alpha_{Q,1} - \omega_2|_p) \ll$$
$$\ll \max(2^{-t(n+1-r_1)}, 2^{-t\sigma}) = 2^{-t\sigma}.$$

Now we estimate $|\alpha_{P,1} - \alpha_{Q,i}|_p \quad (2 \leqslant i \leqslant m)$. Since $r_1 + l_2/T \leqslant n$ it follows that

$$|\alpha_{P,1} - \alpha_{Q,i}|_p \leqslant \max(|\alpha_{P,1} - \alpha_{Q,1}|_p, |\alpha_{Q,1} - \alpha_{Q,i}|_p) \ll \max(2^{-t\sigma}, 2^{-t\rho_i}) \leqslant$$
$$\leqslant \max(2^{-t\sigma}, 2^{-t(l_i-1)/T}) \leqslant 2^{-t(l_i/T-\varepsilon_1)}.$$

Hence

$$\prod_{i=1}^m |\alpha_{P,1} - \alpha_{Q,i}|_p \ll 2^{-t(\sigma+(l_2+\ldots+l_m)/T-(m-1)\varepsilon_1)} = 2^{-t(\sigma+r_1-(m-1)\varepsilon_1)}.$$

Similarly we obtain

$$\prod_{i=1}^m |\alpha_{P,2} - \alpha_{Q,i}|_p \leqslant \prod_{i=1}^m \max\left(|\alpha_{P,2} - \alpha_{P,1}|_p, |\alpha_{P,1} - \alpha_{Q,1}|_p, |\alpha_{Q,1} - \alpha_{Q,i}|_p\right) \leqslant$$

$$\leqslant \max(2^{-t\rho_2}, 2^{-t\sigma}) \prod_{i=2}^m \max(2^{-t\rho_2}, 2^{-t\sigma}, 2^{-t\rho_i}) \ll$$

$$\ll 2^{-t(l_2/T-\varepsilon_1)} \prod_{i=2}^m 2^{-t(l_i/T-\varepsilon_1)} = 2^{-t(l_2/T-\varepsilon_1+(l_2+\ldots+l_m)/T-(m-1)\varepsilon_1)} = 2^{-t(l_2/T+r_1-m\varepsilon_1)}.$$

Let $R(P,Q)$ be the resultant of P and Q, i.e.

$$|R(P,Q)|_p = |H|_p^{2n} \prod_{1 \leqslant i,j \leqslant n} |\alpha_{P,i} - \alpha_{Q,j}|_p.$$

By the previous estimates for $i = 1, 2$ and the trivial estimates $|\alpha_{P,i} - \alpha_{Q,j}|_p \ll p^n$ for $3 \leqslant i \leqslant n$ we get

$$|R(P,Q)|_p \ll 2^{-t(\sigma + r_1 - (m-1)\varepsilon_1 + l_2/T + r_1 - m\varepsilon_1)} \leqslant 2^{-t(\sigma + 2r_1 + l_2/T - (2n-1)\varepsilon_1)} < 2^{-t(2n+\delta')}$$

where $\delta' > 0$. On the other hand we have $|R(P,Q)|_p \gg 2^{-2nt}$ as P and Q have not common roots. The last inequalities lead to a contradiction. □

Proposition 6. *If*

$$2 - \varepsilon/2 < r_1 + l_2/T \leqslant n - 1 + 2n\varepsilon_1 \tag{33}$$

then $\mu(A(\bar{l}, \xi)) = 0$.

Proof. Let

$$\theta = n + 1 - r_1 - l_2/T. \tag{34}$$

Let $[\theta]$ and $\{\theta\}$ be the integral and the fractional parts of θ respectively.

