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On a metrical theorem of W. Schmidt

by

V. Beresnevich and V. Bernik (Minsk)

1. Introduction. Let $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ be $(n+1)$-times continuously differentiable functions. Write

$$W(f_1', \ldots, f_n')(x) = \begin{pmatrix} f_1'(x) & \ldots & f_n'(x) \\ \ldots & \ldots & \ldots \\ f_1^{(n)}(x) & \ldots & f_n^{(n)}(x) \end{pmatrix}, \tag{1}$$

$$w(f_1', \ldots, f_n')(x) = \det W(f_1', \ldots, f_n')(x), \tag{2}$$

$$F_n(x) = a_0 + a_1 f_1(x) + \ldots + a_n f_n(x), \tag{3}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. We denote by $\mathcal{F} = \mathcal{F}_n$ the set of all functions of the form (3). We will suppose that

$$w(f_1', \ldots, f_n')(x) \neq 0 \tag{4}$$

for almost all x. Moreover, μA is the Lebesgue measure of the set A in $\mathbb{R}$. We are interested in the solutions of the inequalities

$$|F(x)| < H^{-n-\varepsilon}, \tag{5}$$

where $H = H(F) = \max(|a_0|, \ldots, |a_n|)$, $F \in \mathcal{F}_n$, $\varepsilon > 0$. For $\varepsilon > 0$ we define

$$\Psi = \Psi_n(\varepsilon) = \{x \in \mathbb{R} : (5) \text{ holds for infinitely many } F \in \mathcal{F}_n\}. \tag{6}$$

In 1964 W. Schmidt proved that $\mu \Psi_2 = 0$ (see [2]). In this article we prove the next case:

Theorem. *For any* $\varepsilon > 0$, $\mu \Psi_3(\varepsilon) = 0$.

We set

$$\sigma(F) = \{x \in \mathbb{R} : |F(x)| < H^{-3-\varepsilon}\}, \tag{7}$$

where $F \in \mathcal{F}_3$. For any finite interval $\Delta \subset \mathbb{R}$ we put

$$\widehat{\Delta} = \{x \in \mathbb{R} : |x - y| \le 2\mu\Delta \text{ for any } y \in \Delta\}. \tag{8}$$

We write $X \ll Y$ for $X = O(Y)$, and $X \asymp Y$ is equivalent to the simultaneous validity of $X \ll Y$ and $Y \ll X$. Moreover, $|A|$ is the number of elements in a finite set A. We denote by $d(\Delta_1, \Delta_2)$ the distance between the centers of two intervals Δ_1, Δ_2. Notice one property of $d(\Delta_1,\Delta_2)$: suppose we have two families of intervals $\Delta_1(t)$ and $\Delta_2(t)$ which satisfy the condition

$$\max(\mu\Delta_1(t),\mu\Delta_2(t)) \underset{t\to\infty}{=} o(d(\Delta_1(t),\Delta_2(t))). \tag{9}$$

Then for any $x_1(t)\in\Delta_1(t)$ and $x_2(t)\in\Delta_2(t)$, we have

$$|x_1(t)-x_2(t)| \asymp d(\Delta_1(t),\Delta_2(t)). \tag{10}$$

The proof is trivial.

Let $1\le m\le n$. We denote by $C(n,m)$ the set of all $\mathbf{J}=(j_1,\ldots,j_m)\in\mathbb{Z}^m$, where $1\le j_1<\ldots<j_m\le n$, and $(f_{j_1},\ldots,f_{j_m})$ is denoted by $\bar f_{\mathbf{J}}$.

2. Auxiliary statements

LEMMA 1. *Let $M\subset\mathbb{R}$ and suppose that every point of M is isolated. Then M is at most countable.*

Lemma 1 is well known. It is an easy exercise.

LEMMA 2. *Let $\varphi:\mathbb{R}\to\mathbb{R}$ be an m-times continuously differentiable function, and $N=\{x\in\mathbb{R}:\varphi(x)=0\}$. Let $\mu N>0$. Then there exists a subset $L\subset N$ such that*

(a) *$N\setminus L$ is at most countable*,
(b) *for any $i\in\{1,\ldots,m\}$ and for any $x\in L$, $\varphi^{(i)}(x)=0$.*

Proof. It is sufficient to prove this lemma for $m=1$. We denote by L the set of all limit points of N. Then $M=N\setminus L$ consists of all isolated points of N. From Lemma 1 it follows that M is at most countable. Since φ is continuous, N is closed. Hence $L\subset N$. Now (b) is easy to obtain by applying the definition of limit points in terms of sequences, Lagrange's formula and the continuity of φ'.

LEMMA 3. *Let $f_i:\mathbb{R}\to\mathbb{R}$ $(1\le i\le n)$ be n-times continuously differentiable functions and $w(f_1',\ldots,f_n')\ne 0$ for almost all $x\in\mathbb{R}$. Then for any $m\in\{1,\ldots,n\}$ and any $\mathbf{J}\in C(n,m)$,*

$$w(\bar f_{\mathbf{J}}')\ne 0 \tag{11}$$

for almost all $x\in\mathbb{R}$.

Proof. Let $m=1$, $1\le j\le n$ and $N=\{x:f_j'(x)=0\}$. Suppose $\mu N>0$. By Lemma 2 there exists $L\subset N$ such that $\mu L=\mu N>0$ and $f_j^{(i)}(x)=0$ for any $i=1,\ldots,n$ and for any $x\in L$. Hence for any $x\in L$ the ith column in $W(f_1',\ldots,f_n')(x)$ is zero. It follows that $w(f_1',\ldots,f_n')=0$ for any $x\in L$. But $\mu L>0$. The contradiction proves the lemma for $m=1$.

