Quantum phase effects for electrically charged particles: converging descriptions via fields and potentials

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Abstract. We analyze the physical meaning of quantum phase effects from a new perspective, related to our recent disclosure of two novel quantum phases for electric/magnetic dipoles – in addition to the previously known Aharonov-Casher and He-McKellar-Wilkens phases, and two novel quantum phases for point-like charged particles – in addition to the electric and magnetic Aharonov-Bohm phases. We show that the obtained complete expression for the quantum phase of a moving charge in an EM field allows to establish its direct link with the interactional electromagnetic momentum in the system "particle and external field" and to better understand the physical meaning of quantum phase effects, which is discussed using a number of particular examples.

1 Introduction

The Aharonov-Bohm (A-B) effect was predicted more than 60 years ago [1, 2] and later confirmed experimentally [3]; nevertheless, discussions about its physical meaning and implications are carried out up to the present time (see, e.g. [4-10]).

In a general consideration, the analysis of the A-B effect involves, as minimum, three entities: a charged particle, a source of EM field, its EM fields/potentials, and in the most general approach, all of them should be quantized (see, e.g., [9]).

In what follows, in our analysis of the A-B effect, we will mainly follow the approximate description applied in [1], where the source of EM field and its fields/potentials are treated as semi-classical, while the moving charge is considered as a quantum particle. Not to have to discuss at this point the limits of applicability of this approximation, we notice that this approach allows one to exclude from consideration any specific sources of EM fields/potentials, and to consider EM fields and potentials in the vicinity of the trajectory of a charged particle as some exogenous parameters. This approach was applied in the very first paper establishing the A-B effect [1] along with the known expression for the Hamiltonian of a quantum charged particle in an EM field (see, e.g. [11])

$$\hat{H}=\frac{\left(-i\hbar\nabla-e\boldsymbol{A}/c\right)^2}{2m}+e\varphi \ . \tag{1}$$

Here $\boldsymbol{A}$ and φ denote for the vector and scalar potentials, correspondingly, defined at the location of the charged particle *e*.

In this case, both the magnetic and electric A-B phases can be simultaneously derived through the Schrödinger equation with Hamiltonian (1), leading to the general expression for the quantum phase of charged particle in the presence of an EM field (see, e.g., [1])

$$\phi=-\frac{1}{\hbar}\int\left(\hat{H}-\hat{H}_0\right)dt \, . \tag{2}$$

Here $\hat{H}_0=-i\hbar\nabla/2m$ stands for the Hamiltonian of the particle in the absence of EM field. Indeed, combining eqs. (1) and (2), we present, with sufficient accuracy of calculations, the A-B phase $\phi_{A\text{-}B}$ for charged particle *e* as the sum

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$$\varphi_{A-B}=\varphi_{\varphi}+\varphi_{A}, \tag{3}$$

where

$$\varphi_{\varphi}=-\frac{1}{\hbar}\int e\varphi\, dt \tag{4}$$

corresponds to the electric A-B phase, and

$$\varphi_{A}=\frac{1}{\hbar c}\int e\boldsymbol{A}\cdot d\boldsymbol{s} \tag{5}$$

stands for the magnetic A-B phase, ***ds***=***v****dt* is the spatial elements, and ***v*** is the velocity of charge.

The magnetic A-B phase (5) was first observed in [3]. The available attempts to experimentally confirm the presence of the electric A-B effect (4) remain unsuccessful due to serious technical difficulties in distinguishing this phase effect from dynamical effects arising from the non-vanishing electric component of the Lorentz force even in optimized experimental setups (see, *e.g.* [12]). At the same time, from a conceptual viewpoint, there are no doubts in reality of both electric (4) and magnetic (5) A-B phases, since they are conjointly derived from the Schrödinger equation for a charged particle in an EM field with the common Hamiltonian (1).

A few decades later since the discovery of the A-B effect, a new quantum phase has been predicted by Aharonov and Casher for a magnetic dipole, moving in an electric field ***E*** [13]

$$\varphi_{mE}=\frac{1}{\hbar c}\int_{L}\left(\boldsymbol{m}_{0}\times\boldsymbol{E}\right)\cdot d\boldsymbol{s}, \tag{6}$$

where $\boldsymbol{m}_0$ is the proper magnetic dipole moment of the dipole.

Next, by the end of the past century, one more quantum phase for a moving electric dipole in the presence of a magnetic field ***B*** has been predicted by He, McKellar and Wilkens [14, 15]

$$\varphi_{pB}=-\frac{1}{\hbar c}\int_{L}\left(\boldsymbol{p}_{0}\times\boldsymbol{B}\right)\cdot d\boldsymbol{s}, \tag{7}$$

where $\boldsymbol{p}_0$ designates the proper electric dipole moment.

Later, the existence of quantum phase effects (6) and (7) has been confirmed experimentally [16, 17]. Usually, the Aharonov-Casher (A-C) phase is associated with the presence of a hidden momentum for a magnetic dipole (see, e.g., [18-20]), while the He-McKellar-Wilkens (HMW) phase for an electric dipole can be associated with the product $\left(\boldsymbol{p}_{0}\times\boldsymbol{B}\right)/c$, which was named in [21] as the hidden momentum for an electric dipole.

It is worth to notice that both quantum phase effects for moving dipoles (6) and (7) have been derived in [13-15] with some particular non-covariant expressions for the Lagrangian of an electric/magnetic dipole in an EM field, which, in general, left open the problem of the possible existence of more quantum phase effects for such dipoles.

This problem was first explored in refs. [21, 22] on the basis of an explicitly covariant expression for the Lagrangian density of a material medium in the presence of an EM field [23]

$$l=M^{\mu\nu}F_{\mu\nu}/2, \tag{8}$$

where $M^{\mu\nu}$ is the magnetization-polarization tensor, and $F_{\mu\nu}$ is the tensor of EM field.

Further integration of the Lagrangian density (8) over the volume of a compact dipole allowed us to obtain a new Lorentz-invariant expression for the Lagrangian,

$$L=\boldsymbol{p}\cdot\boldsymbol{E}+\boldsymbol{m}\cdot\boldsymbol{B}, \tag{8a}$$

where ***p***, ***m*** stand for the electric and magnetic dipole moment, respectively, for a laboratory observer.

Based on eq. (8a), we obtained a new relativistically consistent motional equation and a new Hamilton function for the dipole [21, 22]. Its further generalization to the quantum limit with application of eq. (2) yields the following new expression for the total quantum phase of a moving dipole [21, 22]

$$¢_{\text{dipole}} = \frac{1}{\hbar c}\int (\boldsymbol{m}_0 \times \boldsymbol{E})\cdot d\boldsymbol{s} - \frac{1}{\hbar c}\int (\boldsymbol{p}_0 \times \boldsymbol{B})\cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int ¥ (\boldsymbol{p}_{0//}\cdot \boldsymbol{E})\boldsymbol{v}\cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int ¥ (\boldsymbol{m}_{0//}\cdot \boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{s}, \qquad (9)$$

where $\boldsymbol{p}_{0//}$, $\boldsymbol{m}_{0//}$ denote vector components collinear with the velocity of the dipole $\boldsymbol{v}$.

