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A. Baker's conjecture and Hausdorff dimension

by

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(*Dedicated to the* 60*th birthday of Professor Kálmán Győry*)

Abstract. It this paper we discuss an application of the Hausdorff dimension to the set of very well multiplicatively approximable points $(x,\dots,x^n)$. In 1998 D.Kleinbock and G.Margulis proved A.Baker's conjecture stating that this set is of measure zero. We show that for any natural n multiplicatively approximable points $(x,\dots,x^n)$ to order $1+\varepsilon$ form a set of Hausdorff dimension at least $2/(1+\varepsilon)$. It is conjectured that this number is the exact value of the dimension. We also prove this conjecture for $n=2$.

Introduction. We will use the following notation. The Vinogradov symbol $\ll$ ($\gg$) means '$\le$ ($\ge$) up to a positive constant multiplier'; $a \asymp b$ is equivalent to $a \ll b \ll a$. The Lebesgue measure of $A \subset \mathbb{R}$ is denoted by $|A|$. We denote by $\mathcal{P}_n$ the set of polynomials $P \in \mathbb{Z}[x]$ with $\deg P \le n$. Given a polynomial $P(x) = a_nx^n + \dots + a_1x + a_0 \in \mathbb{Z}[x]$, we define the height of P as $H(P) = \max\{|a_0|,\dots,|a_n|\}$.

Let $\varepsilon > 0$, $n \in \mathbb{N}$ and $S_n(\varepsilon)$ denote the set of $x \in \mathbb{R}$ such that the inequality

$$|P(x)| < H(P)^{-n(1+\varepsilon)} \tag{1}$$

has infinitely many solutions $P \in \mathcal{P}_n$. In 1932 K. Mahler, in his classification of real numbers, conjectured that for any $\varepsilon > 0$ the Lebesgue measure of $S_n(\varepsilon)$ is zero. Mahler's problem was settled by V. Sprindzuk [5] in 1964. The concept of Hausdorff dimension (see [4]) makes it possible to differ sets of measure zero. In particular, this was applied to $S_n(\varepsilon)$. In 1970 A. Baker and W. Schmidt [2] established a lower bound for $\dim S_n(\varepsilon)$, the Hausdorff dimension of $S_n(\varepsilon)$. Later it was proved by V. Bernik [3] that this value is also an upper bound for $\dim S_n(\varepsilon)$ resulting in

$$\dim S_n(\varepsilon) = \frac{n+1}{n+1+n\varepsilon}. \tag{2}$$

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In 1975 A. Baker raised a problem by replacing the right hand side of (1) with the function $\Pi_+(P)^{-1-\varepsilon}$, where $P(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathcal{P}_n$ and $\Pi_+(P) = \prod_{i=1}^n \max(1, |a_i|)$. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, let $M_n(\varepsilon)$ be the set of $x \in \mathbb{R}$ such that the inequality

$$|P(x)| < \Pi_+(P)^{-1-\varepsilon} \tag{3}$$

has infinitely many solutions $P \in \mathcal{P}_n$. A. Baker [1] conjectured that for any $n \in \mathbb{N}$ one has $|M_n(\varepsilon)| = 0$ for any $\varepsilon > 0$.

Notice that Baker's conjecture is stronger than that of Mahler. Indeed, since $H(P)^n \geq \Pi_+(P)$, we have $H(P)^{-(1+\varepsilon)n} \leq \Pi_+(P)^{-(1+\varepsilon)}$. Therefore, if (1) is soluble infinitely often, then so is (3). In particular, it means that

$$S_n(\varepsilon) \subset M_n(\varepsilon). \tag{4}$$

Baker's conjecture was proved by D. Kleinbock and G. Margulis [7] in 1998.