Now suppose the lemma is proved for $m-1$ with $m>1$. We write $N=\{x:w(\bar f_{\mathbf{J}}')(x)=0\}$, where $\mathbf{J}\in C(n,m)$. We denote by $\bar r_i$ the ith derivative of $\bar{f}_{\mathbf{J}}$. Suppose $\mu N > 0$. According to Lemma 2 there exists $L \subset N$ such that $\mu L = \mu N > 0$ and

$$\frac{d^k}{dx^k}(w(\bar{f}'_{\mathbf{J}})) = 0 \tag{12}$$

for all $x \in L$, where $1 \le k \le n-m$. From the inductive assumption it follows that the vectors $\bar{r}_1(x), \ldots, \bar{r}_{m-1}(x)$ are linearly independent for almost all $x \in \mathbb{R}$. Hence we can assume that they are linearly independent for all $x \in L$. Applying (12) with $k = 1, \ldots, n-m$ we find that $\bar{r}_i(x)$ depends linearly on $\bar{r}_1(x), \ldots, \bar{r}_{m-1}(x)$ for all $x \in L$, $1 \le i \le n$. Hence the columns of $W(f'_1, \ldots, f'_n)$ with indices $j_1, \ldots, j_m$ are linearly dependent for all $x \in L$. This contradiction finishes the proof.

Define

$$S = \bigcup_{m=1}^{n} \bigcup_{\mathbf{J} \in C(n,m)} \{x \in \mathbb{R} : w(\bar{f}'_{\mathbf{J}})(x) = 0\}.$$

Since S is closed, $\mathbb{R} \setminus S$ has the form $\bigcup_{k=1}^{\infty}[a_k, b_k]$. From Lemma 3 it follows that $\mu S = 0$. Then

$$\mu\Psi \le \sum_{k=1}^{\infty} \mu(\Psi \cap [a_k, b_k]).$$

In order to prove our theorem it is sufficient to show that if $I = [a, b]$ and $I \cap S = \emptyset$ then $\mu(\Psi \cap I) = 0$. Later on, to simplify the writing, we let I be a fixed closed interval in $\mathbb{R} \setminus S$. We redefine $\sigma(F)$ and Ψ to be the intersection of I with the former sets $\sigma(F)$ and Ψ. Since $w(\bar{f}'_{\mathbf{J}})$ is continuous and not zero over I, for all $\mathbf{J} \in C(n, m)$ with $1 \le m \le n$ and for all $x \in I$ we have

$$|w(\bar{f}'_{\mathbf{J}})(x)| \ge d > 0, \tag{13}$$

where d is a positive constant depending on the functions $f_1, \ldots, f_n$ and the interval I only.

LEMMA 4. *Let $\delta, \nu > 0$. Let φ be an n-times continuously differentiable function on (a, b) satisfying $|\varphi^{(n)}(x)| \ge \delta$ for all $x \in (a, b)$. Then $\mu(\{x \in (a, b) : |\varphi(x)| < \nu\}) \le c(n)(\nu/\delta)^{(1/n)}$.*

This is proved in [1].

LEMMA 5. *Set $\alpha_m = \max\{1, \sup\{|f_j^{(i)}(x)| : x \in I\} : 0 \le i \le m,\ 1 \le j \le n\}$ and $C_1 = d\alpha_n^{-n}/(n+1)!$, where $f_i \in C^{(n)}(\mathbb{R})$ $(1 \le i \le n)$. Then for all $x \in \sigma(F)$ and $H \ge H_0$ we have*

$$\max_{1 \le i \le n}(|F^{(i)}(x)|) \ge C_1 H, \tag{14}$$

where $\sigma(F)$ is defined in (7), *$F \in \mathcal{F}_n$ and $H = H(F)$.*

Proof. We may write the following system of linear equations:

$$(15)\qquad \begin{cases} a_0 + a_1 f_1(x) + \ldots + a_n f_n(x) = F(x), \\ a_1 f_1'(x) + \ldots + a_n f_n'(x) = F'(x), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1 f_1^{(n)}(x) + \ldots + a_n f_n^{(n)}(x) = F^{(n)}(x). \end{cases}$$

The modulus of the determinant of (15) is not less than d. Using Cramer's rule we have, for $i = 0, \ldots, n$,

$$|a_i| \le \frac{1}{d}\alpha_n^n (n+1)! \max\{|F^{(j)}(x)| : 0 \le j \le n\},$$

whence the lemma readily follows.

LEMMA 6. *Let f_i $(1 \le i \le n)$ be $(n+1)$-times continuously differentiable functions. Suppose $C_2 = C_1/(2n\alpha_{n+1})$, $\mu I \le C_2$ and $|F^{(i)}(\kappa)| \ge C_1 H$, where $\kappa \in I$ and $1 \le i \le n$. Then $|F^{(i)}(x)| \ge C_1 H/2$ for all $x \in I$.*

Proof. Assume $x \in I$. By Lagrange's formula, $F^{(i)}(x) = F^{(i)}(\kappa) + F^{(i+1)}(\kappa_1)(x-\kappa)$. Furthermore, $|F^{(i+1)}(\kappa)(x-\kappa)| \le n\alpha_{n+1} C_2 H = C_1 H/2$. Thus $|F^{(i)}(x)| \ge |F^{(i)}(\kappa)| - |F^{(i+1)}(\kappa_1)(x-\kappa)| \ge C_1 H - C_1 H/2$, and the lemma is proved.

Since I is a finite union of intervals of length $\le C_2$, we may suppose without loss of generality that $\mu I \le C_2$.

3. Preliminary remarks. From now on, $n = 3$.

Remark 1. Suppose we have a finite set of conditions according to which $\mathcal{F}$ is divided into subclasses: $\mathcal{F} = \bigcup_{i=1}^N \mathcal{F}^i$. Let a division of $\sigma(F) = \bigcup_{j=1}^M \sigma^j(F)$ into a finite number of intervals be defined for every $F \in \mathcal{F}$, where M is an absolute constant. Define

$$\Psi_{i,j} = \bigcap_{k=1}^{\infty} \bigcup_{F \in \mathcal{F}^i,\, H(F) \ge k} \sigma^j(F).$$

Then

$$(16)\qquad \Psi = \bigcap_{k=1}^{\infty} \bigcup_{F \in \mathcal{F},\, H(F) \ge k} \sigma(F) \subset \bigcup_{i=1}^{N} \bigcup_{j=1}^{M} \Psi_{i,j}.$$

Hence if we prove that $\mu\Psi_{i,j} = 0$ for all $1 \le i \le N$, $1 \le j \le M$, we obtain $\mu\Psi = 0$. In the sequel to simplify the writing we shall impose some conditions, additional indices being omitted.

Remark 2. From Lemmas 5 and 6 it follows that there exists $k \in \{1,2,3\}$ such that $|F^{(k)}(x)| \ge C_1 H/2$. We obtain a covering of I by at most six subintervals such that $F^{(i)}(x)$ is monotone on each subinterval for

$0 \le i \le k-1$, where $F^{(0)} \equiv F$. Therefore by Remark 1 we can assume that $\sigma(F)$ is such an interval.