At first we consider the case $\{\theta\} \geqslant \varepsilon$. We define

$$\beta = [\theta] - 1 + 0,2\{\theta\} - 0,1\varepsilon, \tag{35}$$

$$\sigma_1 = l_2/T + 0,8\{\theta\} + (m+1)\varepsilon_1, \tag{36}$$

$$d = [\theta] - 1. \tag{37}$$

Fix any sufficiently large integer H and divide the set K_0 into the discs of radius $H^{-\sigma_1}$. The number of these discs is estimated by $\ll H^{\sigma_1}$. We shall say that the disc D contains the polynomial $P \in \mathcal{P}_n(H, \bar{l})$ and write $P \prec D$ if there exists a point $\omega_0 \in D$ such that $|P(\omega_0)|_p < H^{-n-1}$.

Let $B_1(H)$ be the collection of discs D such that $\#\{P \in \mathcal{P}_n(H, \bar{l}) : P \prec D\} \leqslant H^{\beta}$. By Lemmas 1 and 5, (35) and (36) we have

$$\sum_{P \in B_1(H)} \mu(\chi(P)) \ll H^{\beta} H^{\sigma_1} H^{-n-1+r_1} = H^{\theta - 1 + r_1 + l_2/T - 0,1\varepsilon + (m+1)\varepsilon_1 - n - 1}.$$

From (34) we get

$$\sum_P \mu(\chi(P)) \ll \sum_H H^{-1-\varepsilon/20} < \infty.$$

By Borel-Cantelli lemma the set of those ω, which belong to $\chi(P)$ for infinitely many $P \in \bigcup_H B_1(H)$, has zero measure.

Let $B_2(H)$ be the collection of the discs that do not belong to $B_1(H)$ and thus contain more than H^{β} polynomials $P \in \mathcal{P}_n(H, \bar{l})$. Let $D \in B_2(H)$. We divide the set $\{P \in \mathcal{P}_n(H, \bar{l}) : P \prec D\}$ into classes as follows. Two polynomials

$$P_1(x) = Hx^n + a_{n-1}^{(1)} x^{n-1} + \ldots + a_1^{(1)} x + a_0^{(1)},$$

$$P_2(x) = Hx^n + a_{n-1}^{(2)} x^{n-1} + \ldots + a_1^{(2)} x + a_0^{(2)}$$

are in one class if

$$a_{n-1}^{(1)} = a_{n-1}^{(2)}, \quad \ldots, \quad a_{n-d}^{(1)} = a_{n-d}^{(2)},$$

where d is defined in (37). It is clear that the number of different classes is less than $(2H+1)^d$ and the number of polynomials under consideration is greater than H^{β}. By the pigeon-hole principle, there exists a class M which contains at least $cH^{\beta-d}$ polynomials where $c > 0$ is a constant independent of H. The classes containing less than $cH^{\beta-d}$ polynomials are considered in a similar way as above, with the Borel-Cantelli arguments.

Further, we denote polynomials from M by $P_1(x),\dots,P_{s+1}(x)$ and consider s new polynomials

$$R_1(x)=P_2(x)-P_1(x),\dots,R_s(x)=P_{s+1}(x)-P_1(x).$$

By (37), we get

$$\deg R_i\leqslant n-d-1=n-[\theta]\quad(1\leqslant i\leqslant s). \tag{38}$$

Using (34), the left-hand side of (33) and the condition $\{\theta\}\geqslant\varepsilon$ we obtain

$$n-d-1=n-[\theta]=n-\theta+\{\theta\}=-1+r_1+l_2/T+\{\theta\}>1+\varepsilon/2>1.$$

Since $n-[\theta]$ is integer then

$$n-[\theta]\geqslant 2. \tag{39}$$

Now we estimate the values $|R_i(\omega)|_p$ $(1\leqslant i\leqslant s)$ when $\omega\in D$. For every polynomial P_i there exists a point $\omega_{0i}\in D$ such that $|P_i(\omega_{0i})|_p<H^{-n-1}$. Let α_{1i} be the root nearest to ω_{0i}. By Lemmas 1 and 5, we get $|\omega_{oi}-\alpha_{1i}|_p\ll H^{-n-1+r_1}$ and

$$|\omega-\alpha_{1i}|_p\leqslant\max(|\omega-\omega_{0i}|_p,|\omega_{0i}-\alpha_{1i}|_p)\ll\max(H^{-\sigma_1},H^{-n-1+r_1})$$

for any $\omega\in D$. It follows from (36) and the right-hand side of (33) that

$$\sigma_1\leqslant n-1-r_1+2n\varepsilon_1+0,8\{\theta\}+(m+1)\varepsilon_1<n+1-r_1.$$

