THE ROYAL SOCIETY

The Khintchine–Groshev theorem for planar curves

By V. V. Beresnevich[1], V. I. Bernik[1], H. Dickinson[2] and M. M. Dodson[2]

[1]*Institute of Mathematics, Academy of Sciences, Surganova 11, 220072 Minsk, Belarus*
[2]*Department of Mathematics, University of York, Heslington, York YO1 5DD, UK*

Received 2 December 1998; accepted 3 March 1999

The analogue of the classical Khintchine–Groshev theorem, a fundamental result in metric Diophantine approximation, is established for smooth planar curves with non-vanishing curvature almost everywhere.

Keywords: metric Diophantine approximation; linear forms; regular systems

1. Introduction

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function. The Khintchine–Groshev theorem (see theorem 12 in Sprindžuk (1979) for details but note that the notation here differs slightly) in the plane asserts that the set of points $\boldsymbol{x} \in \mathbb{R}^2$, which obey the inequality

$$|\boldsymbol{q} \cdot \boldsymbol{x} + q_0| \leqslant \psi(H(\boldsymbol{q}))H(\boldsymbol{q})^{-1}$$

for infinitely many vectors $\boldsymbol{q} = (q_0, q_1, q_2) \in \mathbb{Z}^3$, has zero or full Lebesgue measure according to whether the sum $\sum_{r=1}^{\infty} \psi(r)$ converges or diverges, respectively ($H(\boldsymbol{q}) = \max\{|q_0|, |q_1|, |q_2|\}$, the height of $\boldsymbol{q}$). In this paper the analogue of this theorem is established for smooth planar curves with non-zero curvature almost everywhere.

Theorem 1.1. *Let $I \subset \mathbb{R}$ be an interval and suppose that the functions $f_1, f_2 : I \to \mathbb{R}$ are C^3 and satisfy $f_1'(x)f_2''(x) - f_1''(x)f_2'(x) \neq 0$ for almost all $x \in I$. Then, for almost all $x \in I$ the inequality*

$$|q_2 f_2(x) + q_1 f_1(x) + q_0| < \psi(H(\boldsymbol{q}))H(\boldsymbol{q})^{-1} \tag{1.1}$$

holds for infinitely many or only finitely many integer vectors $\boldsymbol{q}$ according to whether the sum

$$\sum_{r=1}^{\infty} \psi(r) \tag{1.2}$$

diverges or converges, respectively.

Schmidt's theorem on the extremality of planar curves (Schmidt 1964) corresponds to $\psi(r) = r^{1-v}$ with $v > 2$ and is clearly a special case of the above result. The case of convergence was proved in Bernik *et al.* (1998), which we also refer to for historical details. The complementary case of divergence is now proved.

Throughout this article the Lebesgue measure of a measurable set E will be denoted by $|E|$. Since the curvature vanishes only on a set of measure zero we take I, without loss of generality, to be a sufficiently small closed interval with $|I| \leqslant 1$ on which the curvature does not vanish. By the implicit function theorem we can, again without loss of generality, take the curve $\{(f_2(x), f_1(x)) : x \in I\}$ to be of the form $\{(f(x), x) : x \in I\}$. Thus instead of the linear form $q_2 f_2(x) + q_1 f_1(x) + q_0$ we consider

$$F(x) = q_0 + q_1 x + q_2 f(x),$$

where $(q_0, q_1, q_2) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ and $f : I \to \mathbb{R}$ is a smooth function with non-zero second derivative everywhere. We write $H(F) = \max\{|q_2|, |q_1|, |q_0|\}$. Thus it suffices to prove that

$$|F(x)| < \psi(H(F))H(F)^{-1}$$

for infinitely many F for almost all x when the sum (1.2) diverges.

Since I is a closed interval, the constant

$$M = \max_{0 \leqslant i \leqslant 3} \sup_{x \in I} |f^{(i)}(x)| + 1 \tag{1.3}$$

is finite. Also $|f''(x)| \geqslant c > 0$ for all $x \in I$. Let

$$\mathcal{F} = \{F = q_2 f(x) + q_1 x + q_0 : q_2, q_1, q_0 \in \mathbb{Z},\ \boldsymbol{q} \neq \mathbf{0}\}$$

be the family of non-zero F and let

$$\Gamma = \{\gamma \in I : \text{there exists } F \in \mathcal{F},\ F(\gamma) = 0\}. \tag{1.4}$$

If $q_2 \neq 0$, then $F''(x) = q_2 f''(x) \neq 0$, and it follows that F has at most two roots in I and hence that the set Γ is countable. For each $\gamma \in \Gamma$, define the height $h(\gamma)$ of γ to be the positive integer

$$h(\gamma) = \min\{H(F) : F \in \mathcal{F} \text{ with } F(\gamma) = 0\}.$$

The proof of theorem 1.1 is based on the following result, which deals with the approximation of real numbers by elements of Γ.