Remark 3. We define

$$\mathcal{F}(t) = \{F \in \mathcal{F} : 2^t \le H(F) \le 2^{t+1}\}. \tag{17}$$

The number of functions in $\mathcal{F}(t)$ is $\ll 2^{4t}$. Suppose we have $\mu\sigma(F) \ll H^{-4-\xi}$ for some $\xi > 0$. Then

$$\sum_{F \in \mathcal{F}(t)} \mu\sigma(F) \ll 2^{-\xi t}. \tag{18}$$

The convergence of $\sum 2^{-\xi t}$ and the Borel–Cantelli lemma now show that the set of x belonging to infinitely many sets of $\sigma(F)$ has measure zero.

Remark 4. Lemmas 4–6 give the estimate

$$\mu\sigma(F) \ll H^{-(4+\varepsilon)/3}. \tag{19}$$

If $\varepsilon > 8$ then from (19) we get $\mu\sigma(F) \ll H^{-4-\xi}$, where $\xi = (\varepsilon - 8)/3$, and Remark 3 yields the assertion of the Theorem. Therefore below we consider $\varepsilon \le 8$.

Remark 5. If $|F'(x)| \ge H^{1-\varepsilon/2}$ for $x \in \sigma(F)$ then we get the estimate $\mu\sigma(F) \ll H^{-4-\varepsilon/2}$. If $|F'(x)| < H^{-9}$ for $x \in \sigma(F)$ then $\mu\sigma(F) \ll H^{-5}$. If $|F''(x)| < H^{-4}$ then $\mu\sigma(F) \ll H^{-5}$. These estimates readily follow from Lemma 4 with φ equal to F' and F'' respectively. In each of these cases, Remark 3 yields the assertion of the Theorem. Therefore further we may suppose that

$$|F'(x)| < H^{1-\varepsilon/2}, \tag{20}$$

$$|F'(x)| \ge H^{-9}, \quad |F''(x)| \ge H^{-4} \tag{21}$$

for $x \in \sigma(F)$.

Choose a positive parameter

$$\delta = \min\left(\frac{\varepsilon}{20}, \frac{\varepsilon^2}{4(5+\varepsilon)}, \frac{\varepsilon^2}{16(4+\varepsilon)}\right). \tag{22}$$

The conditions

$$H^{(l-1)\delta} \le |F'(x)| < H^{l\delta}, \tag{23}$$

$$H^{(k-1)\delta} \le |F''(x)| < H^{k\delta}, \tag{24}$$

where $k, l \in \mathbb{Z}$, define a subdivision of $\sigma(F)$. If $(l-1)\delta > 1$ or $(k-1)\delta > 1$ then the corresponding element of the subdivision is empty when $H \ge H_0$. From (21) we have $l\delta \ge -9$, $k\delta \ge -4$. Hence the number of different integers (k, l) is finite. We can thus suppose that $\sigma(F)$ is an interval and conditions (23) and (24) hold for all $x \in \sigma(F)$, where k and l are fixed.

4. Proof of the Theorem. The case of large first derivative

PROPOSITION 1. *Let* $(l-1)\delta \geq -1-\varepsilon/4$ *and suppose condition* (23) *holds for* $x \in \sigma(F)$. *Then the measure of those* $x \in I$ *which belong to infinitely many* $\sigma(F)$ *is at most* $\mu\Psi(\varepsilon+\varepsilon/8)$.

Proof. The considered functions F are divided into the subclasses $\mathcal{F}(t)$ defined in (17). Suppose $\eta = 3+3\varepsilon/4+(l-1)\delta$. Using Lemma 4 and (23) we get

$$\mu\sigma(F) \ll H^{-3-\varepsilon-(l-1)\delta}. \tag{25}$$

We define

$$[\Delta]_t = \{F \in \mathcal{F}(t) : \sigma(F)\cap\Delta \neq \emptyset\} \tag{26}$$

for any interval $\Delta \subset I$. For every fixed t we divide I into subintervals I_s^t of length $cn^{-\eta t}$ each, where $c = c(t) \in [1,2]$.

The number of different I_s^t is $\ll 2^{\eta t}$. Now define

$$\mathcal{F}'(t) = \bigcup_s [I_s^t], \tag{27}$$

where the union is taken over those I_s^t for which $|[I_s^t]_t| \leq 2^{(\varepsilon/4-\delta)t}$. We consider

$$\mathcal{F}''(t) = \mathcal{F}(t)\setminus\mathcal{F}'(t), \quad \mathcal{F}' = \bigcup_t \mathcal{F}'(t), \quad \mathcal{F}'' = \bigcup_t \mathcal{F}''(t). \tag{28}$$

Counting the number of functions in $\mathcal{F}'(t)$ and using (25) we get

$$\begin{aligned}\sum_{t\geq 0}\sum_{F\in\mathcal{F}'(t)} \mu\sigma(F) &\ll \sum_{t\geq 0} 2^{\eta t}2^{(\varepsilon/4-\delta)t}2^{(-3-\varepsilon-(l-1)\delta)t}\\ &= \sum_{t\geq 0} 2^{-\delta t} < \infty.\end{aligned}$$

Thus, from the Borel–Cantelli lemma it follows that the set of those $x \in I$ which belong to infinitely many $\sigma(F)$ for $F \in \mathcal{F}'$ has measure zero.

Now consider $x_0 \in I$ belonging to infinitely many $\sigma(F)$ for $F \in \mathcal{F}''$. The choice of η and the estimate (25) show that $\sigma(F) \subset \widehat{I}_s^t$ if $t \geq t_0$ and $F \in [I_s^t]_t$. Thus x_0 belongs to $\widehat{I}_s^t$ for infinitely many t with $|[I_s^t]_t| > 2^{(\varepsilon/4-\delta)t}$. Consider a fixed such interval I_s^t. Let $F \in [I_s^t]_t$ and $\kappa \in \sigma(F)\cap I_s^t$. By Taylor's formula we have

$$F(x) = F(\kappa) + F'(\kappa)(x-\kappa) + \tfrac{1}{2}F''(\kappa_1)(x-\kappa)^2, \tag{29}$$

where κ_1 lies between x and κ. From (5), (23) and the estimate $|x-\kappa| \ll H^{-\eta}$ we get

$$|F(x)| \ll H^{-3-\varepsilon} + H^{l\delta-\eta} + H^{1-2\eta}. \tag{30}$$

The choice of δ and η and the assumption of Proposition 1 imply that the first and third terms on the right hand side (30) are less than the second term. Now using the value of η, and (30), we obtain

$$|F(x)| \ll H^{-3-3\varepsilon/4+\delta} \tag{31}$$

for all $x \in \widehat{I}_s^t$. Analogously we have

$$|F'(x)| \ll H^{l\delta} \tag{32}$$

for all $x \in \widehat{I}_s^t$.