Therefore $|\omega-\alpha_{1i}|_p\ll H^{-\sigma_1}$. By Lemma 5, we have

$$|P_i^{(j)}(\alpha_{1i})(\omega-\alpha_{1i})^j|_p\ll H^{-r_j+(m-j)\varepsilon_1-j\sigma_1}\quad\text{for}\quad 1\leqslant j\leqslant m,$$
$$|P_i^{(j)}(\alpha_{1i})(\omega-\alpha_{1i})^j|_p\ll H^{-j\sigma_1}\quad\text{for}\quad m<j\leqslant n.$$

From (36), (34) and the definition of the r_j $(1\leqslant j\leqslant m)$ we get

$$|P_i'(\alpha_{1i})(\omega-\alpha_{1i})|_p\ll H^{-(n+1-\theta)-0,8\{\theta\}-2\varepsilon_1},$$
$$|P_i^{(j)}(\alpha_{1i})(\omega-\alpha_{1i})^j|_p\ll H^{-(n+1-\theta)-0,8\{\theta\}-(m+1)\varepsilon_1}\ \text{for}\ 2\leqslant j\leqslant n.$$

Using Taylor's formula for $P_i(\omega)$ $(1\leqslant i\leqslant s+1)$ in the disc $|\omega-\alpha_{1i}|_p\ll H^{-\sigma_1}$ and the previous estimates, we obtain

$$|R_i(\omega)|_p\ll H^{-(n+1-\theta)-0,8\{\theta\}-2\varepsilon_1}=H^{-\tau}\quad(1\leqslant i\leqslant s) \tag{40}$$

for any $\omega\in D$. There are the following three cases:

1) Suppose that for each i $(1\leqslant i\leqslant s)$, $R_i(x)=b_iR(x)$ with $b_i\in\mathbb{Z}$. Since the R_i are all different so are the b_i. Let $b=\max\limits_{1\leqslant i\leqslant s}|b_i|=|b_1|$, so that $b>s/2$. As $bH(R)\leqslant 2H$, $s\gg H^{\beta-d}=H^{0,2\{\theta\}-0,1\varepsilon}$ and $\{\theta\}\geqslant\varepsilon$, we get

$$H(R)\ll H^{1-0,2\{\theta\}+0,1\varepsilon}\quad\text{and}\quad b\gg H^{0,2\{\theta\}-0,1\varepsilon}. \tag{41}$$

Using (40) and $H(R_1)=bH(R)$ we have

$$|R_1(\omega)|_p=|b|_p|R(\omega)|_p\ll H(R_1)^{-\tau}=H(R)^{-\tau}b^{-\tau}$$

and

$$|R(\omega)|_p\ll H(R)^{-\tau}|b|^{-\tau}|b|_p^{-1}\leqslant H(R)^{-\tau}b^{-\tau+1}.$$

From this and (41) we find

$$|R(\omega)|_p\ll H(R)^{-\lambda}, \tag{42}$$

where

$$\lambda = \tau + (\tau - 1)(0,2\{\theta\} - 0,1\varepsilon)(1 - 0,2\{\theta\} + 0,1\varepsilon)^{-1}.$$

By the definition of the τ in (40), the condition $\{\theta\} \geqslant \varepsilon$, (38) and (39) we get $\lambda > n - [\theta] + 1 \geqslant \deg R + 1$. It follows from (42) that

$$|R(\omega)|_p \ll H(R)^{-\deg R - 1 - \delta'}$$

for all $\omega \in D$, where $\delta' > 0$. By Sprindžuk's theorem [Spr69, p. 112], the set of ω for which there are infinitely many polynomials R satisfying the previous inequality has zero measure.