One can see that the first and second terms on the rhs of eq. (9) correspond to the A-C phase (6) and the HMW phase (7), while the remaining two terms describe new quantum phases, emerging under the motion of an electric dipole in an electric field

$$¢_{pE} = -\frac{1}{\hbar c^2}\int ¥ (\boldsymbol{p}_{0//}\cdot \boldsymbol{E})\boldsymbol{v}\cdot d\boldsymbol{s}, \qquad (10)$$

and under the motion of a magnetic dipole in a magnetic field

$$¢_{mB} = -\frac{1}{\hbar c^2}\int ¥ (\boldsymbol{m}_{0//}\cdot \boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{s}, \qquad (11)$$

correspondingly.

Thus, we have found four quantum phase effects for a moving electric/magnetic dipole, corresponding to four possible combinations between the pair $\boldsymbol{E}$, $\boldsymbol{B}$ and the pair $\boldsymbol{p}_0$, $\boldsymbol{m}_0$, and this finding makes topical the question about their physical interpretation.

Seeking an answer to this question, we supposed in refs. [24, 25] that, due to the linearity of fundamental equations of quantum mechanics, each quantum phase for a moving dipole can be presented as a superposition of the corresponding quantum phases for all charges composing the dipole.

This idea, being attractive from the physical viewpoint, nevertheless faces substantial difficulties in an immediate attempt of its realization. This is due to the fact that all components of the quantum phase for the dipole, presented by the four terms in eq. (9), explicitly depend on the velocity $\boldsymbol{v}$ of this dipole, and thus, they cannot include the velocity-independent electric A-B phases (4) for the charges of the dipole. At the same time, it is obvious that the remaining magnetic A-B phase (5) for point-like charges cannot simultaneously explain the four quantum phase effects for a moving dipole presented by eq. (9).

We explore this problem in sect. 2 and show that the application of the "superposition principle" to explanation of quantum phase effects to electric/magnetic dipoles allows disclosing two new quantum phase effects for moving charges, next to the phases (4), (5), with the following expression for the velocity-dependent phase component [24-26]:

$$¢_{EM}(\boldsymbol{v}) = \frac{e}{\hbar c}\int \boldsymbol{A}\cdot d\boldsymbol{s} + \frac{e}{\hbar c^2}\int ¨\,\boldsymbol{v}\cdot d\boldsymbol{s} - \frac{e}{\hbar c^3}\int \boldsymbol{v}(\boldsymbol{A}\cdot \boldsymbol{v})\cdot d\boldsymbol{s}. \qquad (12)$$

Here the first term on the rhs corresponds to the magnetic A-B phase (5), while the other two terms define new quantum phase effects for a moving charge, named in [24, 25] respectively as the complementary electric A-B phase

$$¢_{c¨} = \frac{1}{\hbar c^2}\int e\,¨\,\boldsymbol{v}\cdot d\boldsymbol{s}, \qquad (13)$$

and the complementary magnetic A-B phase

$$¢_{cA} = -\frac{e}{\hbar c^3}\int (\boldsymbol{v}\cdot \boldsymbol{A})\boldsymbol{v}\cdot d\boldsymbol{s}. \qquad (14)$$

In sect. 3 we show that the disclosure of the velocity-dependent quantum phase for a charged particle in the form (12) allows us to clarify its physical meaning, where it is directly related to the interactional field momentum for a charged particle in an external EM field – a point, which was not so obvious, when we deal with the magnetic A-B phase (5) taken alone.

This finding allows us to conclude that the wave vector of a moving charged particle $\boldsymbol{k}$ is linearly proportional to the sum of mechanical $\boldsymbol{P}_M$ and electromagnetic $\boldsymbol{P}_{EM}$ momenta. Therefore, $\boldsymbol{P}_{EM}$ does affect the quantum phase of charged particle even in the situation where no net force exerts on the particle and its mechanical momentum $\boldsymbol{P}_M$ remains constant along different paths.

Further, we emphasize that the new expression for the velocity-dependent A-B phase (12) cannot be derived via equation (2) with the standard Hamiltonian (1) and requires its modification to the form [10, 25]

$$\hat{H} = \frac{\left(-i\hbar\nabla - \boldsymbol{P}_{EM}\right)^2}{2M} + e¨ \ , \qquad (15)$$

where $\boldsymbol{P}_{EM}$ denotes the interactional EM field momentum for the system "charged particle in an EM field".

As we further showed in refs. [10, 25], eq. (15) implies an appropriate re-definition of the momentum operator for a charged particle, where instead of its canonical momentum (which, as is known, does not have a real physical meaning [27]), this operator is associated with the sum of the mechanical and interactional field momentum for a charged particle in an EM field [10, 25].

The close link between the disclosure of the complementary electric (13) and complementary magnetic (14) A-B phases for electrically charged particle and the need to redefine the momentum operator makes topical the problem of experimental observation of the phases (13), (14) either directly for relativistic electrons or indirectly, by measuring new quantum phases (10) and (11) for moving dipoles, as discussed further in sect. 3.

Next, in sect. 4, we discuss the problem of non-locality of the A-B effect from a new angle of view and clarify the role of EM fields and potentials in its manifestation. Finally, we conclude in sect. 5.

2 Quantum phase effects for dipoles and new fundamental quantum phases for moving charges

Here, we analyse four quantum phase effects for moving dipoles, presented by eq. (9). In order to simplify subsequent calculations, we will use a weak relativistic limit, corresponding to an accuracy c^{-3}, where eq. (9) can be written in a convenient approximate form [22]

$$¢_{\text{dipole}} \approx \frac{1}{\hbar c}\int(\boldsymbol{m}\times\boldsymbol{E})\cdot d\boldsymbol{s} - \frac{1}{\hbar c}\int((\boldsymbol{p}\times\boldsymbol{B}))\cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int(\boldsymbol{p}\cdot\boldsymbol{E})\boldsymbol{v}\cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int(\boldsymbol{m}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{s}\,, \qquad (16)$$

with all quantities evaluated in a labframe.

Further, in the subsequent derivation of the fundamental quantum phases for point-like charges, which compose the quantum phases for dipoles (16), we have to express all terms of (16) through the scalar ¨ and vector $\boldsymbol{A}$ potentials. This problem has been solved in ref. [25], where the following relationships were obtained:

$$¢_{pB} = -\frac{1}{\hbar c}\int_L\int_V °\boldsymbol{A}\cdot d\boldsymbol{s}dV\,, \qquad (17a)$$

$$¢_{mE} = -\frac{1}{\hbar c^2}\int_L\int_V ¨\,°\boldsymbol{u}\cdot d\boldsymbol{s}dV\,, \qquad (17b)$$

$$¢_{pE} = -\frac{1}{\hbar c^2}\int_L\int_V °\,¨\boldsymbol{v}\cdot d\boldsymbol{s}dV\,, \qquad (17c)$$

$$¢_{mB} = -\frac{1}{\hbar c^3}\int_L\int_V (°\boldsymbol{u}\cdot\boldsymbol{A})\boldsymbol{v}\cdot d\boldsymbol{s}dV\,; \qquad (17d)$$

here ° is the charge density, and $\boldsymbol{u}$ is the flow velocity of carriers of current in a magnetic dipole.