Both a_2 and a_3 range in the interval $[-2^{t+1}, 2^{t+1}]$. We divide it into intervals Δ_j with length $2^{t(1-\varepsilon/8+\delta/2)+2}$. Thus we obtain at most $2^{t(\varepsilon/4-\delta)}$ pairs of intervals $(\Delta_{j_1}, \Delta_{j_2})$. Since by assumption we have $|[I_s^t]_t| > 2^{t(\varepsilon/4-\delta)}$, there exist $F_1, F_2 \in [I_s^t]_t$ whose coefficients a_2 and a_3 belong to one pair of intervals $(\Delta_{j_1}, \Delta_{j_2})$. Consider $R(x) = F_1(x) - F_2(x)$. We obtain

$$|a_i(R)| \le 2^{t(1-\varepsilon/8+\delta/2)+2} \tag{33}$$

for $i = 2, 3$. From (31) and (32) for F_1 and F_2 it follows that

$$|R(x)| \ll 2^{t(-3-3\varepsilon/4+\delta)}, \tag{34}$$

$$|R'(x)| \ll 2^{l\delta t} \tag{35}$$

for all $x \in \widehat{I}_s^t$. From (20) we get $l\delta \le 1 - \varepsilon/2 + \delta < 1 - \varepsilon/8 + \delta/2$. Therefore from (13), (33) and (35) we have $|a_1(R)| \ll 2^{t(1-\varepsilon/8+\delta/2)}$. From this and (34) we obtain $|a_0(R)| \ll 2^{t(1-\varepsilon/8+\delta/2)}$. Thus we conclude that

$$H(R) \ll 2^{t(1-\varepsilon/8+\delta/2)}. \tag{36}$$

The relation

$$|R(x)| \ll H(R)^{-(3+3\varepsilon/4-\delta)/(1-\varepsilon/8+\delta/2)} \tag{37}$$

follows from (34) and (36) for all $x \in \widehat{I}_s^t$. We have

$$\frac{3+3\varepsilon/4-\delta}{1-\varepsilon/8+\delta/2} - (3+\varepsilon) > 3 + 3\varepsilon/4 - \delta - (1-\varepsilon/8+\delta/2)(3+\varepsilon) \ge \varepsilon/8.$$

Therefore

$$|R(x_0)| < H(R)^{-3-\varepsilon-\varepsilon/8}, \tag{38}$$

where $H(R) \ge H_0$ and H_0 is sufficiently large.

R e m a r k 6. Applying Lemma 2 it is easy to show that for every fixed $R \in \mathcal{F}_n$ the measure of $E_n(R) = \{x : R(x) = 0\}$ is zero. Then the union E_n of all $E_n(R)$ with $R \in \mathcal{F}_n$ also has measure zero. If the number of different $R(x)$ in (38) is finite then x_0 is a solution of some equivalent $R(x) = 0$, where R has the form (3).

The inequality (38) and the previous discussion prove Proposition 1.

5. The case of small second derivative

PROPOSITION 2. *Let* $(l-1)\delta < -1-\varepsilon/4, (k-1)\delta \le 1-\varepsilon/2$ *and suppose that conditions* (23) *and* (24) *are valid for* $x \in \sigma(F)$. *Then the measure of the set of* $x \in I$ *belonging to infinitely many* $\sigma(F)$ *is at most* $\mu\Psi(\varepsilon+\varepsilon^2/16)$.

Proof. From (23), (24) and Lemmas 5 and 6 it follows that

$$|F'''(x)| \ge C_1 H/2 \tag{39}$$

for all $x \in I$. By Lemma 4, from (5), (23), (24) and (39) we get six estimates of the measure of $\sigma(F)$. Choosing the optimal estimate we obtain

$$\mu\sigma(F) \ll H^{-\nu}, \tag{40}$$

where

$$\nu = \max\bigg(3+\varepsilon+(l-1)\delta, \frac{3+\varepsilon+(k-1)\delta}{2}, \frac{4+\varepsilon}{3}, -l\delta+(k-1)\delta, \frac{-l\delta+1}{2}, -k\delta+1\bigg).$$

Suppose $\eta = \nu-\varepsilon/8$. We divide all the functions $F \in \mathcal{F}_3$ under consideration into the subclasses $\mathcal{F}(t)$ defined in (17). For every fixed integer t we divide I into subintervals I_s^t of length $c2^{-\eta t}$ each, where $c = c(t) \in [1,2]$. The number of different I_s^t is $\ll 2^{\eta t}$. The classes $\mathcal{F}'(t)$ and $\mathcal{F}''(t)$ are defined in the same way as in (27) and (28), with the union in (27) taken over those I_s^t for which $|[I_s^t]_t| \le 2^{t(\varepsilon/8-\delta)}$. The classes $\mathcal{F}'$ and $\mathcal{F}''$ are defined as above. Counting the number of functions in $\mathcal{F}'(t)$ and using (40) we get

$$\sum_{t\ge 0}\sum_{F\in\mathcal{F}'(t)} \mu\sigma(F) \ll \sum_{t\ge 0} 2^{\eta t}2^{(\varepsilon/8-\delta)t}2^{-\nu t} = \sum_{t\ge 0} 2^{-\delta t} < \infty.$$

Thus the Borel–Cantelli lemma shows that the set of those $x \in I$ which belong to infinitely many $\sigma(F)$ for $F \in \mathcal{F}'$ has zero measure.