Theorem 1.2. *For almost all $x \in \mathbb{R}$ the inequality*

$$|x - \gamma| < h(\gamma)^{-2}\psi(h(\gamma)) \tag{1.5}$$

has infinitely many or only finitely many solutions $\gamma \in \Gamma$ according to whether the sum (1.2) diverges or converges, respectively.

Using the Borel–Cantelli lemma it is not difficult to prove that if $\sum_{h=1}^{\infty} \psi(h) < \infty$ then, for almost all $x \in I$, inequality (1.5) has at most finitely many solutions $\gamma \in \Gamma$.

The proof of theorem 1.2 in the case of divergence is based on some facts concerning the distribution of Γ. To investigate this distribution, the concept of regular systems, introduced by Baker & Schmidt (1970) in their study of Hausdorff dimension and Diophantine approximation, is used.

Definition 1.3. Let Ω be a countable set of real numbers and $N : \Omega \to \mathbb{R}^+$ be a function. The pair (Ω, N) is called *a regular system* on an interval I if there exists a constant $C_1 = C_1(\Omega, N, I) > 0$ such that for any finite interval $J \subset I$ there exists a sufficiently large number $T_0 = T_0(\Omega, N, J) > 0$ such that for any $T \geqslant T_0$ there are $\gamma_1, \dots, \gamma_t$ in $\Omega \cap J$ such that

$$N(\gamma_i) \leqslant T \quad (1 \leqslant i \leqslant t), \tag{1.6}$$

$$|\gamma_i - \gamma_j| \geqslant T^{-1} \quad (1 \leqslant i < j \leqslant t), \tag{1.7}$$

$$t \geqslant C_1|J|T. \tag{1.8}$$

In order to establish theorem 1.2 the following refinement of the lower-bound part of theorem 3 in Baker (1978) will be proved.

Theorem 1.4. *Let $N(\gamma) = h(\gamma)^3$ for each $\gamma \in \Gamma$ (defined in (1.4)). Then (Γ, N) is a regular system on I.*

2. Proof of theorem 1.4

We begin with a brief outline of the proof. Let the interval $J = [a, b] \subset I$ and the sufficiently large integer Q be given. The intervals

$$\{x \in J : |x - \gamma| \ll Q^{-3}\},$$

where γ runs over $\Gamma \cap J$ with $h(\gamma) \ll Q$, will be shown to cover a subset $G(J, Q)$ of J having measure $|G(J, Q)| \geqslant \frac{1}{2}|J|$ in order to deduce that (Γ, N) is a regular system. This will be done by finding, for each $x \in G(J, Q)$, a function $F \in \mathcal{F}$ such that $H(F) \ll Q$, $|F(x)| < \varepsilon_Q$ and $|F'(x)| \gg Q$ for some ε_Q satisfying $Q^{-2} \ll \varepsilon_Q \leqslant Q^{-2}$. Indeed, it will be proved (see lemma 2.1) that this function F has a root γ approximating x with error Q^{-3}, where $\ll b$ means $a \leqslant cb$ for some constant $c > 0$.

It will also be shown that if $|F'(x)| \ll Q$, then $H(F) \leqslant Q$; this ensures that the condition $H(F) > Q$ guarantees that $|F'(x)| \gg Q$. In addition, we will restrict ourselves to points $x \in J$, which are at least ε_Q from the boundary of J, so that $\gamma \in \Gamma \cap J$. We define $G(J, Q)$ to be the set of points x in $(a + \varepsilon_Q, b - \varepsilon_Q)$ such that for any $F \in \mathcal{F}$ satisfying $|F(x)| < \varepsilon_Q$ we have $H(F) > Q$. We will choose ε_Q so that the set $B(J, Q) = J \setminus G(J, Q)$ has measure at most $\frac{1}{2}|J|$ for Q sufficiently large. Thus the first step is to show that a suitable ε_Q exists and to obtain an upper bound for $|B(J, Q)|$.

The cases of large and small derivatives are considered separately. From now on let $Q \in \mathbb{N}$, $\varepsilon > 0$ and $J = [a, b]$ be a subinterval of I. Let

$$\mathcal{F}(Q) = \{F \in \mathcal{F} : H(F) \leqslant Q\}$$

and let $B_J(Q, \varepsilon)$ be the set of $x \in J$ for which there exists a function $F \in \mathcal{F}(Q)$ such that

$$|F(x)| < \varepsilon, \qquad |F'(x)| \geqslant 2|J|^{-1}. \tag{2.1}$$

For any $F \in \mathcal{F}(Q)$ define $\sigma(F)$ as the set of all the solutions of (2.1) belonging to J. It is necessary to show that $|B_J(Q, \varepsilon)|$ is relatively small.