In ref. [25], we proposed to clarify the physical meaning of eqs. (17a-d) through the "superposition principle" for quantum phases, using the simplest model of an electric dipole – two mechanically bound charges *–e* and *+e*, separated by a small distance ***d***; and using the simplest model of a magnetic dipole – a small electrically neutral conducting loop carrying a steady current.

First, we consider the HMW phase and A-C phases, which have been confirmed experimentally and thus, the application of "superposition principle" for these phases can be considered as its practical validation.

So, we apply the adopted model of electric dipole to the HMW phase (17a) and obtain:

$$¢_{pB} = -\frac{1}{\hbar c}\int_V\int_L °\mathbf{A}\cdot d\mathbf{s}dV = -\frac{e}{\hbar c}\left(\int_{L_+}\mathbf{A}(\mathbf{r}+\mathbf{d})\cdot d\mathbf{s} - \int_{L_-}\mathbf{A}(\mathbf{r})\cdot d\mathbf{s}\right), \tag{18}$$

where L_+ (L_-) stands for the paths of a positive (negative) charge of the dipole, and ***r*** is the spatial coordinate of the dipole.

Eq. (18) signifies that the HMW phase represents the algebraic sum of the magnetic A-B phases (2) for each charge constituting the dipole. This result demonstrates the applicability of the "superposition principle for quantum phases" to the explanation of the origin of quantum phase effects for moving dipoles. Moreover, the experimental confirmation of the HMW phase mentioned above [8], along with observations of magnetic A-B phase (see, e.g. [3]) does confirm the correctness of this principle.

This result makes further application of the superposition principle for quantum phase to be important and interesting, and further we address the A-C phase, which has also been confirmed by experiments [6,7].

Using the model of magnetic dipole specified above, we adopt that the positive charges of the frame of the loop of the dipole are immovable in its proper frame, and their phase is defined as

$$\left(¢_{mE}\right)_{positive} = -\frac{1}{\hbar c^2}\int_V\oint ¨\, °_+\mathbf{v}\cdot d\mathbf{s}dV\ . \tag{19a}$$

Considering the contribution of the negative charged (the carries of current), we designate their flow velocity through ***u***, and obtain their phase as

$$\left(¢_{mE}\right)_{negative} = -\frac{1}{\hbar c^2}\int_V\oint ¨\, °_-(\mathbf{u}+\mathbf{v})\cdot d\mathbf{s}dV\ . \tag{19b}$$

Summing equations (19a) and (19b) at $°_+ = -°_- = °$, we obtain the phase (17b).

Equations (19a-b) indicate that the motion of a point-like charge in the field of a scalar potential ¨ yields the phase (13), which has been named in refs. [21, 22] as a "complementary electric A-B phase". In what follows, we will supply it with the subscript "*c*".

Next, we consider the phase $¢_{pE}$, which for the adopted model of an electric dipole takes the form:

$$¢_{pE} = -\frac{1}{\hbar c^2}\int_V\int_L °\, ¨\,\mathbf{v}\cdot d\mathbf{s}dV = -\frac{e}{\hbar c^2}\int_{L_+} ¨\,(\mathbf{r}+\mathbf{d})\mathbf{v}\cdot d\mathbf{s} + \frac{e}{\hbar c^2}\int_{L_+} ¨\,(\mathbf{r})\mathbf{v}\cdot d\mathbf{s}\ . \tag{20}$$

This equation shows that the observed quantum phase $¢_{pE}$ for an electric dipole can be presented as a superposition of complementary electric A-B phases $¢_{c¨}$ (13), and once again confirms the necessity of introducing this phase to clarify the physical meaning of quantum phase effects for moving dipoles.

Finally, we consider the remaining phase $¢_{mB}$ (17d) for a magnetic dipole, which for the model of the magnetic dipole specified above can be written by analogy with eqs. (19a-b):

$$(\varphi_{mB})_{positive} = -\frac{1}{\hbar c^2}\int_V \int \lambda_+ (\boldsymbol{v}\cdot\boldsymbol{A})\boldsymbol{v}\cdot d\boldsymbol{s}dV\,, \tag{21a}$$

$$(\varphi_{mB})_{negative} = -\frac{1}{\hbar c^2}\int_V \int \phi\,\lambda_- ((\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{A})(\boldsymbol{u}+\boldsymbol{v})\cdot d\boldsymbol{s}dV\,. \tag{21b}$$

Summing eqs. (21a) and (21b) at $\lambda_+ = -\lambda_-$, and taking into account that for a closed loop the integrals $\int_V \int_L (\boldsymbol{u}\cdot\boldsymbol{A})\boldsymbol{u}\cdot d\boldsymbol{s}dV$, $\int_V \int_L (\boldsymbol{v}\cdot\boldsymbol{A})\boldsymbol{u}\cdot d\boldsymbol{s}dV$ are vanishing, we arrive at the phase (17d).

Thus, we have shown that the phase φ_{mB} for a magnetic dipole moving in a magnetic field can be presented as a superposition of new quantum phases for point-like charges (14), which has been named in refs. [21, 22], as "complementary magnetic A-B phase".

Thus, the idea to present the quantum phases for a moving dipole through a superposition of fundamental phases for point-like charges, composing the dipole, allowed us to disclose the new phase effects for moving charges (13) and (14), and to establish a relationship between quantum phase effects for charges and dipoles as shown in Fig. 1.

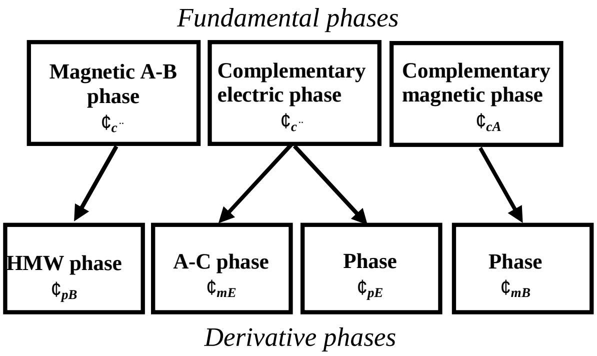

FIG. 1 adapted from [25]: Relationship between velocity-dependent quantum phases for charged particles and for moving dipoles, established through the superposition principle for quantum phases.

3 Quantum phase effects for a charge particle through the interactional EM momentum and possible ways for their observations

In this section, we first explore the physical meaning of the velocity-dependent quantum phase (12) for a moving charge in an EM field, which, as assumed in [25], should be closely related to the interactional EM field momentum for the system "charged particle in an EM field" [27]

$$\boldsymbol{P}_{EM} = \frac{1}{4\pi c}\int_V (\boldsymbol{E}\times\boldsymbol{B}_e)dV + \frac{1}{4\pi c}\int_V (\boldsymbol{E}_e\times\boldsymbol{B})dV\,, \tag{22}$$

where $\boldsymbol{E}$, $\boldsymbol{B}$ are the external electric and magnetic fields, while $\boldsymbol{E}_e$, $\boldsymbol{B}_e$ are the electric and magnetic fields of the charged particle e.