Now consider $x_0 \in I$ belonging to infinitely many $\sigma(F)$ for $F \in \mathcal{F}''$. The choice of η and the estimate (40) give $\sigma(F) \subset \widehat{I}_s^t$ if $t \ge t_0$ and $F \in [I_s^t]_t$. Thus x_0 belongs to $\widehat{I}_s^t$ for infinitely many t and $|[I_s^t]_t| > 2^{(\varepsilon/4-\delta)t}$. Consider a fixed such interval I_s^t. Let $F \in [I_s^t]_t$ and $\kappa \in \sigma(F) \cap I_s^t$. From (24) and Taylor's formula we obtain

$$\begin{aligned}|F''(x)| &= |F''(\kappa)+F'''(\kappa_1)(x-\kappa)| \le |F''(\kappa)| + |F'''(\kappa_1)(x-\kappa)| \\ &\ll H^{k\delta} + H^{1-\eta} \le H^{k\delta} + H^{k\delta+\varepsilon/8} \le 2\cdot H^{k\delta+\varepsilon/8},\end{aligned}$$

where κ_1 lies between x and κ. Analogously we get estimates for $F(x)$ and $F'(x)$ using (23), (24) and Taylor's formula. Thus

$$|F(x)| \ll H^{-3-\varepsilon+3\varepsilon/8+\delta}, \tag{41}$$

$$|F'(x)| \ll H^{l\delta+2\varepsilon/8+\delta}, \tag{42}$$

(43) $$|F''(x)| \ll H^{k\delta+\varepsilon/8}$$

for all $x \in \widehat{I}_s^t$. The coefficient a_3 ranges over the interval $[-2^{t+1}, 2^{t+1}]$. We divide it into intervals Δ_j of length $2^{t(1-\varepsilon/8+\delta)+2}$. There are at most $2^{t(\varepsilon/8-\delta)}$ intervals Δ_j. Since by assumption we have $|[I_s^t]_t| > 2^{t(\varepsilon/8-\delta)}$ there exist $F_1, F_2 \in [I_s^t]_t$ whose coefficients a_3 belong to one Δ_j. Consider $R(x) = F_1(x) - F_2(x)$. Then

(44) $$|a_3(R)| \le 2^{t(1-\varepsilon/8+\delta)+2}.$$

It is clear that conditions (41)–(43) apply to $R(x)$ if we substitute 2^t for H. It is not difficult to verify that $l\delta + 2\varepsilon/8 + \delta \le 1 - \varepsilon/8 + \delta$ and $k\delta + \varepsilon/8 \le 1 - \varepsilon/8 + \delta$. From conditions (42) and (43) for F_1 and F_2 it follows that

(45) $$|R'(x)| \ll 2^{t(1-\varepsilon/8+\delta)}, \quad |R''(x)| \ll 2^{t(1-\varepsilon/8+\delta)}.$$

By (44) and (45),

(46) $$|a_1(R)f_1'(x) + a_2(R)f_2'(x) \ll 2^{t(1-\varepsilon/8+\delta)},$$
$$|a_1(R)f_1''(x) + a_2(R)f_2''(x)| \ll 2^{t(1-\varepsilon/8+\delta)}.$$

From (46) we obtain $|a_i(R)| \ll 2^{t(1-\varepsilon/8+\delta)}$ $(i = 1, 2)$ because $|w(f_1', f_2')| \ge d > 0$ according to (13). From (41) for F_1 and F_2 it follows that

(47) $$|R(x)| \ll 2^{t(-3-\varepsilon+3\varepsilon/8+\delta)}$$

and from (47) we find $|a_0(R)| \ll 2^{t(1-\varepsilon/8+\delta)}$. Hence

(48) $$H(R) \ll 2^{t(1-\varepsilon/8+\delta)}.$$

Observe that
$$\frac{3-\varepsilon-3\varepsilon/8-\delta}{1-\varepsilon/8+\delta} - (3-\varepsilon) > 3+\varepsilon-3\varepsilon/8-\delta-(1-\varepsilon/8+\delta)(3+\varepsilon) \ge \varepsilon^2/16.$$

Thus from (47) and (48) we obtain

$$|R(x_0)| < H(R)^{-3-\varepsilon-\varepsilon^2/16}$$

with $H(R) \ge H_0$, where H_0 is sufficiently large. The last inequality together with Remark 6 finishes the proof of Proposition 2.

6. The last case. Let $\gamma > 0$. Set

$$\mathcal{G} = \{F = a_0 + a_1 f_1 + a_2 f_2 : (a_0, a_1, a_2) \in \mathbb{Z}^3 \setminus \{0\}\}.$$

For $F \in \mathcal{G}$ consider the system

(49) $$|F(x)| < H^{-1-\gamma}, \quad |F'(x)| < H^{-\gamma/2},$$

where $H = H(F) = \max(|a_0|, |a_1|, |a_2|)$. The set of its solutions is denoted by $\sigma^*(F)$. Define

(50) $$\Omega(\gamma) = \{x \in I : (49) \text{ is valid for infinitely many } F \in \mathcal{G}\}.$$

Now we return to our problem. By Remark 1 we can assume that any $F \in \mathcal{F}_3$ has $a_3 \geq |a_i|$ $(1 \leq i \leq 3)$.

PROPOSITION 3. *Let* $(l-1)\delta < -1-\varepsilon/4, (k-1)\delta > 1-\varepsilon/2$ *and suppose that conditions* (23) *and* (24) *are valid throughout* $\sigma(F)$. *Moreover*, *let* $a_3 \geq |a_i|$ $(1 \leq i \leq 3)$ *for* $F \in \mathcal{F}_3$. *Then the measure of the set of* $x \in I$ *belonging to infinitely many* $\sigma(F)$ *is at most* $\mu\Omega(\varepsilon/5)$.