2) Suppose that some of polynomials R_i are reducible. By (38) we have (40) with $\tau \geqslant \deg R_i + \delta$ where $\delta = 1 - 0,2\{\theta\} + \varepsilon_1 > 0$. Then Lemma 2 shows that the set of ω for which there are infinitely many such polynomials has zero measure.
3) Suppose that all polynomials R_i are irreducible and that at least two are relatively prime (otherwise use case 1). Then Lemma 6 can be used on two of polynomials, R_1 and R_2, say. We have $\deg R_i \leqslant n - [\theta]$ $(i = 1, 2)$. It follows from (40), (34) and (36) that

$$\tau = n + 1 - \theta + 0,8\{\theta\} + 2\varepsilon_1 = r_1 + l_2/T + 0,8\{\theta\} + 2\varepsilon_1,$$
$$\tau - \sigma_1 = r_1 - (m-1)\varepsilon_1 = (l_2 + \ldots + l_m)/T - (m-1)/T \geqslant T^{-1} > 0,$$
$$\tau + 2(\tau - \sigma_1) = 3r_1 + l_2/T + 0,8\{\theta\} - 2(m-2)\varepsilon_1,$$
$$2(n - [\theta]) + \delta = -2 + 2r_1 + 2l_2/T + 2\{\theta\} + \delta.$$

As $r_1 \geqslant l_2/T$ then $\tau + 2(\tau - \sigma_1) > 2(n - [\theta]) + \delta$ if $0 < \delta < \varepsilon$. The last inequality contradicts Lemma 6.

In the case of $\{\theta\} < \varepsilon$ we set

$$\beta = [\theta] - 1 + \varepsilon, \quad \sigma_1 = l_2/T + \{\theta\} + (m+1)\varepsilon_1 - (1,5 + \varepsilon')\varepsilon, \quad \varepsilon' = \varepsilon/(9n+2) \quad d = [\theta] - 1$$

and apply the same arguments as above. □

Proposition 7. *If*

$$\varepsilon \leqslant r_1 + l_2/T \leqslant 2 - \varepsilon/2 \tag{43}$$

then $\mu(A(\bar{l}, \xi)) = 0$.

Proof. All polynomials $P(\omega) = H\omega^n + a_{n-1}\omega^{n-1} + ... + a_1 x + a_0 \in \mathcal{P}_n(H, \bar{l})$ corresponding to the same vector $\overline{a} = (a_{n-1}, \ldots, a_2)$ are grouped together into a class $\mathcal{P}_n(H, \bar{l}, \overline{a})$. Let

$$B(P) = \{\omega \in K_0 \cap S(\alpha_1) : |\omega - \alpha_1|_p \leqslant H^{-n-1}|P'(\alpha_1)|_p^{-1}\},$$
$$B_1(P) = \{\omega \in K_0 \cap S(\alpha_1) : |\omega - \alpha_1|_p \leqslant H^{-2+\varepsilon'}|P'(\alpha_1)|_p^{-1}\},$$

where $\varepsilon' = \varepsilon/6$. It is clear that $B(P) \subset B_1(P)$,

$$\mu B(P) = c_1(p) H^{-n-1}|P'(\alpha_1)|_p^{-1}, \qquad \mu B_1(P) = c_2(p) H^{-2+\varepsilon'}|P'(\alpha_1)|_p^{-1}$$

and

$$\mu B(P) = c_3(p) H^{-n+1-\varepsilon'} \mu B_1(P), \tag{44}$$

where $c_i(p) > 0 \quad (i = 1, 2, 3)$ are the constants dependent on p. Now we estimate $|P(\omega)|_p$ when $P \in \mathcal{P}_n(H, \bar{l}, \overline{a})$ and $\omega \in B_1(P)$. It follows from the definition of $B_1(P)$ that