Lemma 2.1 shows that if the height $H(F)$ of F exceeds M (given in (1.3)), then within a small interval the derivative of F is bounded away from zero. Recall that without loss of generality $|I| \leqslant 1$.

Lemma 2.1. *Fix $Q > M$ and $0 < \varepsilon < Q^{-2}$. Then for any $F \in \mathcal{F}(Q)$ such that $\sigma(F) \neq \emptyset$, at least one of the following statements is true for any $x_0 \in \sigma(F)$.*

(1) There exists a number $\gamma \in J$ such that $F(\gamma) = 0$ and

$$|F'(\gamma)| \geqslant |F'(x_0)|/2 \geqslant |J|^{-1},$$
$$|x_0 - \gamma| < \frac{2\varepsilon}{|F'(\gamma)|}. \tag{2.2}$$

(2) $\min\{|x_0 - a|, |x_0 - b|\} \leqslant \varepsilon$.

Proof. Fix a function $F \in \mathcal{F}(Q)$ such that $\sigma(F) \neq \emptyset$. Then choose $x_0 \in \sigma(F)$. We may assume without loss of generality that $|x_0 - a| > \varepsilon$ and $|x_0 - b| > \varepsilon$ as otherwise the lemma is true. Then for any x such that $|x - x_0| \leqslant \varepsilon$ we have $x \in J$. By the mean value theorem (MVT), $F'(x) = F'(x_0) + F''(x_1)(x - x_0)$, where x_1 is a point between x and x_0. It is readily verified from (1.3) that

$$|F''(x_1)| = |q_2 f''(x_1)| \leqslant MH(F) \leqslant MQ.$$

Hence $|F''(x_1)(x - x_0)| \leqslant MQ\varepsilon \leqslant MQ^{-1} \leqslant |J|^{-1}$ since $Q > M$ and $|J| \leqslant |I| \leqslant 1$. Since $|F'(x_0)| \geqslant 2|J|^{-1}$, we have

$$|F'(x)| \geqslant |F'(x_0)| - |F''(x_1)(x - x_0)| \geqslant \tfrac{1}{2}|F'(x_0)|. \tag{2.3}$$

Thus, by continuity, F' does not change sign in the interval $[x_0 - \varepsilon, x_0 + \varepsilon]$. Further, by the MVT, for any x satisfying $|x - x_0| \leqslant \varepsilon$ we have $F(x) = F(x_0) + F'(x_2)(x - x_0)$, where $x_2 = x_2(x)$ is a point between x and x_0. Putting $x = x_0 \pm \varepsilon$ gives

$$|F'(x_2)(x - x_0)| \geqslant \tfrac{1}{2}\varepsilon|F'(x_0)| \geqslant \varepsilon.$$

Moreover, one of the values of $F'(x_2)(x - x_0)$ is positive and the other is negative. Since $|F(x_0)| < \varepsilon$, the expression $F(x) = F(x_0) + F'(x_2)(x - x_0)$ has different signs at points $x_0 \pm \varepsilon$. It follows that there exists a point $\gamma \in [x_0 - \varepsilon, x_0 + \varepsilon] \subset J$ such that $F(\gamma) = 0$ and, as we have already proved, $|F'(\gamma)| \geqslant \frac{1}{2}|F'(x_0)| \geqslant |J|^{-1}$. Next, by Taylor's formula,

$$F(x_0) = [F'(\gamma) + \tfrac{1}{2}F''(x_3)(x_0 - \gamma)](x_0 - \gamma). \tag{2.4}$$

Using the same method as for (2.3) above, it can be shown that

$$|F'(\gamma) + \tfrac{1}{2}F''(x_3)(x_0 - \gamma)| \geqslant \tfrac{1}{2}|F'(\gamma)|.$$

Together with (2.4) this gives (2.2) and lemma 2.1 is proved. ∎

Next, an estimate for $|B_J(Q, \varepsilon)|$ is obtained.

Lemma 2.2. *Let $Q > Q_1 = \max\{3, M, |J|^{-1}\}$ and $\varepsilon > 0$. Then*

$$|B_J(Q, \varepsilon)| \leqslant 35\varepsilon Q^2|J|.$$

Proof. First note that if $\varepsilon \geqslant Q^{-2}$ there is nothing to prove; we therefore assume that $\varepsilon < Q^{-2}$. Consider the non-empty interval $J' = [a + \varepsilon, b - \varepsilon]$. Given $F \in \mathcal{F}(Q)$, define $\sigma'(F) = \sigma(F) \cap J'$ and $\sigma''(F) = \sigma(F) \setminus \sigma'(F)$. Since $\sigma'' \subset ([a, a + \varepsilon] \cup [b - \varepsilon, b])$ it is readily verified that

$$\left| \bigcup_{F \in \mathcal{F}(Q)} \sigma''(F) \right| \leqslant 2\varepsilon. \tag{2.5}$$