Proof. We have $|F(x)| \geq H^{1-\varepsilon/2}$. By Lemma 4 we get $\mu\sigma(F) \ll H^{-2-\varepsilon/4}$. Define $\eta = 1+\varepsilon/8$. Divide the collection of $F \in \mathcal{F}_3$ under consideration into the subclasses $\mathcal{F}(t) = \{F \in \mathcal{F}_3 : a_3(F) = t\}$. It is clear that $H(F) \asymp t$ for $F \in \mathcal{F}(t)$. Fix t and divide I into subintervals I_s^t of length $ct^{-\eta}$ each, where $c = c(t) \in [1,2]$. The number of different I_s^t is $\ll t^\eta$. The classes $\mathcal{F}'(t)$ and $\mathcal{F}''(t)$ are defined as in (27) and (28), with the union in (27) taken over those I_s^t for which $|[I_s^t]_t| \leq 1$. The classes $\mathcal{F}'$ and $\mathcal{F}''$ are defined as above. Counting the number of functions in $\mathcal{F}'(t)$ and estimating the measure of $\sigma(F)$ we get

$$\sum_{t\geq 1}\sum_{F\in\mathcal{F}'(t)} \mu\sigma(F) \ll \sum_{t\geq 1} t^\eta t^{-2-\varepsilon/4} = \sum_{t\geq 1} t^{-1-\varepsilon/8} < \infty.$$

Thus, the Borel–Cantelli lemma shows that the set of those $x \in I$ which belong to infinitely many $\sigma(F)$ for $F \in \mathcal{F}'$ has measure zero.

Now consider $x_0 \in I$ belonging to infinitely many $\sigma(F)$ with $F \in \mathcal{F}''$. The choice of η implies that $\sigma(F) \subset \widehat{I}_s^t$ if $t \geq t_0$, where $F \in [I_s^t]_t$. Thus x_0 belongs to $\widehat{I}_s^t$ for infinitely many t with $|[I_s^t]_t| \geq 2$. Consider a fixed such interval I_s^t. Let $F \in [I_s^t]_t$ and $\kappa \in \sigma(F) \cap I_s^t$. By Taylor's formula we have $F'(x) = F'(\kappa) + F''(\kappa_1)(x-\kappa)$. Hence

$$|F'(x)| \ll H^{-\varepsilon/8}. \tag{51}$$

Analogously we find

$$|F(x)| \ll H^{-1-\varepsilon/4} \tag{52}$$

for all $x \in \widehat{I}_s^t$. There exist different $F_1, F_2 \in [I_s^t]_t$. Consider $R = F_1 - F_2$. Then $R \in \mathcal{G}$ and $H(R) \ll t$. From (51) and (52) it follows that

$$|R(x)| < H(R)^{-1-\varepsilon/5}, \quad |R'(x)| < H(R)^{-\varepsilon/10},$$

whenever $H(R) \geq H_0$. Thus Proposition 3 is proved.

PROPOSITION 4. *For any* $\gamma > 0$, $\mu\Omega(\gamma) = 0$.

Proof. We shall consider only those $F \in \mathcal{G}$ for which $\sigma^*(F) \neq \emptyset$. As in the proof of Lemmas 5 and 6, for all $x \in I$ we obtain

$$|F''(x)| \geq C_3 H, \tag{53}$$

where $F \in \mathcal{G}$, $H = H(F)$ and C_3 is a fixed positive constant. Moreover, from the condition $|a_i| = o(H)$, where $1 \leq i \leq 2$, we would get a contradiction.

Therefore we assume that

(54) $$\min(|a_1|,|a_2|) \geq C_4 H,$$

where $H = H(F)$ with $F = a_0 + a_1 f_1 + a_2 f_2$. Now we deal with the inequalities

(55) $$|F(x)| < H^{-1-\gamma},$$

(56) $$|F'(x)| < H^{-\gamma/2}$$

with $F \in \mathcal{G}$. Using Lemma 4 and condition (53) we find that for (55) the measure of the solution set is $\ll H^{-1-\gamma/2}$, and similarly for (56). Thus

(57) $$\mu\sigma(F) \ll H^{-1-\gamma/2},$$

where $\sigma'(F)$ denotes the union of the solution sets for (55) and (56). Since $\sigma^*(F) \neq \emptyset$ we can assume that $\sigma'(F)$ is an interval. Moreover, $\sigma^*(F) \subset \sigma'(F)$.

Condition (53) implies the monotonicity of $F'(x)$ in $I = [a,b]$. Consider those $F \in \mathcal{G}$ which have a nonvanishing derivative on all I. Either a or b necessarily belongs to $\sigma'(F)$ because F' is monotonic. Thus there exist $C_5 > 0$ and H_0 such that for any $H \geq H_0$ and for all $F \in \mathcal{G}$ with $H(F) \geq H$ we have

$$\sigma(F) \subset [a, a + C_5 H^{-1-\gamma/2}] \cup [b - C_5 H^{-1-\gamma/2}, b].$$

Hence $\mu\Omega(\gamma) \ll H^{-1-\gamma/2}$ and $\mu\Omega(\gamma) = 0$.

The remaining case is when $F'(x)$ has a root $\kappa = \kappa(F) \in I$ for $F \in \mathcal{G}$.

We use the following notations: $\mathbf{A} = (a_0, a_1, a_2)$ is a vector; $F_{\mathbf{A}} = a_0 + a_1 f_1 + a_2 f_2$; $\mathbf{F}(x) = (1, f_1(x), f_2(x)) \in \mathbb{R}^3$; $(\mathbf{A}, \mathbf{B})$ is the scalar product of the vectors $\mathbf{A}$ and $\mathbf{B}$; $\mathbf{A} \times \mathbf{B}$ is their vector product. Set $g(x) = f_2'(x)/f_1'(x)$. Then

(58) $$g'(x) = \frac{f_1''(x) f_2'(x) - f_2''(x) f_1'(x)}{(f_1'(x))^2}.$$

From (13) and (58) it follows that $g'(x) \neq 0$ for all $x \in I$. Hence $g'(x) \asymp 1$. Let $F_{\mathbf{A}}, F_{\mathbf{B}} \in \mathcal{G}$, and let $\kappa_{\mathbf{A}}$ and $\kappa_{\mathbf{B}}$ be the roots of $F'_{\mathbf{A}}$ and $F'_{\mathbf{B}}$ respectively. Obviously $g(\kappa_{\mathbf{A}}) = a_1/a_2$ and $g(\kappa_{\mathbf{B}}) = b_1/b_2$.