In order to determine the relationship between the velocity-dependent phase (12) and the interactional EM field momentum (22), we have, first of all, to express $\boldsymbol{P}_{EM}$ through the scalar ϕ and the vector $\boldsymbol{A}$ potentials of the external EM field. Adopting the case of a macroscopic source of external EM fields, where the fields $\boldsymbol{E}$, $\boldsymbol{B}$ can be taken as constant vectors in the vicinity of a charged particle, we obtain [25, 26]

$$\boldsymbol{P}_{EM} = \frac{e\boldsymbol{A}}{c} + \frac{\boldsymbol{v}e\phi}{c^2} - \frac{e\boldsymbol{v}(\boldsymbol{A}\cdot\boldsymbol{v})}{c^3}\,. \tag{23}$$

Comparison of eqs. (12) and (23) straightforwardly yields a relationship between the velocity-dependent quantum phase (12) and the interaction EM field momentum (23):

$$¢_{EM}(\boldsymbol{v}) = \frac{1}{\hbar}\int \boldsymbol{P}_{EM} \cdot d\boldsymbol{s} . \tag{24}$$

This equation could not be revealed earlier, when the complementary electric and magnetic A-B phases were unknown. We will show below that this equation, recently found in [26], opens new perspectives in understanding the physical meaning of quantum phase effects for moving charges in the presence of an EM field.

Next, we remind that the quantum phase for a freely moving particle, which we designate as $¢_{free}(\boldsymbol{v})$, is defined by de Broglie relationship [11], *i.e.*

$$¢_{free}(\boldsymbol{v}) = \frac{1}{\hbar}\int \boldsymbol{P}_{M} \cdot d\boldsymbol{s} , \tag{25}$$

where $\boldsymbol{P}_M$ denotes the mechanical momentum of the particle.

Hence, the total velocity-dependent phase of charged particle is given as the sum of eqs. (24) and (25):

$$¢(\boldsymbol{v}) = ¢_{EM}(\boldsymbol{v}) + ¢_{free}(\boldsymbol{v}) = \frac{1}{\hbar}\int (\boldsymbol{P}_{M} + \boldsymbol{P}_{EM}) \cdot d\boldsymbol{s} . \tag{26}$$

Using the general expression for the phase in terms of the wave vector $\boldsymbol{k}$,

$$¢(\boldsymbol{v}) = \int \boldsymbol{k} \cdot d\boldsymbol{s} , \tag{27}$$

and comparing eqs. (26) and (27), we obtain: $\boldsymbol{k} = (\boldsymbol{P}_M + \boldsymbol{P}_{EM})/\hbar$.

Therefore, the wavelength of a charged particle moving in the EM field is equal to [26]

$$\bar{\lambda} = \frac{\hbar}{|\boldsymbol{P}_M + \boldsymbol{P}_{EM}|} . \tag{28}$$

This equation shows that the de Broglie wavelength of a charged particle depends not only on its mechanical momentum $\boldsymbol{P}_M$, but rather on the modulus of the vector sum of the mechanical and EM momenta[2]. This means, in particular, that at constant $\boldsymbol{P}_M$, when no net force acts on the charge, its de Broglie wavelength may nevertheless change due to a variation of the interactional EM field momentum $\boldsymbol{P}_{EM}$.

As we will see in the next section, eq. (28) plays an important role in further clarification of the physical meaning of quantum phase effects for charged particles, moving in an EM field.

Now we emphasize the key point, which is that the new phases (13) and (14) could in no way be obtained using the Schrödinger equation with the commonly used Hamiltonian (1); the latter, as is known, yields only electric (2) and magnetic (3) A-B phases. Moreover, the Dirac equation and the Klein-Gordon equation too, fail yielding complementary A-B phases (13) and (14), when the standard definition (1) for the momentum operator is implied [10].

The incapacity of the known fundamental equations of quantum mechanics to derive the complementary electric (13) and magnetic (14) A-B phases becomes especially crucial in the situation, where the A-C phase for a moving magnetic dipole is revealed in the experiment [16], which, in the view of an equivalent presentation of the A-C phase through the scalar potential (17b), can also be considered as an experimental confirmation of the complementary electric A-B phase. These facts do validate the entire approach based on the derivation of quantum phase effects for dipoles (9) through the covariant Lagrangian (8a) with a further application of the "superposition principle" for quantum phases, and endorse the existence of both complementary electric (13) and magnetic (14) A-B phases.

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[2]We notice that eq. (28) also holds for the bound electron in hydrogenlike atoms with the appropriate expression for the interactional EM field momentum [28, 29].

In this situation, one can realize that the disclosure of new phases (13) and (14) requires, in general, to modify the Hamiltonian of a charged particle in view of eq. (2), and such a modification should have a general character and equality applicable to both the weak relativistic limit (corresponding to the Schrödinger equation for a charged particle in an EM field) and to the general relativistic case (covered by the Dirac and the Klein-Gordon equations). As is known, the covariance of the latter equations is ensured, amongst other things, by the known fact that the scalar potential and vector potential entering into these equations constitute a four-vector. One more four-vector, characterizing the properties of the EM field is the combination of the interactional EM energy U_{EM} and the interactional EM momentum $\boldsymbol{P}_{EM}$ of the system “charged particle in an EM field”. In the general case the interactional EM energy is presented as the sum of the electric and magnetic components, and, as is known (see, e.g. [27]), the corresponding Hamiltonian does not include the component of the interactional magnetic energy. From the physical viewpoint, this reflects the known result of classical electrodynamics as to the magnetic force does not do work. Taking further into account that the interactional electric energy is already presented in all fundamental equations of quantum mechanics as the term $e\ddot{}$, one can thus replace the term with the vector potential ($e\boldsymbol{A}/c$) by the interactional EM momentum $\boldsymbol{P}_{EM}$ without violating the covariance of these equations.

Such a replacement can be formally made by re-defining the momentum operator for the system “charged particle in an EM field”, where instead of the old customary definition of this operator through the canonical momentum of a charges particle $\boldsymbol{P}_c$, i.e.

$$\boldsymbol{P}_c = \boldsymbol{P}_M + \frac{e\boldsymbol{A}}{c} \to \hat{\boldsymbol{P}} = -i\hbar\nabla \, , \tag{29}$$

we associate this operator with the vector sum of the mechanical momentum $\boldsymbol{P}_M$ and the interactional EM momentum $\boldsymbol{P}_{EM}$ [10, 25], i.e.

$$\boldsymbol{P}_M + \boldsymbol{P}_{EM} \to \hat{\boldsymbol{P}} = -i\hbar\nabla \, . \tag{30}$$

Thus, using the new momentum operator (30) we obtain the Hamiltonian of the Schrödinger equation in the form (15) instead of eq. (1) derived with the momentum operator (29).