Now we proceed to estimate the measure of the union of $\sigma'(F)$ over $\mathcal{F}(Q)$. Fix q_1 and q_2 not both zero and such that $|q_1|, |q_2| \leqslant Q$ and consider $R(x) = q_2 f(x) + q_1 x$. There exists a cover of J consisting of two intervals $[w_{i-1}, w_i]$, $i = 1, 2$ such that R' is monotonic (it has at most one turning point) and of constant sign in each one, one of which could be just one point. For any function $F(x) = R(x) + q_0 \in \mathcal{F}(Q)$, define the sets

$$Z_i(F) = \{\gamma \in [w_{i-1}, w_i] : F(\gamma) = 0,\ |F'(\gamma)| \geqslant |J|^{-1}\}, \quad i = 1, 2,$$

with $Z(F) = Z_1(F) \cup Z_2(F)$ and

$$\hat{Z}_i(R) = \bigcup_{\substack{F = R + q_0 \\ |q_0| \leqslant Q}} Z_i(F), \quad i = 1, 2$$

with $\hat{Z}(R) = \hat{Z}_1(R) \cup \hat{Z}_2(R)$. Finally, let $\sigma(\gamma, F)$ denote the set

$$\left\{ x \in J : |x - \gamma| < \frac{2\varepsilon}{|F'(\gamma)|} \right\}.$$

For any $F \in \mathcal{F}(Q)$, lemma 2.1 implies that

$$\sigma'(F) \subset \bigcup_{\gamma \in Z(F)} \sigma(\gamma, F).$$

Since the derivatives of $F = R + q_0$ and R coincide, $\sigma(\gamma, R) = \sigma(\gamma, F)$. Ordering the elements in the sets $\hat{Z}_i(R)$, $i = 1, 2$ as follows,

$$\hat{Z}_i(R) = \{\gamma_i^{(1)}, \dots, \gamma_i^{(k_i)}\},$$

we have

$$\begin{aligned} \left| \bigcup_{F = R + q_0 \in \mathcal{F}(Q)} \sigma'(F) \right| &\leqslant \left| \bigcup_{F = R + q_0 \in \mathcal{F}(Q)} \bigcup_{\gamma \in Z(F)} \sigma(\gamma, F) \right| \\ &\leqslant \sum_{\gamma \in \hat{Z}(R)} |\sigma(\gamma, R)| \leqslant \sum_{i=1}^{2} \sum_{j=1}^{k_i} |\sigma(\gamma_i^{(j)}, R)| \\ &\leqslant \sum_{i=1}^{2} \sum_{j=1}^{k_i} \frac{4\varepsilon}{|R'(\gamma_i^{(j)})|}. \end{aligned} \tag{2.6}$$

Choose i such that $k_i > 1$, and consider two sequential roots $\gamma_i^{(j)}$ and $\gamma_i^{(j+1)}$ of $R + q_0^{i,j}$ and $R + q_0^{i,j+1}$ say, respectively. Without loss of generality assume that R' is positive and increasing on (w_{i-1}, w_i). Using the MVT and the monotonicity of R' we find that

$$\begin{aligned} 1 \leqslant |q_0^{i,j} - q_0^{i,j+1}| &= |R(\gamma_i^{(j)}) - R(\gamma_i^{(j+1)})| \\ &= |R'(\tilde{\gamma}_i^{(j)})||\gamma_i^{(j)} - \gamma_i^{(j+1)}| \leqslant |R'(\gamma_i^{(j+1)})||\gamma_i^{(j)} - \gamma_i^{(j+1)}|, \end{aligned}$$

where $\tilde{\gamma}_i^{(j)}$ is a point between $\gamma_i^{(j)}$ and $\gamma_i^{(j+1)}$. It follows that

$$\frac{1}{|R'(\gamma_i^{(j+1)})|} \leqslant \gamma_i^{(j+1)} - \gamma_i^{(j)}, \quad j = 1, \dots, k_i - 1.$$

Summing this over all $j = 1, \dots, k_i - 1$ gives

$$\sum_{j=1}^{k_i-1} \frac{1}{|R'(\gamma_i^{(j+1)})|} \leqslant \sum_{j=1}^{k_i-1} (\gamma_i^{(j+1)} - \gamma_i^{(j)}) = \gamma_i^{(k_i)} - \gamma_i^{(1)} \leqslant w_i - w_{i-1},$$

further implying that

$$\sum_{j=1}^{k_i} \frac{1}{|R'(\gamma_i^{(j)})|} \leqslant w_i - w_{i-1} + \frac{1}{|R'(\gamma_i^{(1)})|} \leqslant w_i - w_{i-1} + |J|.$$