We have

$$|a_1/a_2 - b_1/b_2| = |g(\kappa_{\mathbf{A}}) - g(\kappa_{\mathbf{B}})| = |g'(\tau)(\kappa_{\mathbf{A}} - \kappa_{\mathbf{B}})| \asymp |(\kappa_{\mathbf{A}} - \kappa_{\mathbf{B}})|,$$

where τ lies between $\kappa_{\mathbf{A}}$ and $\kappa_{\mathbf{B}}$. We obtain

(59) $$|a_1/a_2 - b_1/b_2| \asymp |\kappa_{\mathbf{A}} - \kappa_{\mathbf{B}}|.$$

We divide the considered $F \in \mathcal{G}$ into the classes

(60) $$G(t) = \{F \in \mathcal{G} : 2^t \leq H(F) \leq 2^{t+1}\}$$

and choose the parameters α and β as follows:

$$0 < \alpha < \gamma/4, \tag{61}$$

$$\alpha/2 < \beta < \alpha. \tag{62}$$

For every t we divide I into intervals I_s^t of length $c2^{t(-1-\gamma/2+\alpha)}$ each, where $c = c(t) \in [1,2]$. Let

$$[I_s^t]_t = \{F \in \mathcal{G}(t) : \sigma(F) \cap I_s^t \neq \emptyset\}. \tag{63}$$

If $F \in [I_s^t]_t$, then by Taylor's formula, (55) and (56), we get

$$|F(x)| \ll 2^{t(-1-\gamma+2\alpha)}, \tag{64}$$

$$|F'(x)| \ll 2^{t(-\gamma/2+\alpha)} \tag{65}$$

for all $x \in \widehat{I}_s^t$.

Consider the following four types of intervals:

1) I_s^t is called of *type A* if $|[I_s^t]_t| \leq 2^{\alpha t/2}$.

2) I_s^t is called of *type B* if for any distinct $F_1, F_2 \in [I_s^t]_t$,

$$d(F_1, F_2) \leq 2^{t(-1-\gamma/2+\beta)}, \tag{66}$$

where $d(F_1, F_2) = d(\sigma(F_1), \sigma(F_2))$.

3) I_s^t is called of *type C* if there exist $F_{\mathbf{A}}, F_{\mathbf{B}}, F_{\mathbf{C}} \in [I_s^t]_t$ such that

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} \neq 0 \tag{67}$$

with $\mathbf{A} = (a_0, a_1, a_2)$, $\mathbf{B} = (b_0, b_1, b_2)$ and $\mathbf{C} = (c_0, c_1, c_2)$.

4) If I_s^t is not of type A, B or C, then it is called of *type D*.

ASSERTION 1. *The measure of those $x \in I$ which belong to infinitely many $\sigma(F)$, where $F \in [I_s^t]_t$ and I_s^t is a type A or B interval, is equal to zero.*

Proof. Counting the number of F for type A intervals I_s^t with a fixed t we get

$$\sum_{F \in \mathcal{G}(t)} \mu\sigma(F) \ll 2^{t(-1-\gamma/2)} 2^{t(1+\gamma/2-\alpha)} 2^{\alpha t/2} = 2^{-\alpha t/2}.$$

The Borel–Cantelli lemma finishes the proof in this case. Let I_s^t be a type B interval. By (66) there exists an interval Δ_s^t of length $\ll 2^{t(-1-\gamma/2+\beta)}$ such that

$$\bigcup_{F \in [I_s^t]_t} \sigma(F) \subset \Delta_s^t.$$

Then counting the number of intervals I_s^t we get

$$\sum_{t\geq 0}\sum_{s}\mu\Big(\bigcup_{F\in[I_s^t]_t}\sigma(F)\Big)\ll\sum_{t\geq 0}2^{t(1+\gamma/2-\alpha)}2^{t(-1-\gamma/2+\beta)}$$
$$\ll\sum_{t\geq 0}2^{-(\alpha-\beta)t}<\infty.$$

The Borel–Cantelli lemma finishes the proof.

Now if $x_0\in I$ belongs to infinitely many $\sigma(F)$, where $F\in[I_s^t]_t$ with I_s^t an interval of type C or D, then x_0 belongs to $\widehat{I}_s^t$ for infinitely many t, where I_s^t is a type C or D interval.

ASSERTION 2. *The measure of those $x\in I$ which belong to infinitely many $\widehat{I}_s^t$, where I_s^t is a type C interval, is equal to zero.*

Proof. We consider a type C interval I_s^t. There exist $F_{\mathbf{A}},F_{\mathbf{B}},F_{\mathbf{C}}\in[I_s^t]_t$ satisfying (67). For rational integers p_1, p_2, p_3 such that $|p_i|\leq 2^{t/3}$ $(i=1,2,3)$, we consider expressions of the form

$$p_1a_2+p_2b_2+p_3c_2. \tag{68}$$

Their values belong to some interval $[-C_62^{t+t/3},C_62^{t+t/3}]$, where C_6 is a constant independent of t. The number of different expressions of the form (68) is $\asymp 2^t$. Dirichlet's principle implies the existence of two different expressions of the form (68) with difference $\ll 2^{t/3}$. Let $p_{10}a_2+p_{20}b_2+p_{30}c_2$ denote this difference. It is obvious that

$$|p_{10}|+|p_{20}|+|p_{30}|\neq 0. \tag{69}$$

We define $R(x)=p_{10}F_{\mathbf{A}}(x)+p_{20}F_{\mathbf{B}}(x)+p_{30}F_{\mathbf{C}}(x)$. From (69) and (67) we have $R(x)\not\equiv 0$. Moreover, $R(x)=a_0(R)+a_1(R)f_1+a_2(R)f_2$ and

$$|a_2(R)|\ll 2^{t/3}. \tag{70}$$

The estimates (64), (65) and the definition of R yield

$$|R(x)|\ll 2^{t(-2/3-\gamma+2\alpha)}, \tag{71}$$
$$|R'(x)|\ll 2^{t(1/3-\gamma/2+\alpha)} \tag{72}$$

for all $x\in\widehat{I}_s^t$. The exponents in (71) and (72) are less than $t/3$. Hence from (70) we obtain $H(R)\ll 2^{t/3}$ and then from (71) we have

$$|R(x)|\ll H(R)^{-2-(3\gamma-6\alpha)}. \tag{73}$$

The exponent satisfies the inequality $-2-(3\gamma-6\alpha)<-2$. Therefore the proof is finished by Schmidt's theorem.

Consider a type D interval I_s^t. It has the following properties:

(a) $|[I_s^t]_t|>2^{\alpha t/2}$;

(b) there exist $F_{\mathbf{A}},F_{\mathbf{B}}\in[I_s^t]_t$ such that $d(F_{\mathbf{A}},F_{\mathbf{B}})>2^{t(-1-\gamma/2+\beta)}$;

(c) for any $F_{\mathbf{A}}, F_{\mathbf{B}}, F_{\mathbf{C}} \in [I_s^t]_t$ condition (67) does not hold.

