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Quantum phase effects for electrically charged particles and redefinition of the momentum operator

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Abstract. We analyze the physical meaning of quantum phases for moving electric/magnetic dipoles through a superposition of quantum phases for point-like charges of the dipole, and in this way we disclose two new quantum phases for moving charges, next to the well-known magnetic and electric Aharonov-Bohm phases. We find that a consistent description of the full set of quantum phase effects for charged particles requires to modify the standard definition of the momentum operator via the canonical momentum of a particle towards a more inclusive definition, where this operator is associated with the vector sum of mechanical and interactional electromagnetic (EM) momenta for a charged particle in the EM field. Some principal implications of this result are discussed.

1. Introduction

It is known that in the middle of 20th century, Aharonov and Bohm predicted two quantum phase effects for electrons in an electromagnetic (EM) field [1,2]: the electric Aharonov-Bohm (A-B) effect, characterized by a phase

$$\delta_\varphi = -\frac{e}{\hbar}\int \varphi dt \tag{1}$$

as well as the magnetic Aharonov-Bohm (A-B) effect with a phase

$$\delta_A = \frac{e}{\hbar c}\int_L \boldsymbol{A}\cdot d\boldsymbol{s}\,. \tag{2}$$

Hereinafter φ, $\boldsymbol{A}$ stand for scalar and vector potentials, correspondingly, $d\boldsymbol{s}=\boldsymbol{v}dt$ is the path element of the charge e along the line L, c is the light velocity in vacuum, and $\hbar$ is the reduced Planck constant.

A few decades later, Aharonov and Casher predicted a quantum phase effect for a moving magnetic dipole $\boldsymbol{m}$ in the presence of an electric field $\boldsymbol{E}$, which is given by the equation [3]

$$\delta_{mE} = \frac{1}{\hbar c}\int_L \left(\boldsymbol{m}\times\boldsymbol{E}\right)\cdot d\boldsymbol{s}\,, \tag{3}$$

while He, McKellar [4] and Wilkens [5] derived a corresponding quantum phase for a magnetic dipole $\boldsymbol{p}$, moving in a magnetic field $\boldsymbol{B}$,

$$\delta_{pB} = -\frac{1}{\hbar c}\int_L (\boldsymbol{p}\times\boldsymbol{B})\cdot d\boldsymbol{s}\,. \tag{4}$$

At the same time, neither the authors of [3], nor the authors of [4,5] provided a reasonable physical interpretation of equations (3) and (4).

This problem became especially topical after a direct observation of both the Aharonov-Casher (A-C) phase [6,7] and the He-McKellar-Wilkens (HMW) phase [8], and our team seems to have been the first to pin down the actual physical meaning of the phases (3), (4) (see section 2). In this way, we, first of all, derived the classical motional equation for dipoles in an EM field on the basis of an explicitly covariant expression for their Lagrangian. Then, going to the quantum limit, we have revealed two more quantum phase effects for moving dipoles [9,10], next to the A-C and HMW phases.

Further on, in our subsequent publications [11,12] we proposed a new way of understanding the physical meaning of quantum phases for electric/magnetic dipoles as a superposition of quantum phases for point-like charges composing the dipoles. Such a "superposition principle" for quantum phases naturally follows from the linearity of fundamental equations of quantum mechanics, and its application to the description of quantum phase effects for moving dipoles allowed us to find two new fundamental quantum phases for moving charges – next to the known electric and magnetic A-B phases – which we named as complementary electric and complementary magnetic A-B phases, correspondingly [11,12].

Thus, the total quantum phase for a charged particle in an EM field represents the sum of four components, and in section 3 we find a direct relationship between the velocity-dependent quantum phases and the interactional EM field momentum $\boldsymbol{P}_{EM}$ for the system "charged particle in an EM field", which could not be found earlier, when only the magnetic A-B phase (2) was associated with the moving charge. This new result allows us to arrive at a generalized de Broglie relationship, where the wavelength of a moving charged particle is defined by the modulus of the vector sum of mechanical $\boldsymbol{P}_M$ and electromagnetic $\boldsymbol{P}_{EM}$ momenta. This finding naturally explains the phase shifts of the wave functions of a moving charge due to variation of $\boldsymbol{P}_{EM}$ even if the mechanical momentum $\boldsymbol{P}_M$ remains constant in the absence of any net force on the particle.

In section 4, we emphasize that the full set of quantum phase effects we obtained for electric charges cannot be deduced via the Schrödinger equation for a charged particle in an EM field, as well as using the fundamental equations of relativistic quantum mechanics, if the standard definition of the momentum operator in an EM field is applied, i.e.

$$\hat{\boldsymbol{p}} \to \hat{\boldsymbol{p}} - \frac{e\boldsymbol{A}}{c}, \tag{5}$$

where $\hat{\boldsymbol{p}} = -i\hbar\nabla$ is the momentum operator in the absence of EM field.

Indeed, in the non-relativistic case, the corresponding Hamiltonian for a charged particle with mass m and charge e

$$\hat{H} = \frac{(-i\hbar\nabla - e\boldsymbol{A}/c)^2}{2m} + e\varphi \tag{6}$$

yields only A-B phases (1), (2). Moreover, we emphasize that relativistic quantum mechanics also fails to describe the complementary electric and magnetic A-B phases with the customary definition of the momentum operator (5).