Summing the last inequality over all i gives

$$\sum_{i=1}^{2}\sum_{j=1}^{k_i} \frac{1}{|R'(\gamma_i^{(j)})|} \leqslant \sum_{i=1}^{2}(w_i - w_{i-1} + |J|) = w_2 - w_0 + 2|J| \leqslant 3|J|$$

and hence, by (2.6),

$$\left| \bigcup_{F \in \mathcal{F}(Q)} \sigma'(F) \right| \leqslant \sum_{q_2=0}^{Q} \sum_{q_1=-Q}^{Q} 12\varepsilon|J| = 12\varepsilon|J|(Q+1)(2Q+1).$$

The last estimate together with (2.5) gives the required result and completes the proof. ∎

Let $\varepsilon_Q = \frac{1}{280}Q^{-2}$ and $B_1(J,Q) = B_J(Q,\varepsilon_Q)$. Then by lemma 2.2 $|B_1(J,Q)| \leqslant \frac{1}{8}|J|$ when $Q > Q_1$ for some Q_1 sufficiently large.

Now we turn to the case of small derivatives. Consider the set of $x \in J$ such that

$$|F(x)| < \varepsilon_Q, \qquad |F'(x)| < 2|J|^{-1} \tag{2.7}$$

for some F in $\mathcal{F}$. This set will be divided into two, the first for which $H(F)$ is large and the second for which $H(F)$ is small; both will be shown to have small measure. The following lemma will be needed.

Lemma 2.3. *Let J be a finite interval. For almost all $x \in J$ the system*

$$|F(x)| < H(F)^{-2}, \qquad |F'(x)| < 2|J|^{-1} \tag{2.8}$$

has at most finitely many solutions $F \in \mathcal{F}$.

This lemma follows from a result in Beresnevich (1996) but can also be proved by using the following.

Lemma 2.4. *Given $\delta > 0$, for almost all x the system*

$$|F(x)| < H(F)^{-1-\delta}, \qquad |F'(x)| < H(F)^{-\delta}$$

has at most finitely many solutions $F \in \mathcal{F}$.

Lemma 2.4 is proved in Beresnevich & Bernik (1996). In addition lemma 2.3 can be obtained from lemma 2.4 by adapting the argument in § 2 of Beresnevich & Bernik (1996).

For any function $F \in \mathcal{F}$, denote the set of points $x \in J$ that satisfies the system of inequalities (2.8) by $\tau_2(F)$ and let

$$B_2(J, Q') = \bigcup_{\substack{F \in \mathcal{F} \\ H(F) \geqslant Q'}} \tau_2(F).$$

By lemma 2.3, $|B_2(J, Q')| \to 0$ as $Q' \to \infty$. Hence there exists a number Q_2 such that $|B_2(J, Q_2)| \leqslant \frac{1}{8}|J|$. It is straightforward to verify that if x satisfies system (2.7) for some $F \in \mathcal{F}(Q)$ with $H(F) \geqslant Q_2$, then $x \in B_2(J, Q_2)$ for any $Q > Q_2$. This leaves the case $H(F) \leqslant Q_2$. Let $\tau_3(F, Q) = \{x \in J : |F(x)| < \varepsilon_Q\}$. It is easy to see that $|\tau_3(F, Q)| \to 0$ as $Q \to \infty$. Then it follows that $|B_3(J, Q)| \to 0$ as $Q \to \infty$, where

$$B_3(J, Q) = \bigcup_{F \in \mathcal{F}(Q_2)} \tau_3(F, Q).$$

Thus, there exists Q_3 such that for any $Q \geqslant Q_3$ we have $|B_3(J, Q)| \leqslant \frac{1}{8}|J|$.

Define the set $B(J, Q) = B_1(J, Q) \cup B_2(J, Q_2) \cup B_3(J, Q) \cup [a, a + \varepsilon_Q] \cup [b - \varepsilon_Q, b]$. Then from above

$$|B(J, Q)| \leqslant \tfrac{1}{2}|J|$$

for Q sufficiently large.