Further, using the explicit presentation of the EM field momentum (23) via the EM potentials, we arrive at a new expression for the Hamiltonian

$$H = \frac{1}{2M}\left(\boldsymbol{P} - \frac{e\boldsymbol{A}}{c} - \frac{\boldsymbol{v}e\ddot{}}{c^2} + \frac{e\boldsymbol{v}(\boldsymbol{A}\cdot\boldsymbol{v})}{c^3}\right)^2 + e\ddot{} \, , \tag{31}$$

where all variables are considered as operators.

Assuming the Coulomb gauge, where the operators $\boldsymbol{v}$ and $\boldsymbol{A}$ commutate with each other (see, e.g. [11]) and adopting the weak relativistic limit, where the equation $\boldsymbol{P} = M\boldsymbol{v}$ can be used, we obtain with an accuracy c^{-3}:

$$H = -\frac{\hbar^2}{2M}\Delta + e\ddot{} - \frac{e\boldsymbol{A}\cdot\boldsymbol{v}}{c} - \frac{e\ddot{}\, v^2}{c^2} + \frac{ev^2(\boldsymbol{A}\cdot\boldsymbol{v})}{c^3} \, . \tag{32}$$

Here we have neglected the term $e^2A^2/2Mc^2$ in comparison with other terms of eq. (37), which is justified in any practical situation.

Next, substituting Hamiltonian (32) into eq. (2), we obtain the total phase of a charged particle associated with its motion in the EM field, as

$$\text{¢}_{EM} = -\frac{e}{\hbar}\int \ddot{}\, dt + \frac{e}{\hbar c}\int \boldsymbol{A}\cdot d\boldsymbol{s} + \frac{e}{\hbar c^2}\int \ddot{}\,\boldsymbol{v}\cdot d\boldsymbol{s} - \frac{e}{\hbar c^3}\int \boldsymbol{v}(\boldsymbol{A}\cdot\boldsymbol{v})\cdot d\boldsymbol{s} \, . \tag{33}$$

The first and second terms on the rhs stand for the electric and magnetic A-B phases, while the third and fourth terms describe the complementary electric and magnetic A-B phases, correspondingly.

By such a way, the total phase (33) derived from the Schrödinger equation with a new definition of the momentum operator (2), contains all the fundamental quantum phases for point-like charges indicated in Fig. 1, which are harmonized with the corresponding quantum phases for the moving dipoles entering into Eq. (16) through the superposition principle for quantum phases [24, 25] resulting from the linearity of the fundamental equations of quantum mechanics.

Further on, we emphasize that both definitions of the momentum operator (29) and (30) represent, in fact, alternative basic postulates of quantum mechanics, which can only be verified experimentally. In this regard, the problem of observing complementary A-B phases (13) and (14) acquires a fundamental significance.

Discussing possible ways of experimental observation of complementary A-B phases, we notice that they are v^2/c^2 times smaller than the corresponding electric (4) and magnetic (5) A-B phases at all other equal conditions. Hence, in the non-relativistic case, the electric and magnetic A-B phases strongly dominate over the complementary A-B phases, and the latter become practically not observable.

Experimental evidences of the need to redefine the momentum operator (2) can be obtained, at least in principle, by measuring the magnetic A-B phase for relativistic electrons, moving at a velocity v tending to c in a region with a non-vanishing vector potential $\boldsymbol{A}$, and ¨=0. Then, one can see from Eq. (12) that the total phase (the sum of the magnetic A-B phase and the complementary magnetic A-B phase) tends to zero at $v \to c$. Therefore, the decrease of the total magnetic A-B phase with an increase of the velocity of relativistic electrons should be interpreted as an unambiguous proof towards the existence of the complementary magnetic A-B phase. In turn, this result should once again demonstrate the need to re-define the momentum operator in the form (2).

A similar argumentation is applicable to the electric A-B phase for relativistic electrons moving in a region with a non-vanishing scalar potential ¨ and $\boldsymbol{A}$=0, where the total phase (the sum of the electric A-B phase and the complementary electric A-B phase) tends to zero when $v \to c$.

Nowadays, both kinds of such experiments with relativistic electrons look technically complicated, and their detailed discussion lies outside the scope of the present paper.

In the non-relativistic limit, the existence of complementary electric (13) and complementary magnetic (14) A-B phases can be confirmed indirectly through the corresponding measurement of quantum phases for moving dipoles, as indicated in Fig. 1, provided that the validity of the superposition principle for quantum phases is demonstrated in experiments.

From this angle of view, the recent experiment [17], where the HMW phase ¢$_{pB}$ for electric dipoles has been observed, acquires a fundamental importance as the practical validation of the superposition principles for quantum phases, given that the HMW phase is composed from the known magnetic A-B phases summarized over all charges of the electric dipole (see eq. (18)).

This fundamentally important result, along with the successful observation of the A-C phase [16], unambiguously indicates the existence of the complementary electric A-B phase (13). Since the latter phase can be derived from the Schrödinger equation only with a new definition of the momentum operator (2), we can already consider the result [16] *as the first experimental confirmation of the new form (2) for the momentum operator.*

Further experimental evidence with respect to the existence of complementary A-B phases (13), (14) can be obtained by measuring the quantum phases ¢$_{pE}$ (17c) and ¢$_{mB}$ (17d), which are directly related to the complementary A-B phases as indicated in Fig. 1.

In particular, the phase ¢$_{pE}$ (17c) can be revealed in the quantum interference of molecules with a large electric dipole moment, and in ref. [22] we suggested, as an example, a molecule BaS, which meets this requirement. The corresponding numerical estimation gives a typical value δ_{pE} ≈10 mrad under realistic experimental conditions [22], which is comparable with the A-C and HMW phases revealed in refs. [16, 17].

The measurement of the phase δ_{mB} can be carried out using neutron interferometry, and under realistic experimental conditions it can achieve a value near 30 mrad [22].

The estimated values of the phases δ_{pE} and δ_{mB} are comparable with the A-C and HMW phases observed in experiments [16, 17], which makes their measurement topical, provided that they are reliable distinguished from the Stark phase [30] (for δ_{pE}) and the Zeeman phase [31] (for δ_{mB}).

At the same time, we once again highlight the importance of experiments for the direct measurement of the complementary electric (13) and complementary magnetic (14) A-B phases; until such experiments are not performed, the existence of complementary phases should be taken as a conjecture.

We also emphasize that new measurements of quantum phase effects for charges and dipoles moving in an EM field represent an important source of information regarding the choice of the correct definition of the momentum operator.

This claim is gained by the fact that realizations of the A-B effect, e.g., in gravitational field (see, e.g., [32]), or in photonic systems [33] cannot contribute to the solution of this problem. This is related to the fact that in the phase factor for the wave function $e^{i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}$, the presence of gravity affects the time rate t and thus influences on the term ωt, whereas the velocity-dependent complementary electric and magnetic A-B phases are determined by the variation of the wave vector $\boldsymbol{k}$.

Under observation of the photonic A-B effect (see, e.g. [33] and references therein), both the frequency ω and the wave vector $\boldsymbol{k}$ of photons are modulated, though the physical mechanisms of such modulation substantially differ from the mechanism of variation of quantum phase for electrically charged particles, which is analyzed in the next section.