As in the case of $S_n(\varepsilon)$, it is also of interest to determine the Hausdorff dimension of $M_n(\varepsilon)$. We will use the following properties [4]:

1) $\dim A \leq \dim B$ for any $A, B \subset \mathbb{R}$ with $A \subset B$;

2) $\dim A = \sup_{i=1,2,\ldots} \dim A_i$, where $A = \bigcup_{i=1}^{\infty} A_i$ and $A_i \subset \mathbb{R}$.

Conjectures and results. First of all, notice that

$$M_k(\varepsilon) \subset M_n(\varepsilon) \text{ for any } k, n \in \mathbb{N} \text{ with } k < n. \tag{5}$$

It follows from (4) and (5) that $S_1(\varepsilon) \subset M_n(\varepsilon)$ for any $n \in \mathbb{N}$. Therefore, we have $\dim M_n(\varepsilon) \geq \dim S_1(\varepsilon)$. Now applying (2) gives

Theorem 1. *For any $n \in \mathbb{N}$ and any $\varepsilon > 0$*

$$\dim M_n(\varepsilon) \geq \frac{2}{2+\varepsilon}. \tag{6}$$

Conjecture H1. For any $n \in \mathbb{N}$ and $\varepsilon > 0$ one has

$$\dim M_n(\varepsilon) = \frac{2}{2+\varepsilon}.$$

This conjecture is trivial for $n = 1$. Indeed, it is easy to notice that for any $\delta > 0$ we have the inclusion $M_1(\varepsilon) \subset S_1(\varepsilon - \delta)$. Therefore, for any δ with $0 < \delta < \varepsilon$ we have $\dim M_1(\varepsilon) \leq \dim S_1(\varepsilon - \delta)$. By (2), we conclude that

$\dim M_1(\varepsilon) \leq 2/(2+\varepsilon-\delta)$. Since $\delta \in (0,\varepsilon)$ is arbitrary, we have $\dim M_1(\varepsilon) \leq 2/(2+\varepsilon)$. In this paper we also prove the conjecture for $n = 2$.

Theorem 2. *For any $\varepsilon > 0$ we have*

$$\dim M_2(\varepsilon) = \frac{2}{2+\varepsilon}.$$

Proof of Theorem 2. By (6), it is sufficient to show that $\dim M_2(\varepsilon) \leq 2/(2+\varepsilon)$. Let $\{I_k\}_{k=1}^{\infty}$ be a collection of closed intervals such that $\mathbb{R} \setminus \{0\} = \bigcup_{k=1}^{\infty} I_k$. The existence of such a collection is easily verified. Then, $M_2(\varepsilon) = \{0\} \bigcup (\bigcup_{k=1}^{\infty} M_2(\varepsilon) \bigcap I_k)$. Since $\dim\{0\} = 0$, by property 2 of Hausdorff dimension above, we have $\dim M_2(\varepsilon) \leq \sup_{k=1,2,\ldots} \dim(M_2(\varepsilon) \bigcap I_k)$. Therefore, it is sufficient to show that $\dim(M_2(\varepsilon) \bigcap I_k) \leq 2/(2+\varepsilon)$ for any k. Let I be one of the intervals I_k. There is no loss of generality in assuming that $I = [a,b]$ with $0 < a < b < \infty$.

Let $x \in I$ and $P(t) = a_2t^2 + a_1t + a_0 \in \mathcal{P}_2$ be a solution of (3). It follows from (3) that

$$\begin{aligned} |a_0| = \; & |P(x) - a_2x^2 - a_1x| \leq \Pi_+(P)^{-1-\varepsilon} + |a_2|x^2 + |a_1|x \leq \\ & 1 + |a_2|b^2 + |a_1|b \leq (1+b+b^2)\max\{|a_1|,|a_2|\}. \end{aligned}$$

Therefore, we have

$$\max\{|a_1|,|a_2|\} \leq H(P) \leq (1+b+b^2)\max\{|a_1|,|a_2|\}. \tag{7}$$

Now define the constant

$$C = \min(a, 1/2)/(1+b+b^2). \tag{8}$$

Let $M_2^1(\varepsilon, I)$ be the subset of $M_2(\varepsilon) \cap I$ consisting of $x \in I$ such that there are infinitely many $P \in \mathcal{P}_2$ satisfying

$$\begin{cases} |P(x)| < \Pi_+(P)^{-1-\varepsilon}, \\ |P'(x)| < CH(P). \end{cases} \tag{9}$$

Let $x \in I$ and $P(t) = a_2t^2 + a_1t + a_0$ be a solution of (9). We have the following two possibilities:

1) $|a_2| \geq |a_1|$;
2) $|a_1| \geq |a_2|$.