$|P'(\alpha_1)(\omega-\alpha_1)|_p < H^{-2+\varepsilon'}$. By the right-hand side of (43) and the definition of the r_j $(2\leqslant j\leqslant m)$ we have

$$jr_1 - r_j = (j-1)r_1 + r_1 - r_j = (j-1)r_1 + (l_2+\ldots+l_j)/T \leqslant (j-1)(2-\varepsilon/2).$$

From this, Lemma 5 and the definition of $B_1(P)$ we find

$$|P^{(j)}(\alpha_1)(\omega-\alpha_1)^j|_p < H^{-r_j+(m-j)\varepsilon_1}H^{-(2-\varepsilon')j+jr_1} \leqslant H^{-(2-\varepsilon')j+(j-1)(2-\varepsilon/2)+(m-j)/\varepsilon_1} =$$

$$= H^{-2-(j-1)\varepsilon/2+(m-j)\varepsilon_1+\varepsilon' j} \leqslant H^{-2-\delta}$$

for $2\leqslant j\leqslant m$, where $\delta>0$ if $\varepsilon_1\leqslant \varepsilon/(2n)$. By the right-hand side of (43) and the definition of the r_1 we have $r_1 < (2-\varepsilon/2)(1-1/j)$. From this, Lemma 5 and the definition of $B_1(P)$ we find

$$|P^{(j)}(\alpha_1)(\omega-\alpha_1)^j|_p \ll |\omega-\alpha_1|_p^j < H^{-j(2-\varepsilon'-r_1)} < H^{-2-\varepsilon/3}$$

for $m<j\leqslant n$. By Taylor's formula and the previous estimates we get

$$|P(\omega)|_p \ll H^{-2+\varepsilon'} \tag{45}$$

for any $\omega\in B_1(P)$. Further we use essential and inessential domains introduced by Sprindžuk [Spr69]. The disc $B_1(P)$ is called *inessential* if there exists a polynomial $Q\in \mathcal{P}_n(H,\bar{l},\bar{a})$ such that $\mu(B_1(P)\cap B_1(Q)) \geqslant \frac{1}{2}\mu B_1(P)$ and *essential* otherwise.

Let the disc $B_1(P)$ be inessential and $D=B_1(P)\cap B_1(Q)$. Then

$$\mu D \geqslant \tfrac{1}{2}\mu B_1(P) = c_4(p)H^{-2+\varepsilon'}|P'(\alpha_1)|_p^{-1}$$

where $c_4(p)>0$ is a constant dependent on p. By (45) the difference $R(\omega)=P(\omega)-Q(\omega)=b_1\omega+b_0$, where $\max(|b_0|,|b_1|)\leqslant 2H$, satisfies

$$|R(\omega)|_p = |b_1\omega - b_0|_p \ll H^{-2+\varepsilon'} \tag{46}$$

for any $\omega\in B_1(P)$. Note that $b_1\neq 0$ since if $b_1=0$, then $|b_0|_p \ll H^{-2+\varepsilon'}$. It is contradicted to $|b_0|_p\geqslant |b_0|^{-1}\gg H^{-1}$. It follows from (46) that

$$|\omega - b_0/b_1|_p \ll H^{-2+\varepsilon'}|b_1|_p^{-1}. \tag{47}$$

Let $D_1=\{\omega\in K_0\cap S(\alpha_1) : \text{the inequality (47) holds}\}$. Then $D\subseteq D_1$ and $\mu D_1 = c_5(p)H^{-2+\varepsilon'}|b_1|_p^{-1}$, where $c_5(p)>0$ is a constant dependent on p. We have

$$c_4(p)H^{-2+\varepsilon'}|P'(\alpha_1)|_p^{-1} \leqslant \mu D \leqslant \mu D_1 \ll H^{-2+\varepsilon'}|b_1|_p^{-1}.$$