4 On the physical meaning of quantum phase effects for charged particles in an EM field

Analyzing the physical meaning of quantum phase effects for charged particles with the inclusion of complementary electric and complementary magnetic A-B phases, we address the expression for the total quantum phase (33), which can be presented in the equivalent form

$$\phi_{EM}=-\frac{e}{\hbar}\int\varphi\,dt+\phi_{EM}(\boldsymbol{v})=-\frac{e}{\hbar}\int\varphi\,dt+\frac{1}{\hbar}\int\boldsymbol{P}_{EM}\cdot d\boldsymbol{s} \qquad (34)$$

with taking into account eqs. (12) and (24).

Further on, using the known expression for the interactional electric energy for a charged particle e located in a static electric field $\boldsymbol{E}$ [27]

$$e\varphi=\frac{1}{4\pi}\int_V\boldsymbol{E}\cdot\boldsymbol{E}_e dV \qquad (35)$$

(where $\boldsymbol{E}_e$ stands for the electric field of the particle), and the expression for the interactional EM field momentum (22), we reveal that the total phase (34) can be expressed exclusively in terms of external electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields, without explicit introducing the corresponding scalar and vector potentials:

$$\phi_{EM}=-\frac{1}{4\pi\hbar}\int\int_V\boldsymbol{E}\cdot\boldsymbol{E}_e dVdt+\frac{1}{4\pi c\hbar}\int\int_V(\boldsymbol{E}\times\boldsymbol{B}_e)\cdot d\boldsymbol{s}dV+\frac{1}{4\pi c\hbar}\int\int_V(\boldsymbol{E}_e\times\boldsymbol{B})\cdot d\boldsymbol{s}dV. \qquad (36)$$

Thus, we express the quantum phase ϕ_{EM} for a charged particle exclusively in terms of external electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields, without explicit introducing the corresponding scalar and vector potentials.

The new form (36) for the total quantum phase suggests to renew the discussion about the manifestations of quantum phase effects and their physical meaning, which is carried out below.

In the subsequent analysis, we first address the recent discussion about the possibility of an alternative understanding of the A-B effect through the non-local interaction of gauge-independent quantities, proposed in [6], instead of its common interpretation through local gauge-dependent quantities, such as scalar and vector potentials [1, 2].

In particular, in [6] it was assumed that even in the case when the net force on the electron on behalf of a source of the EM field is equal to zero, the local action of the field of the electron on the charged particles of a source is capable to explain the non-vanishing A-B phase due to the mechanism of quantum entanglement, described through the overall wave function for the electron and the source. In support of this claim, some particular examples are considered in [6], dealing with the electric A-B effect and with the magnetic A-B effect, correspondingly, which yield correct expressions for quantum phases through quantum entanglement.

Addressing the magnetic A-B effect, Vaidman suggested introducing into the Mach-Zehnder interferometer a solenoid of a special configuration, consisting of two non-conducting cylinders of radius *r*, large length *L* and opposite charges *Q* and –*Q*, which are homogeneously distributed on their surfaces. It is assumed that the cylinders rotate in opposite directions without friction, and thus, when an electron enters one arm of the circumference co-axial to the solenoid, it induces torques of opposite sign on each cylinder. The torque, being integrated for each cylinder, induces a phase in the electron wave function, which numerically coincides with the observed A-B phase (for details of calculations, see [6]).

Thus, according to [6], the magnetic A-B effect can be interpreted in term of a gauge-independent interaction through an electric field with the introducing of non-local quantum entanglement of an electron and a solenoid.

However, as pointed out in ref. [34], Vaidman missed the principal problem of how a phase generated at a solenoid becomes detectable on an electron. The authors of [34] show that the latter problem can be consistently solved when we abandon the approximation of classical field, and consider a quantizied EM field. This way, they show that "…*the A-B phase is mediated locally by the entanglement between the charge and the photon*…", and predict a gauge-invariant value for the phase difference at each point along the electron's path. This leads the authors of [34] to conclude that for any two points – which they designate as $\boldsymbol{r}_L$ and $\boldsymbol{r}_R$ – belonging to two branches of the electron path, there exists a gauge-independent phase difference that is measurable, at least in principle.

At the same time, the authors of ref. [34] do not comment on the work by Aharonov *et al.* [7], where the entire idea of Vaidman to explain the origin of the magnetic A-B effect by the quantum entanglement of the electron and the source was strongly criticized. In particular, the authors of [7] proposed to introduce an electromagnetic superconducting shield between the solenoid and the electron, which screens the solenoid from the electric and magnetic fields of the electron, when it passes near the solenoid along a semicircular orbit. Consequently, any torque on each cylinder of the solenoid disappears, and no phase emerges in the wave function of the electron and cylinders. At the same time, the vector potential $\boldsymbol{A}$ generated by the solenoid at the location of the electron remains practically unchanged in the presence of a shield, so that the magnetic A-B effect is present even in the absence of any torque on the cylinders of the solenoid due to the moving electron [7].

This observation shows that, as minimum, the idea by Vaidman to explain the magnetic A-B phase via quantum entanglement cannot be universal. Based on this result, the authors of [7] came to the conclusion that the original interpretation of the magnetic A-B effect through the local action of the vector potential of the solenoid on the electron [1, 2] remains in force.

We can add that the criticism of Aharonov *et. al.* [7] regarding the approach by Vaidman [6] remains also in force with respect to the work [34], where no comments were made with respect to the case of an electrically shielded solenoid.

Now, we suggest to reanalyze the magnetic A-B effect on the basis of eq. (36), which for an infinitely long solenoid and an orbiting electron, is simplified to the form

$$\text{¢} = \frac{1}{4\text{®} c\hbar}\int\int_V \left(\boldsymbol{E}_e \times \boldsymbol{B}\right)\cdot d\boldsymbol{s}dV \,, \quad (37)$$

where $\boldsymbol{B}$ stands for the magnetic field of the solenoid. One can show [26] that eq. (37) can be presented in the form

$$\varphi = \frac{e}{c\hbar\gamma^2}\int_s \boldsymbol{A}\cdot d\boldsymbol{s} \approx \frac{e}{c\hbar}\int_s \boldsymbol{A}\cdot d\boldsymbol{s} \tag{38}$$

for a non-relativistic electron, which coincides with the standard expression for the magnetic A-B effect.

Further, we emphasize that involvement of a superconducting shield between the solenoid and the electron leads to the emergence of an electric $\boldsymbol{E}_{shield}=-\boldsymbol{E}_e$ and a magnetic $\boldsymbol{B}_{shield}=-\boldsymbol{B}_e$ fields in its inner volume, including the location of the solenoid. Thus, any force on the solenoid due to a moving electron (resulting from the time variation of the "hidden momentum" of the solenoid (see, e.g., [18-20]), and not mentioned in [6]) and any torque on its cylinders disappears. At the same time, the emergence of the fields $\boldsymbol{E}_{shield}$, $\boldsymbol{B}_{shield}$ does not affect the interactional EM field momentum for the solenoid and the orbiting electron

$$\boldsymbol{P}_{EM} = \frac{1}{4\pi c}\int_V (\boldsymbol{E}_e \times \boldsymbol{B})dV , \tag{39}$$

and does not affect the quantum phase (37).