Define the constant $L = \max\{M, \sup_{x \in J} |x|\} > 1$ (by the definition of M) and fix a point x in $J \setminus B(J, Q)$. Consider the system

$$|q_2 f(x) + q_1 x + q_0| \leqslant \varepsilon_Q, \qquad |q_2 f'(x) + q_1| \leqslant 840 L^2 Q, \qquad |q_2| \leqslant \frac{1}{3L^2} Q. \tag{2.9}$$

By Minkowski's linear-forms theorem, there exists a non-zero integer solution (q_0, q_1, q_2) of the system (2.9). From now on we assume that $F(x) = q_2 f(x) + q_1 x + q_0$ where (q_0, q_1, q_2) is the solution of (2.9). By working backwards in (2.9) starting with the third inequality it can be readily verified that the system (2.9) implies that $H(F) \leqslant 841 L^3 Q$. If

$$|q_2 f'(x) + q_1| \leqslant \frac{1}{3L^2} Q,$$

then, by (2.9), $H(F) \leqslant Q$. In this situation the point x would belong to $B(J, Q)$ contradicting $x \in J \setminus B(J, Q)$. Hence

$$|F'(x)| \geqslant \frac{1}{3L^2} Q.$$

From now on Q will be assumed to be sufficiently large. By lemma 2.1 there exists a root $\gamma \in J$ of the function F such that

$$|x - \gamma| \leqslant \tfrac{4}{280} Q^{-2} \left(\frac{1}{3L^2} Q \right)^{-1} < \tfrac{1}{20} L^2 Q^{-3}.$$

Therefore, by definition, $h(\gamma) \leqslant H(F) \leqslant 841 L^3 Q$. Thus, for any $x \in J \setminus B(J, Q)$ there exists $\gamma \in \Gamma \cap J$ such that $h(\gamma) \leqslant 841 L^3 Q$ and $|x - \gamma| < \frac{1}{20} L^2 Q^{-3}$.

Fix a maximal collection $\hat{\Gamma} = \hat{\Gamma}(J, Q) = \{\gamma_1, \dots, \gamma_t\} \subset \Gamma \cap J$ satisfying the following conditions:

$$h(\gamma_i) \leqslant 841L^3Q \quad \text{and} \quad |\gamma_i - \gamma_j| \geqslant \tfrac{1}{20}L^2Q^{-3} \quad \text{for } i \neq j.$$

Then for any $\gamma \in \Gamma \cap J$ such that $h(\gamma) \leqslant 841L^3Q$ there exists γ_i in $\hat{\Gamma}$ such that

$$|\gamma - \gamma_i| \leqslant \tfrac{1}{20}L^2Q^{-3}.$$

Hence for any $x \in J \setminus B(J, Q)$ there exists $\gamma_i \in \hat{\Gamma}$ such that

$$|x - \gamma_i| \leqslant \tfrac{1}{10}L^2Q^{-3}.$$

The set $J \setminus B(J, Q)$ is covered by the union of the intervals

$$K_i = \{x \in J : |x - \gamma_i| \leqslant \tfrac{1}{10}L^2Q^{-3}\} \quad \text{for } \gamma_i \in \hat{\Gamma},$$

with $|K_i| \leqslant \frac{1}{5}L^2Q^{-3}$ and $i = 1, \dots, t$. Thus,

$$\tfrac{1}{2}|J| \leqslant |J \setminus B(J, Q)| \leqslant \tfrac{1}{5}tL^2Q^{-3}$$

so that $t \geqslant 2L^{-2}Q^3|J|$. Taking $T = (841QL^3)^3$ gives (1.6)–(1.8) and completes the proof of the theorem.

3. Proof of theorem 1.2

For any $\gamma \in \Gamma$ define

$$\sigma(\gamma) = \{x \in I : |x - \gamma| < h(\gamma)^{-2}\psi(h(\gamma))\}.$$

Let $\Gamma(\psi)$ denote the set of $x \in \mathbb{R}$, which belongs to infinitely many intervals $\sigma(\gamma)$. Our aim is to prove that if $\sum_{h=1}^{\infty}\psi(h) = \infty$ then $\Gamma(\psi)$ has full measure. Without loss of generality we can assume that

$$\psi(h) \leqslant \tfrac{1}{2}h^{-1} \quad \text{for all } h. \tag{3.1}$$

For each k let $\varphi(k) = 2^k\psi(2^k)$. The monotonicity and divergence of ψ imply that

$$\sum_{k=1}^{\infty} \varphi(k) = \infty. \tag{3.2}$$

The following two lemmas will be needed. The first follows from the Lebesgue density theorem and the second is lemma 5 in Sprindžuk (1979, ch. 1). They can also be found in Harman (1998) as lemmas 1.6 and 2.3, respectively.