Consider the first one. It follows from (7) and (9) that

$$|2x + a_1/a_2| \le CH(P)/|a_2| \le C(1 + b + b^2) \le a.$$

Since $x \ge a$, we have $|a_1/a_2| = |2x - (2x + a_1/a_2)| \ge |2x| - |2x + a_1/a_2| \ge 2a - a = a$. Therefore, we obtain $a \le |a_1/a_2| \le 1$.

Consider the other possibility: $|a_1| \ge |a_2|$. It follows from (7) and (9) that

$$|2xa_2/a_1 + 1| \le CH(P)/|a_1| \le C(1 + b + b^2) \le 1/2.$$

Hence, $|2xa_2/a_1| = |1-(2xa_2/a_1+1)| \ge 1-|2xa_2/a_1+1| \ge 1-1/2 = 1/2$. Since $x \le b$, we have $|a_2/a_1| \ge 1/(4b)$. Therefore, we obtain $1/(4b) \le |a_2/a_1| \le 1$.

As a result we conclude that $|a_1| \asymp |a_2|$ for both the possibilities. Moreover, by (7), we have $|a_1| \asymp |a_2| \asymp H(P)$. Therefore, $P_+(P) \asymp H(P)^2$ and the first inequality of (9) implies that

$$|P(x)| \ll H(P)^{-2(1+\varepsilon)}. \tag{10}$$

Now if $x \in M_2^1(\varepsilon, I)$, then inequality (10) holds for infinitely many polynomials $P \in \mathcal{P}_2$ and for any $\delta > 0$ the inequality $|P(x)| < H(P)^{-2(1+\varepsilon-\delta)}$ has infinitely many solutions $P \in \mathcal{P}_2$. It follows that $M_2^1(\varepsilon, I) \subset S_2(\varepsilon - \delta)$ for any δ with $0 < \delta < \varepsilon$. By (2), we obtain

$$\dim M_2^1(\varepsilon, I) \le \dim S_2(\varepsilon - \delta) = \frac{3}{3 + 2(\varepsilon - \delta)}.$$

Since $\delta \in (0, \varepsilon)$ is arbitrary, we get

$$\dim M_2^1(\varepsilon, I) \le \frac{3}{3 + 2\varepsilon} < \frac{2}{2 + \varepsilon}. \tag{11}$$

Now we consider the set $M_2^2(\varepsilon, I) = (M_2(\varepsilon) \cap I) \setminus M_2^1(\varepsilon, I)$. It is easy to verify that for any $x \in M_2^2(\varepsilon, I)$ the system

$$\begin{cases} |P(x)| < \Pi_+(P)^{-1-\varepsilon}, \\ |P'(x)| \ge CH(P) \end{cases} \tag{12}$$

holds for infinitely many polynomials $P \in \mathcal{P}_2$. Given a polynomial $P \in \mathcal{P}_2$, let $\sigma(P)$ denote the set of $x \in I$ satisfying (12). It is easy to notice that $\sigma(P)$ is a union of at most three intervals, say $\sigma^i(P)$ with $i = 1, 2, 3$. Also if $x \in M_2^2(\varepsilon, I)$ then x belongs to $\sigma^i(P)$ for infinitely many different polynomials $P \in \mathcal{P}_2$.