One should mention that eq. (39) still does not determine the total interactional EM field momentum in the presence of a superconducting shield; we have also to take into account the components of the interactional EM field momentum, determined by the vector products $\boldsymbol{E}_e \times \boldsymbol{B}_{shield}$ and $\boldsymbol{E}_{shield} \times \boldsymbol{B}_e$, which, in general, are not equal to zero inside the shield. At the same time, due to the equalities $\boldsymbol{E}_{shield}=-\boldsymbol{E}_e$ and $\boldsymbol{B}_{shield}=-\boldsymbol{B}_e$ in the inner volume of the shield, we find that in any practical situation the sum $\boldsymbol{E}_e \times \boldsymbol{B}_{shield} + \boldsymbol{E}_{shield} \times \boldsymbol{B}_e = -2\boldsymbol{E}_e \times \boldsymbol{B}_e$ is much smaller than the product $\boldsymbol{E}_e \times \boldsymbol{B}$ in eq. (39), so that it can be neglected.

Thus, eqs. (37) and (38) remain applicable to the description of the magnetic A-B effect regardless of the presence (or absence) of any shielding of the solenoid. The physical meaning of these equations is closely related to the generalized de Broglie relationship (28), which indicates the dependence of the de Broglie wavelength not only on the mechanical momentum of a charged particle, but also on the interactional EM field momentum for the system "charged particle and a source of EM field". The situation looks as if a charged particle instantly "senses" the interactional EM field momentum at any fixed time moment through variation of its de Broglie wavelength (28), even if such an EM field momentum is, in general, distributed over the entire free space.

Here, it is important to emphasize that the latter observation still does not violate classical causality due to the known fact that the velocity-dependent (bound) EM field always represents a function of the state of its source charges [35], which in the particular case of a uniformly moving charge is expressed by the Heaviside solution of Maxwell equations [27, 36]. According to this solution, the electric and magnetic fields measured at the four-point (t, $\boldsymbol{r}$) generated by a uniformly moving source charge located at the four-point (t, $\boldsymbol{r}_e$), can be presented as the corresponding functions of the difference of the present coordinates $|\boldsymbol{r}(t)-\boldsymbol{r}_e(t)|$ and the charge velocity $\boldsymbol{v}$. However, this still does not violate the classical causality, and does not imply any kind of non-local interaction, because for a uniformly moving charge, its electric and magnetic fields at each spatial point $\boldsymbol{r}$ at a given time moment t can always be expressed through the retarded spatial coordinates of the charge, since at constant $\boldsymbol{v}$, the present and retarded coordinates are unambiguously related to each other (see, e.g., [27, 36]). Therefore, the possibility of describing the EM fields of a uniformly moving charge in the present time coordinates should be considered only as a mathematical artifact [27, 36].

Accordingly, the interactional EM field momentum (22) for the system "uniformly moving charged particle and the source of the EM field" can be expressed both through the present and the retarded coordinates of the charge. The first way is definitely convenient from a mathematical viewpoint; nevertheless, the physically meaningful presentation of the interactional EM momentum is to be achieved via retarded coordinates.

The latter option becomes especially important in the analysis of the physical meaning of the generalized de Broglie relationship (28), which naturally explains the phase shifts of the wave functions of a moving charge by the change of its wavelengths versus $\boldsymbol{P}_{EM}$, even if the mechanical momentum $\boldsymbol{P}_M$ in eq. (28) remains constant (no net force of the charge) Here, it is obvious that the calculation of the spatial integrals in eq. (22), defining $\boldsymbol{P}_{EM}$ in the present time coordinates, is convenient from the mathematical viewpoint. Nevertheless, in the analysis of the physical meaning of eq. (28), one should keep in mind that, at the fundamental level, the interactional EM field momentum for a moving charged particle in the presence of an external EM field represents the function of its retarded spatial coordinated, where the electric $\boldsymbol{E}_e$ and magnetic $\boldsymbol{B}_e$ fields of the charge, being evaluated in different present coordinates $\boldsymbol{r}$ and a fixed time moment t in the integral (22), correspond to the suitable retarded coordinates $\boldsymbol{r}'$ taken at different retarded times t'. Here, we specially emphasize that any event (t, $\boldsymbol{r}$) is always related to the corresponding event (t', $\boldsymbol{r}'$) via the time-like space-time interval, which implies no non-locality.

This result means, in particular, that at a constant or at slowly varied velocity of a charged particle, where the radiation losses of the particle are negligible, and its bound EM field represents the function of state [35] (which is always the case in observations of A-B phases), the generalized de Broglie relationship (28), being closely related to eq. (12) for a velocity-dependent quantum phase, does not mean the violation of classical causality. Therefore, the equivalent presentation of the velocity-dependent quantum phase in terms of the electric and magnetic fields in the last two terms of eq. (36) cannot be interpreted as a manifestation of a non-local effect, even if the interactional EM field momentum determining this phase is distributed over the entire space.

Realizing this fact, it becomes obvious that a formal mathematical operation – where the interactional EM field momentum (22), defined as the function of EM fields, is modified to eq. (23) expressed via the scalar ¨ and the vector $\boldsymbol{A}$ potentials – cannot affect the physical meaning of the velocity-dependent component of the A-B phase (24), and only highlights its essence as a local effect, regardless of a particular presentation of interactional EM field momentum either through the EM fields (22) distributed over the entire space, or through the EM potentials at the location points of the moving charge (23).

Next, we consider the electric A-B phase, which corresponds to the first term on the rhs of eq. (36). We would like to re-emphasize that in the absence of any force on the moving charge – which, in fact, represents the necessary condition for observation of quantum phase effects – the electric field $\boldsymbol{E}_e$ of a uniformly moving electron consists of the velocity-dependent (bound) component alone, which, as we have mentioned above, represents the function of state [35]. Therefore, the volume integral $\int_V \boldsymbol{E}\cdot\boldsymbol{E}_e dV$ in eq. (36) admits it evaluation in the present spatial coordinates. At the same time, one should keep in mind that for a uniformly moving electron the present time coordinates of its electric field $\boldsymbol{E}_e$ are unambiguously linked with the corresponding retarded coordinates and thus, the presentation of the electric A-B phase as a function of the fields $\boldsymbol{E}_e$, $\boldsymbol{E}$ in the first term of (36) does not contradicts classical causality and, therefore, does not means any violation of locality in the manifestation of this effect. This is especially clearly seen through Eq. (34), where the scalar potential ¨ is defined at the location point of the source charge e. Thus, like in the case of the velocity-dependent quantum phase, which allows two mathematically equivalent presentations through EM fields and potentials on the basis of eqs. (22), (23), the electric A-B phase also admits two mathematically equivalent presentations through electric fields and scalar potential through eqs. (36), (34), and can be interpreted as a local effect, even if the integration of the product $\boldsymbol{E}\cdot\boldsymbol{E}_e$ in the first term on the rhs of this equation is carried out over the entire free space at a fixed present time moment.