Hence

$$|b_1|_p \ll |P'(\alpha_1)|_p. \tag{48}$$

From (48) and Lemma 5 we get

$$|b_1|_p \ll |P'(\alpha_1)|_p \ll H^{-r_1+(m-1)\varepsilon_1}.$$

Since $r_1\geqslant l_2/T$ the left-hand side of (43) implies $r_1\geqslant \varepsilon/2$. Now we find $|b_1|_p \ll H^{-\varepsilon/3}$ for $\varepsilon_1\leqslant\varepsilon/(2n)$. It follows from (46) that $|b_0|_p\ll H^{-\varepsilon/3}$. Suppose that s is defined by the inequalities $p^s\leqslant H < p^{s+1}$. We have $H^{\varepsilon/3}\asymp p^{[s\varepsilon/3]}$ for sufficiently large H. Hence $b_1\asymp p^{[s\varepsilon/3]}b_{11}$ and $b_0\asymp p^{[s\varepsilon/3]}b_{01}$ where b_{11}, b_{01} are integers. We have

$$b_1\omega + b_0 \asymp p^{[s\varepsilon/3]}(b_{11}\omega + b_{01}) \qquad \text{with} \qquad \max(|b_{11}|,|b_{01}|) \ll H^{1-\varepsilon/3}. \tag{49}$$

Let $R_1(\omega) = b_{11}\omega + b_{01}$. Then $H(R_1) \ll H^{1-\varepsilon/3}$. It follows from (46) and (49) that

$$|b_{11}\omega + b_{01}|_p \ll p^{s\varepsilon/3} H^{-2+\varepsilon'} \ll H^{-2+\varepsilon'+\varepsilon/3} = H(R_1)^{-2-\varepsilon/(6-2\varepsilon)}.$$

Using Khintchine's theorem in $\mathbb{Q}_p$ [Spr69, p. 94], we get that the set of ω belonging to infinitely many discs $B_1(P)$ has zero measure.

Let the disc $B_1(P)$ be essential. By the property of p-adic valuation every point $\omega \in K_0$ belong to no more than one essential disc. Hence

$$\sum_{P \in \mathcal{P}(H,\bar{l},\bar{a})} \mu B_1(P) \leqslant p^n.$$

It follows from (44) that

$$\sum_H \sum_{P \in \mathcal{P}(H,\bar{l})} \mu B(P) = \sum_H \sum_{\bar{a}} \sum_{P \in \mathcal{P}(H,\bar{l},\bar{a})} \mu B(P) \ll$$

$$\ll \sum_H H^{n-2} \sum_{P \in \mathcal{P}(H,\bar{l},\bar{a})} H^{-n+1-\varepsilon'} \mu B_1(P) \ll \sum_H H^{-1-\varepsilon'} < \infty.$$

The Borel-Cantelli lemma completes the proof. □

8. Proof of Proposition 4

First of all we impose some reasonable limitation on the disc K_0 that appear in the statement of Proposition 4. To this end we notice the following two facts.

Remark 1. Let $\omega_0, \theta_0 \in \mathbb{Q}_p$. It is a simple matter to verify that if (Γ, N) is a regular system in a disc K_0 then $(\tilde{\Gamma}, \tilde{N})$ is regular in $\theta_0 K_0 + \omega_0$, where $\tilde{\Gamma} = \{\delta_0\gamma + \omega_0 : \gamma \in \Gamma\}$, $\tilde{N}(\delta_0\gamma + \omega_0) = N(\gamma)$ and $\theta_0 K_0 + \omega_0 = \{\theta_0\omega + \omega_0 : \omega \in K_0\}$.

Remark 2. One more observation is that if $c > 0$ is a constant and (Γ, N) is a regular system in a disc K_0 then (Γ, cN) is also a regular system in K_0.