Lemma 3.1. *Let $A \subset I$ be a measurable set. If there exists a positive constant $C_2 < 1$ such that for any interval $J \subset I$ the inequality $|A \cap J| \geqslant C_2|J|$ holds, then the set A has full measure.*

Lemma 3.2. *Let $E_i \subset I$ be a sequence of measurable sets and let E be the set of points x belonging to infinitely many E_i. If the sum $\sum_{i=1}^{\infty}|E_i|$ diverges, then*

$$|E| \geqslant \limsup_{N \to \infty} \frac{\left(\displaystyle\sum_{i=1}^{N} |E_i|\right)^2}{\displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N} |E_i \cap E_j|}.$$

Fix any interval $J \subset I$. By theorem 1.4, there exist positive constants C_1 and $k_0 = k_0(J)$ such that for any $k \geqslant k_0$ there exists a collection

$$\Gamma_k(J) = \{\gamma_1 < \cdots < \gamma_{t_k}\} \subset \Gamma \cap J$$

satisfying the following conditions (taking $T = 2^{3k}$ and $N(\gamma) = h(\gamma)^3$):

$$h(\gamma) \leqslant 2^k \quad \text{for all } \gamma \in \Gamma_k(J), \tag{3.3}$$

$$|\gamma - \beta| \geqslant 2^{-3k} \quad \text{for any numbers } \beta, \gamma \in \Gamma_k(J) \quad \text{with } \gamma \neq \beta, \tag{3.4}$$

$$C_1 2^{3k}|J| \leqslant t_k \leqslant 2^{3k}|J|. \tag{3.5}$$

Moreover, $\Gamma_k(J)$ can be chosen so that the distance between any $\gamma \in \Gamma_k(J)$ and the boundary of J is more than 2^{-3k}. From now on, unless otherwise stated, $\gamma \in \Gamma_k(J)$. Let

$$E_k = \bigcup_{\gamma \in \Gamma_k(J)} \{x \in I : |x - \gamma| < 2^{-2k}\psi(2^k)\} = \bigcup_{\gamma \in \Gamma_k(J)} E_k(\gamma),$$

say, and consider the set $E(J) = \bigcap_{N=k_0}^{\infty} \bigcup_{k=N}^{\infty} E_k$. The monotonicity of ψ together with (3.3) implies that $E_k(\gamma) \subset \sigma(\gamma)$. It follows that $E(J) \subset \Gamma(\psi) \cap J$, whence

$$|\Gamma(\psi) \cap J| \geqslant |E(J)|. \tag{3.6}$$

It is readily verified that

$$|E_k(\gamma)| = 2 \cdot 2^{-2k}\psi(2^k) = 2 \cdot 2^{-3k}\varphi(k). \tag{3.7}$$

By (3.1) and (3.4), the intersection $E_k(\gamma) \cap E_k(\beta)$ is empty if $\gamma \neq \beta$. Thus, $|E_k| = t_k|E_k(\gamma)|$ and hence, by (3.5) and (3.7), we have

$$2C_1\varphi(k)|J| \leqslant |E_k| \leqslant 2\varphi(k)|J|. \tag{3.8}$$

It follows that

$$2C_1|J| \sum_{k=k_0}^{N} \varphi(k) \leqslant \sum_{k=k_0}^{N} |E_k| \leqslant 2|J| \sum_{k=k_0}^{N} \varphi(k), \tag{3.9}$$

and so from (3.2) that $\sum_{k=k_0}^{\infty} |E_k| = \infty$.

We proceed to estimate the measures of the intersections E_k and E_l. In general $|E_k \cap E_l|$ will not be comparable with $|E_k||E_l|$, but 'on average' suitable estimates hold. Fix, as we may by (3.2), a number $N_0 > k_0$ such that

$$\sum_{k=k_0}^{N_0} \varphi(k) > 1. \tag{3.10}$$

Fix k and l such that $k_0 \leqslant k < l \leqslant N$, where $N > N_0$. For any $\gamma \in \Gamma_k(J)$,

$$E_l \cap E_k(\gamma) = \bigcup_{\beta \in \Gamma_l(J)} E_l(\beta) \cap E_k(\gamma). \tag{3.11}$$

The number of different $\beta \in \Gamma_l(J)$ satisfying $E_l(\beta) \cap E_k(\gamma) \neq \emptyset$ is less than or equal to

$$2 + |E_k(\gamma)|/2^{-3l} \leqslant 2 + 2 \cdot 2^{3(l-k)}\varphi(k)$$

from (3.7). Using this, (3.7) and (3.11) give

$$|E_l \cap E_k(\gamma)| \leqslant 4 \cdot 2^{-3l}\varphi(l)(1 + 2^{3(l-k)}\varphi(k))$$

and therefore from (3.5)

$$|E_l \cap E_k| \leqslant 4|J|\varphi(l)\varphi(k) + 4|J|2^{-3(l-k)}\varphi(l). \tag{3.12}$$

Since $E_k \cap E_l = E_l \cap E_k$, we have

$$\sum_{l=k_0}^{N}\sum_{k=k_0}^{N} |E_l \cap E_k| = \sum_{l=k_0}^{N} |E_k| + 2\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1} |E_l \cap E_k|. \tag{3.13}$$