The common interpretation of the electric A-B effect in terms of local interaction was questioned in [6]. Like in the case of the magnetic A-B effect, Vaidman argued [6] that that the electric A-B effect can be understood through the quantum entanglement of the source of the electric field/potential and the electron due to the force exerted by the electron on a source of electric field in a situation, where the reactive force of the source on the electron is equal to zero.

As an example, the author [6] proposed to consider a special configuration of the Mach-Zehnder interferometer, where an electron *e*, propagating along on its arm, falls into some "potential hole" and spends some time *t*=² there. Simultaneously, two source particles with the same charge *Q* are rapidly approaching the electron from opposite sides and stop at equal distances *r* from the electron, remaining at rest during the time interval *T*<². In this configuration, the net force acting on the electron due to both source charges is always equal to zero, whereas the force on each source charge due to the electron has the equal value and opposite sign for each charge.

Therefore, as Vaidman shows [6], the overall wave function acquires a phase

$$¢ = \frac{-2eQT}{r\hbar}, \tag{40}$$

which is equal to the phase of the electric A-B effect for the considered configuration.

This approach by Vaidman was again criticized by Aharonov et al. [7] on the basis of a number of particular examples, where a moving electron creates no force on the source of electric field/potential, but the electric A-B phase definitely emerges. Thus, examples [7] show that the idea by Vaidman [6] to explain the origin of the electric A-B effect by the quantum entanglement of the source of the electric field and the electron is at least not universal.

What is more, one can straightforwardly show that eq. (40) obtained in [6] for the electric phase does agree with our general expression for the quantum phase (36) and corresponds to its first term on the rhs, when we deal with a static or quasistatic case.

Thus, the general expression for the quantum phase (36) remains well applicable to all problems dealing with both the electric and magnetic A-B effect, presented in [7] for demonstration of the failure of quantum entanglement [6] to explain these effects.

5. Conclusion

1. In sect. 1, 2 and partially in sect. 3, we shortly presented, for the convenience of the readers, our earlier results [10, 21, 22, 24-26] obtained in the study of quantum phase effects for electrically charged particles, which can be summarized as follows:

- On the basis of the explicitly covariant expression (8) for the Lagrangian density of the material medium in an EM field [21, 22], we derived the complete quantum phase (9) for a moving electric/magnetic dipole, which contains two previously known A-C and HMW phases (the first and second terms on the rhs of eq. (9)), and two new terms described by the third and the fourth terms on the rhs of eq. (9).

- In the analysis of quantum phase effects (9) for the moving dipoles, due to the linearity of the Schrödinger equation, we adopted that each quantum phase for a moving dipole should be composed of the corresponding quantum phases for point-like charges of the dipole [21, 22]. In this way, we have shown that the application of the "superposition principle for quantum phases" to eq. (9) allows us to disclose two new quantum phases for point-like charges, next to the known electric and magnetic A-B phases, which we named as complementary electric and magnetic A-B phases, correspondingly.

- The disclosure of complementary A-B phases does require to re-define the momentum operator of an electrically charged particle in an EM field through the vector sum of the mechanical momentum $\boldsymbol{P}_M$ and the interactional EM field momentum $\boldsymbol{P}_{EM}$ for the system "charged particle in an EM field" [10, 25], instead of the previous definition of this operator through the canonical momentum of a charged particle.

2. In the analysis of the physical meaning of quantum phase effects for electrically charged particles, we distinguished the velocity-independent quantum phase of electric A-B effect $¢_e$ (see

eq. (3)) and the velocity-dependent component of quantum phase $¢_{EM}(\boldsymbol{v})$ defined by eq. (12). We have shown in sect. 3 that the phase $¢_{EM}(\boldsymbol{v})$ is directly related to the interactional EM field momentum for the system "charged particle in an external EM field" through eq. (24). This finding, along with the known expression (25) for the phase of a freely moving particle, allows us to arrive at the generalized de Broglie relationship (28) for the wavelength of a charged particle, moving in an EM field. The obtained relationship (28) indicates that the de Broglie wavelength of the moving charge is inversely proportional to the modulus of the vector sum of the mechanical momentum $\boldsymbol{P}_M$ of the particle and the interactional EM field momentum $\boldsymbol{P}_{EM}$ (22). Thus, in the particular case of a constant $\boldsymbol{P}_M$, when no force acts on the particle, its wavelength can nevertheless change due to variation of the EM field momentum $\boldsymbol{P}_{EM}$.

3. The latter finding occurs crucial in the analysis of physical meaning of quantum phase effects for moving charges. According to the customary viewpoint, the quantum phase of a charged particle is affected by the direct local action of the scalar ¨ and vector $\boldsymbol{A}$ potentials at its location, and the explanation of this effect is based on the claim that a charged particle "feels" EM potentials at its location. A recent attempt to interpret the A-B effect in terms of the non-local action of gauge-independent quantities in the system "charge and source of EM field" [6] and the further development of this idea for a quantized EM field [34] still encounter difficulties in their extension to the case when the elongated solenoid is electrically shielded [7]. Now we clarify the physical meaning of quantum phase effects for charged particles through the dependence of their de Broglie wavelength on the interactional EM field momentum.

4. For the system "charged particle in an EM field" the interactional EM field momentum $\boldsymbol{P}_{EM}$ can be expressed either in terms of the EM fields (22), or in terms of EM potentials (23). Despite the mathematical equivalence of eqs. (22) and (23), one should emphasize that, from the physical viewpoint, the interactional EM field momentum is distributed, in general, over the entire free space, whereas the scalar potential ¨ and the vector potential $\boldsymbol{A}$ entering into eq. (23) are defined at the location point of the charge.

In this situation, it is important to emphasize that for the motion of a charge with a constant or slowly varying velocity – which is always the case in observation of A-B phases – the radiation losses of the charge are negligible, and its electric $\boldsymbol{E}_e$ and magnetic $\boldsymbol{B}_e$ fields in eq. (22) for interactional EM field momentum consist only of velocity-dependent (bound) components. The latter, as is known [27, 35, 36], can be expressed through the present spatial coordinates of the charge. This fact, however, does not affect the local origin of quantum phase effects, because at constant $\boldsymbol{v}$, the present and retarded coordinates of a moving charge are unambiguously related to each other, so that the evaluation of both integrals on the rhs of eq. (22), defining the interactional EM field momentum in the present time coordinates, is equivalent to its evaluation at suitable retarded coordinates $\boldsymbol{r}'$ at different retarded moments t', and each event $(t, \boldsymbol{r})$ is related to the corresponding event $(t', \boldsymbol{r}')$ via a time-like space-time interval in a full agreement with classical causality. This observation once again confirms the local character of quantum phase effects, whether they are expressed through mathematically equivalent eq. (33) (dealing with potentials ¨, $\boldsymbol{A}$), or eq. (36) (dealing with the fields $\boldsymbol{E}$, $\boldsymbol{B}$).

Finally, we considered a number of particular examples [6, 7] aimed to calculate the A-B phases in various situations, and confirmed the universal character of mathematically equivalent eqs. (33) and (36) in the description of quantum phase effects for charged particles in EM fields.

Data availability statement

All data generated or analyzed during this study are included in this published article.

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