The proofs are easy and left as exercises. Now we notice that for any disc K_0 in $\mathbb{Q}_p$ we can choose two numbers $\omega_0, \theta_0 \in \mathbb{Q}$ such that $\theta_0\mathbb{Z}_p + \omega_0 = K_0$. It is clear that the map $\omega \mapsto \theta_0\omega + \omega_0$ sends $\mathbb{A}_{p,n}$ to itself. Moreover, there is a constant $c_1 > 0$ such that for any $\alpha \in \mathbb{Z}_p \cap \mathbb{A}_{p,n}$ one has $H(\theta_0\alpha + \omega_0) \leqslant c_1 H(\alpha)$. Hence, if we will succeed to prove Proposition 4 for the disc $\mathbb{Z}_p$ then in view of the Remarks above it will be proved for K_0. Thus without loss of generality we assume that $K_0 = \mathbb{Z}_p$.

In the proof of Proposition 4 we will refer to the following statement known as Hensel's Lemma (see [BD99, p. 134]).

Lemma 8. *Let P be a polynomial with coefficients in $\mathbb{Z}_p$, let $\xi = \xi_0 \in \mathbb{Z}_p$ and $|P(\xi)|_p < |P'(\xi)|_p^2$. Then as $n \to \infty$ the sequence*

$$\xi_{n+1} = \xi_n - \frac{P(\xi_n)}{P'(\xi_n)}$$

tends to some root $\alpha \in \mathbb{Z}_p$ of the polynomial P and

$$|\alpha - \xi|_p \leqslant \frac{|P(\xi)|_p}{|P'(\xi)|_p^2} < 1.$$

Proposition 8. *Let $\delta > 0$, $Q \in \mathbb{R}_{>1}$. Given a disc $K \subset \mathbb{Z}_p$, let*

$$E(\delta, Q, K) = \bigcup_{P\in\mathbb{Z}[x],\ \deg P\leqslant n,\ H(P)\leqslant Q} \{\omega \in K : |P(\omega)|_p < \delta Q^{-n-1}\}. \tag{50}$$

Then there is a positive constant c such that for any finite disc $K \subset \mathbb{Z}_p$ there is a sufficiently large number Q_0 such that $\mu(E(\delta, Q, K)) \leqslant c\delta\mu(K)$ for all $Q \geqslant Q_0$.

Proof. The set $E(\delta, Q, K)$ can be expressed as follows

$$E(\delta, Q, K) \subset E_1(\delta, Q, K, 1/3) \bigcup E_3(Q, K) \bigcup E_4(),$$

where $E_1(\delta, Q, K, 1/3)$ is introduced in Proposition 1,

$$E_3(Q, K) = \bigcup_{P\in\mathbb{Z}[x],\ \deg P\leqslant n,\ H(P)\geqslant \log Q} \chi(P),$$

$\chi(P)$ is the set of solutions of (5) lying in K with $\xi = 1/3$ and $C = \delta$,

$$E_4(Q, K) = \bigcup_{P\in\mathbb{Z}[x],\ \deg P\leqslant n,\ H(P)\leqslant \log Q} \{\omega \in K : |P(\omega)|_p < \delta Q^{-n-1}\}.$$

By Proposition 2,

$$\mu(E_3(Q, K)) \to 0 \text{ as } Q \to \infty. \tag{51}$$

By Proposition 1,

$$\mu(E_1(\delta, Q, K, 1/3)) \leqslant c_1\delta\mu(K) \text{ for sufficiently large } Q. \tag{52}$$

Now to estimate $\mu(E_4(Q, K))$ we first estimate the measure of $\{\omega \in K : |P(\omega)|_p < \delta Q^{-n-1}\}$ for a fixed P. If $\alpha_{\omega,P}$ is the nearest root to ω then $|a_n(\omega - \alpha_{\omega,P})^n|_p < Q^{-n-1}$. Since $|a_n|_p \geqslant Q^{-1}$, we get $|\omega - \alpha_{\omega,P}|_p < Q^{-1}$. It follows that

$$\mu\{\omega \in K : |P(\omega)|_p < \delta Q^{-n-1}\} \ll Q^{-1}.$$

Hence $\mu(\mu(E_4(Q, K))) \ll (\log Q)^{n+1} Q^{-1} \to 0$ as $Q \to \infty$. Combining this with (51) and (52) completes the proof. □