The double sum on the right-hand side of (3.13) is estimated with the help of (3.12):

$$\begin{aligned} 2\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1} |E_l \cap E_k| &\leqslant 8|J|\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1} \varphi(l)\varphi(k) + 8|J|\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1} 2^{-3(l-k)}\varphi(l) \\ &\leqslant 8|J|\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1} \varphi(l)\varphi(k) + 8|J|\sum_{l=k_0+1}^{N} \varphi(l)\sum_{k=k_0}^{l-1} 2^{-3(l-k)} \\ &< 8|J|\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1} \varphi(l)\varphi(k) + 2|J|\sum_{l=k_0+1}^{N} \varphi(l). \end{aligned} \tag{3.14}$$

Thus, from (3.10), (3.9) and (3.14), we conclude that

$$\begin{aligned} \sum_{l=k_0}^{N}\sum_{k=k_0}^{N} |E_l \cap E_k| &\leqslant 4|J|\sum_{k=k_0}^{N} \varphi(k) + 8|J|\sum_{l=k_0+1}^{N}\sum_{k=k_0}^{l-1} \varphi(l)\varphi(k) \\ &\leqslant 4|J|\left(\sum_{k=k_0}^{N} \varphi(k)\right)^2 + 4|J|\sum_{l=k_0}^{N}\sum_{k=k_0}^{N} \varphi(l)\varphi(k) = 8|J|\left(\sum_{k=k_0}^{N} \varphi(k)\right)^2. \end{aligned}$$

This estimate and (3.9) gives

$$\frac{\left(\sum_{k=k_0}^{N} |E_k|\right)^2}{\left(\sum_{k=k_0}^{N}\sum_{l=k_0}^{N} |E_k \cap E_l|\right)} \geqslant \frac{(2C_1|J|)^2\left(\sum_{k=k_0}^{N} \varphi(k)\right)^2}{8|J|\left(\sum_{k=k_0}^{N} \varphi(k)\right)^2} = \tfrac{1}{2}C_1^2|J|.$$

It follows that $|E(J)| \geqslant \frac{1}{2}C_1^2|J|$ from lemma 3.2 and from (3.6) that $|\Gamma(\psi) \cap J| \geqslant \frac{1}{2}C_1^2|J|$. This holds for any finite interval J. By lemma 3.1 the proof of theorem 1.2 is complete.

4. Proof of theorem 1.1

Let $\mathcal{F}(\psi)$ denote the set of real numbers x satisfying the inequality (1.1) for infinitely many $F \in \mathcal{F}$. Define $\psi_1(h) = \psi(h)/(M+1)$. It is clear that $\psi_1(h)$ is monotonic and that the sum $\sum_{h=1}^{\infty}\psi_1(h)$ diverges. By theorem 1.2, the set $\Gamma(\psi_1)$ has full measure. Given $\gamma\in\Gamma$, define the interval $\sigma_1(\gamma)=\{x\in I:|x-\gamma|<h(\gamma)^{-2}\psi_1(h(\gamma))\}$. Then,

$$\Gamma(\psi_1)=\bigcap_{k=1}^{\infty}\bigcup_{\gamma:h(\gamma)>k}\sigma_1(\gamma).$$

Given $\gamma\in\Gamma$, let F_γ be the unique function in $\mathcal{F}$ with $F(\gamma)=0$ and $h(\gamma)=H(F)$. By the MVT,

$$F_\gamma(x)=F'(\tilde{x})(x-\gamma),$$

where $\tilde{x}$ is a point between γ and x. Thus $|F_\gamma(x)|\leqslant H(F)(M+1)|x-\gamma|$. Let $x\in\sigma_1(\gamma)$. Then

$$|F_\gamma(x)|\leqslant H(F)(M+1)h(\gamma)^{-2}\psi_1(h(\gamma))=H(F)^{-1}\psi(H(F)).$$

Thus for any $\gamma\in\Gamma$ such that $\sigma_1(\gamma)\neq\emptyset$, F_γ is a solution of (1.1) when $x\in\sigma_1(\gamma)$. It follows that if $x\in\Gamma(\psi_1)$, then the inequality (1.1) has infinitely many solutions, whence x belongs to $\mathcal{F}(\psi)$. Thus $\Gamma(\psi_1)\subset\mathcal{F}(\psi)$. It follows that $\mathcal{F}(\psi)$ has full measure and the proof is complete.

The natural question of extending this result to curves and indeed to manifolds in higher dimensions is much more difficult and probably requires deeper arguments.

V.V.B. is supported by a NATO/RS grant and H.D. is supported by the EPSRC (grant no. GR/K56407). We are also grateful to the Royal Society for providing money, which enabled V.I.B. to visit York, where this work was carried out. Finally, we thank the two referees for correcting some references and making some helpful comments.

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