By (c) there exists a plane with the normal $\mathbf{N}_s$ such that $(\mathbf{N}_s, \mathbf{A}) = 0$ for any $F_{\mathbf{A}} \in [I_s^t]_t$. Let $F_{\mathbf{A}}, F_{\mathbf{B}} \in [I_s^t]_t$ and $d(F_{\mathbf{A}}, F_{\mathbf{B}}) > 2^{t(-1-\gamma/2+\beta)}$. Then, using (54) and (59), we obtain

$$|a_1b_2 - a_2b_1| \gg 2^{t(1-\gamma/2+\beta)}. \tag{74}$$

By definition $\mathbf{A} \times \mathbf{B} = (a_1b_2 - a_2b_1, a_2b_0 - a_0b_2, a_0b_1 - a_1b_0)$. Then from (74) we have

$$|\mathbf{A} \times \mathbf{B}| \gg 2^{t(1-\gamma/2+\beta)}. \tag{75}$$

Moreover,

$$\mathbf{N}_s = \pm \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}.$$

It is known that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{B}, \mathbf{A})\mathbf{C} - (\mathbf{C}, \mathbf{A})\mathbf{B}, \tag{76}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^3$. It is obvious that $(\mathbf{F}(x), \mathbf{A}) = F_{\mathbf{A}}(x)$. Then for $x \in \widehat{I}_s^t$ we find

$$\begin{aligned} \mathbf{F}(x) \times \mathbf{N}_s &= \pm\, \mathbf{F}(x) \times \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \pm \frac{1}{|\mathbf{A} \times \mathbf{B}|}((\mathbf{A}, \mathbf{F}(x))\mathbf{B} - (\mathbf{B}, \mathbf{F}(x))\mathbf{A}) \\ &= \pm\, \frac{1}{|\mathbf{A} \times \mathbf{B}|}(F_{\mathbf{A}}(x)\mathbf{B} - F_{\mathbf{B}}(x)\mathbf{A}). \end{aligned}$$

Further, using the estimates (64), (75) and $|\mathbf{A}| \ll 2^t$, $|\mathbf{B}| \ll 2^t$, we get

$$|\mathbf{F}(x) \times \mathbf{N}_s| \ll 2^{t(-1+\gamma/2-\beta)} 2^t 2^{t(-1-\gamma+2\alpha)}.$$

Thus we have

$$|\mathbf{F}(x) \times \mathbf{N}_s| \ll 2^{t(-1-\gamma/2+2\alpha-\beta)} \tag{77}$$

for all $x \in I_s^t$.

Assertion 3. *The measure of those $x \in I$ which belong to infinitely many $\widehat{I}_s^t$, where I_s^t is a type D interval, is equal to zero.*

Proof. A type D interval I_s^t is called a *subtype D_1 interval* if there does not exist a type D interval I_h^t $(s \neq h)$ such that

$$2^{t(-1-\gamma/2+3\alpha/2)} \le d(I_s^t, I_h^t) \le 2^{-1-\gamma/2+2\alpha}. \tag{78}$$

The other type D intervals are *subtype D_2 intervals*. The number of subtype D_1 intervals is $\ll 2^{t(1+\gamma/2-3\alpha/2)}$. Hence

$$\sum_{t\ge 0} \sum_s \mu \widehat{I}_s^t \ll \sum_{t \ge 0} 2^{\alpha t/2} < \infty.$$

The Borel–Cantelli lemma finishes the proof in this case. Further, let I_s^t be a subtype D_2 interval. There exists a type D interval I_h^t satisfying (78). Let

$\Delta^t_{s,h}$ denote the smallest interval containing both I^t_s and I^t_h. From (75) we get

$$\mu\Delta^t_{s,h} \ll 2^{t(-1-\gamma/2+2\alpha)}. \tag{79}$$

If there exist $F_{\mathbf{A}}, F_{\mathbf{B}}, F_{\mathbf{C}} \in [\Delta^t_{s,h}]_t$ such that (67) holds then we obtain a bigger type C interval. The choice of α and (79) yield the following fact: the set of those $x \in I$ which belong to infinitely many such intervals has measure zero as in the proof of Assertion 2.

In the last case the normals coincide: $\mathbf{N} = \mathbf{N}_s = \mathbf{N}_h$. Using (13) and (77) for $x \in I^t_s$ and $y \in I^t_h$ we find

$$\begin{aligned}|x-y| \asymp |f_1(x) - f_1(y)| &\le |\mathbf{F}(x) \times \mathbf{F}(y)| \\ &\ll |\mathbf{F}(x) \times \mathbf{N}| + |\mathbf{F}(y) \times \mathbf{N}| \ll 2^{t(-1-\gamma/2+2\alpha-\beta)}.\end{aligned}$$

The last inequality and (78) give

$$2^{t(-1-\gamma/2+3\alpha/2)} \ll |x-y| \ll 2^{t(-1-\gamma/2+2\alpha-\beta)}. \tag{80}$$

The choice of β in (62) shows that $(-1-\gamma/2+3\alpha/2) > (-1-\gamma/2+2\alpha-\beta)$. Hence inequality (80) is contradictory for t large. Assertion 3 is proved. Thus Proposition 4 is proved.

7. Completion the proof of the Theorem. Let $\lambda = \min(\varepsilon/8, \varepsilon^2/16)$. Applying Propositions 1–4 at most $[8/\lambda] + 1$ times we get

$$\mu\Psi_3(\varepsilon) \le \mu\Psi_3(\varepsilon_1),$$

where $\varepsilon_1 > 8$. By Remark 4 the proof of the Theorem is complete.

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References

[1] A. Piartly, *Diophantine approximations on submanifolds of Euclidean space*, Funktsional. Anal. i Prilozhen. 3 (4) (1969), 59–62 (in Russian).

[2] W. M. Schmidt, *Metrische Sätze über simultane Approximation abhängiger Grössen*, Monatsh. Math. 63 (1964), 154–166.

Institute of Mathematics
Academy of Sciences of Belarus
220072 Minsk, Belarus
E-mail: imanb%imanb.belpak.minsk.by@demos.su

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