Fix $P \in \mathcal{P}_2$ and $x, y \in \sigma^i(P)$. By the Mean Value Theorem, we have $P(x) - P(y) = P'(\theta)(x - y)$, where θ is a point between x and y. Since $\sigma^i(P)$ is an interval, $\theta \in \sigma^i(P)$ and, therefore, $|P'(\theta)| \geq CH(P)$. Hence,

$$|x - y| \leq \frac{|P(x)| + |P(y)|}{|P'(\theta)|} \leq \frac{2\Pi_+(P)^{-1-\varepsilon}}{CH(P)}.$$

Thus,

$$|\sigma^i(P)| \ll \Pi_+(P)^{-1-\varepsilon} \cdot H(P)^{-1}. \tag{13}$$

Let $2/(2 + \varepsilon) < \rho < 1$. We have the following inequality

$$\sum_{P \in \mathcal{P}_2} \sum_{i=1}^{3} |\sigma^i(P)|^\rho \ll \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{2^k \leq |a_1| < 2^{k+1}} \sum_{2^l \leq |a_2| < 2^{l+1}} \sum_{a_0} \sum_{i=1}^{3} |\sigma^i(P)|^\rho, \tag{14}$$

where $P(x) = a_2 x^2 + a_1 x + a_0$. If $2^k \leq |a_1| < 2^{k+1}$ and $2^l \leq |a_2| < 2^{l+1}$ then, by (13), $|\sigma^i(P)| \ll 2^{-(1+\varepsilon)(k+l)-\max\{k,l\}}$. Moreover, by (7), the number of different a_0 such that $\sigma(P) \neq \emptyset$ is $\ll 2^{\max\{k,l\}}$. Now it follows from (14) that

$$\begin{aligned}
&\textstyle\sum_{P \in \mathcal{P}_2} \sum_{i=1}^{3} |\sigma^i(P)|^\rho \ll \\
&\textstyle\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{2^k \leq |a_1| < 2^{k+1}} \sum_{2^l \leq |a_2| < 2^{l+1}} 2^{\max\{k,l\}} \cdot (2^{-(1+\varepsilon)(k+l)-\max\{k,l\}})^\rho \ll \\
&\textstyle\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2^k \cdot 2^l \cdot 2^{\max\{k,l\}} \cdot (2^{-(1+\varepsilon)(k+l)-\max\{k,l\}})^\rho = \\
&\textstyle\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2^{(1-\rho(1+\varepsilon))(k+l)+(1-\rho)\max\{k,l\}} \leq \\
&\textstyle\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2^{(1-\rho(1+\varepsilon))(k+l)+(1-\rho)(k+l)} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2^{(2-\rho(2+\varepsilon))(k+l)} = \\
&\textstyle\left(\sum_{k=0}^{\infty} 2^{(2-\rho(2+\varepsilon))k}\right) \cdot \left(\sum_{l=0}^{\infty} 2^{(2-\rho(2+\varepsilon))l}\right).
\end{aligned}$$

Since $\rho > 2/(2 + \varepsilon)$, we have $2 - \rho(2 + \varepsilon) < 0$. It is now easy to see that the sum $\sum_{l=0}^{\infty} 2^{(2-\rho(2+\varepsilon))l}$ converges. Therefore, we have

$$\textstyle\sum_{P \in \mathcal{P}_2} \sum_{i=1}^{3} |\sigma^i(P)|^\rho < \infty \tag{15}$$

for any ρ with $2/(2 + \varepsilon) < \rho < 1$. By Lemma 4 in [4,pp. 94], the Hausdorff dimension of the set consisting of $x \in I$, which belongs to infinitely many intervals $\sigma^i(P)$, is at most ρ. This set is exactly $M_2^2(\varepsilon, I)$. Since $\rho \in (2/(2 + \varepsilon), 1)$ is arbitrary, we have $\dim M_2^2(\varepsilon, I) \leq 2/(2 + \varepsilon)$. Combining this and (11) completes the proof of Theorem 2. $\square$

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