Proof of Proposition 4. Fix any disc $K \subset \mathbb{Z}_p$ and let $Q > 0$ be a sufficiently large number. Let $\omega \in K$. Consider the system

$$\begin{cases} |P(\omega)|_p < \delta^2 C Q^{-n-1}, & P(\omega) = a_n\omega^n + \cdots + a_1\omega + a_0, \\ |a_j| \leqslant \delta^{-1} Q, & j = \overline{0, n}, \\ |a_j|_p \leqslant \delta, & j = \overline{2, n}. \end{cases} \tag{53}$$

By Dirichlet's principle, it easy to show that there is an absolute constant $C > 0$ such that for any $\omega \in K$ the system (53) has a non zero solution $P \in \mathbb{Z}[x]$. Fix such a solution P.

If $|P'(\omega)| < \delta$, then, by (53),

$$|a_1|_p = |P'(\omega) - \textstyle\sum_{k=2}^n k a_k \omega^{k-1}|_p \leqslant \max\{|P'(\omega)|_p, |2a_2\omega^1|_p, \ldots, |na_n\omega^{n-1}|_p\} < \delta.$$

Also, if Q is sufficiently large, then

$$|a_0|_p = |P(\omega) - \textstyle\sum_{k=1}^n a_k \omega^k|_p \leqslant \max\{|P(\omega)|_p, |a_1\omega^1|_p, \ldots, |a_n\omega^n|_p\} < \delta.$$

Therefore, the coefficients of P have a common multiple d with $\delta/p \leqslant |d|_p < \delta$. It follows that $d^{-1} \leqslant \delta$. Define $P_1 = P/d \in \mathbb{Z}[x]$. Obviously $H(P_1) \leqslant Q$. Also, by (53),

$$|P_1(\omega)|_p = |P(\omega)|_p |d|_p^{-1} \leqslant |P(\omega)|_p \times \delta^{-1} p < \delta C p Q^{-n-1}.$$

This implies $\omega \in E(\delta Cp, Q, K)$. By Proposition 8, $\mu(E(\delta Cp, Q, K)) \leqslant c\delta Cp\mu(K)$ for sufficiently large Q. Put $\delta = (2cpC)^{-1}$. Then $\mu(K \smallsetminus E(\delta Cp, Q, K)) \geqslant \frac{1}{2}\mu(K)$. If now we take $\omega \in K \smallsetminus E(\delta Cp, Q, K)$ then we get

$$|P'(\omega)|_p \geqslant \delta.$$

By Hensel's lemma there is a root $\alpha \in \mathbb{Z}_p$ of P such that $|\omega - \alpha|_p < CQ^{-n-1}$. If Q is sufficiently large then $\alpha \in K$. The height of this α is $\leqslant \delta^{-1}Q$.

Let $\alpha_1, \dots, \alpha_t$ be the maximal collection of algebraic numbers in $K \cap \mathbb{A}_{p,n}$ satisfying $H(\alpha_j) \leqslant \delta^{-1}Q$ and

$$|\alpha_i - \alpha_j|_p \geqslant Q^{-n-1} \quad (1 \leqslant i < j \leqslant t).$$

By the maximality of this collection, $|\omega - \alpha_j|_p < CQ^{-n-1}$ for some j. As ω is arbitrary point of $E(\delta Cp, Q, K)$, we get

$$E(\delta Cp, Q, K) \subset \bigcup_{j=1}^{t} \{\omega \in \mathbb{Z}_p : |\omega - \alpha_j|_p < CQ^{-n-1}\}.$$

Next,

$$\frac{1}{2}\mu(K) \leqslant \mu(E(\delta Cp, Q, K)) \ll Q^{-n-1} t,$$

whence $t \gg Q^{n+1}\mu(K)$. Taking $T = \delta^{-n-1}Q^{n+1}$ one readily verifies the definition of regular systems. The proof is completed.

□

Acknowledgements. The research was supported by Belorussian Fond of Fundamental research (Project 00-249) and by INTAS (project 00-429).

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