Therefore, the disclosure of complementary A-B phases requires re-defining the momentum operator in a wider form, where the term $e\boldsymbol{A}/c$ in equation (5) is replaced by the EM field momentum $\boldsymbol{P}_{EM}$ for the system "charged particle in an EM field". Hence, instead of equation (5), we get [12,13]

$$\hat{\boldsymbol{p}} \to \hat{\boldsymbol{p}} - \boldsymbol{P}_{EM}, \tag{7}$$

and below we will show that equation (7) reduces to (5) only in the non-relativistic limit, where terms of order $(v/c)^2$ and higher are neglected.

Insofar as the Schrödinger equation for a charged particle in the presence of an EM field does contain the terms of order $(v/c)^2$, then it should deal with the momentum operator (7) instead of the old definition (5), and the same re-definition of the momentum operator (7) should be also applied in relativistic quantum mechanics.

In section 5 we discuss the obtained results and indicate their principal implications.

2. Going from classical to quantum description of electric/magnetic dipoles. Quantum phases for dipoles and point-like charges

In our previous papers (e.g., [9,10]) we have shown that all previously known expressions for the force on electric and magnetic dipoles are relativistically inconsistent. In order to obtain a Lorentz-invariant expression for the force, we have used the covariant expression for the Lagrangian density for a magnetized/polarized medium in the presence of an EM field [14]

$$l_{\text{int}} = \frac{1}{2} M^{\alpha\beta} F_{\alpha\beta}, \tag{8}$$

where $M^{\alpha\beta}$ is the magnetization-polarization tensor and $F_{\alpha\beta}$ is the tensor of EM field [14].

Integrating (8) over the volume of a point-like dipole, we arrive at the following Lagrangian [9]:

$$L = -\frac{Mc^2}{\gamma} + \boldsymbol{p} \cdot \boldsymbol{E} + \boldsymbol{m} \cdot \boldsymbol{B}. \tag{9}$$

Further substitution of (9) into the Euler-Lagrange equation

$$\frac{\partial L}{\partial \boldsymbol{r}} = \frac{d}{dt} \frac{\partial L}{\partial \boldsymbol{v}},$$

yields a relativistic expression for the total force on the dipole [9, 10]:

$$\begin{gathered} \boldsymbol{F} = \frac{d}{dt}(\gamma M \boldsymbol{v}) = \nabla(\boldsymbol{p} \cdot \boldsymbol{E}) + \nabla(\boldsymbol{m} \cdot \boldsymbol{B}) + \frac{d}{dt} \frac{\gamma(\boldsymbol{p}_{0//} \cdot \boldsymbol{E})\boldsymbol{v}}{c^2} + \frac{d}{dt} \frac{\gamma(\boldsymbol{m}_{0//} \cdot \boldsymbol{B})\boldsymbol{v}}{c^2} + \\ \frac{d}{dt} \frac{1}{c}(\boldsymbol{p}_0 \times \boldsymbol{B}) - \frac{d}{dt} \frac{1}{c}(\boldsymbol{m}_0 \times \boldsymbol{E}), \end{gathered} \tag{10}$$

where the subscript "$_0$" stands for the proper electric and magnetic dipole moments.

Further, we address to the known expression for the Hamilton function $H = \partial L / \partial \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{L}$, which, in the quantum limit, determines the corresponding Hamiltonian $\hat{H}$. Hence, the total quantum phase of a dipole in the presence of an EM field takes the form [9,10]:

$$\begin{gathered} \delta = \frac{1}{\hbar} \int \hat{H} dt = -\frac{1}{\hbar c^2} \int \gamma(\boldsymbol{p}_{0//} \cdot \boldsymbol{E})\boldsymbol{v} \cdot d\boldsymbol{s} - \frac{1}{\hbar c^2} \int \gamma(\boldsymbol{m}_{0//} \cdot \boldsymbol{B})\boldsymbol{v} \cdot d\boldsymbol{s} + \\ \frac{1}{\hbar c} \int (\boldsymbol{m}_0 \times \boldsymbol{E}) \cdot d\boldsymbol{s} - \frac{1}{\hbar c} \int (\boldsymbol{p}_0 \times \boldsymbol{B}) \cdot d\boldsymbol{s} - \frac{1}{\hbar} \int (\boldsymbol{p} \cdot \boldsymbol{E}) dt - \frac{1}{\hbar} \int (\boldsymbol{m} \cdot \boldsymbol{B}) dt. \end{gathered} \tag{11}$$

In our subsequent analysis, we omit the Stark phase [15] and Zeeman phase [16], which emerge for resting dipoles, and focus on the quantum phases for moving dipoles, presented by the first four terms on the rhs of equation (11).

As shown in [9,10], to the accuracy of calculations c^{-3} (which can be adopted as a weak relativistic limit), the sum of the remaining terms in equation (11) yields

$$\delta \approx -\frac{1}{\hbar c^2}\int (\boldsymbol{p}\cdot\boldsymbol{E})\boldsymbol{v}\cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int (\boldsymbol{m}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{s} + \frac{1}{\hbar c}\int (\boldsymbol{m}\times\boldsymbol{E})\cdot d\boldsymbol{s} - \frac{1}{\hbar c}\int ((\boldsymbol{p}\times\boldsymbol{B}))\cdot d\boldsymbol{s}, \quad (12)$$

where all quantities are evaluated in a laboratory frame.

The third and fourth terms on the rhs of equation (12) stand for the A-C (3) and HMW (4) phases, respectively, while the first and second terms correspond to previously unknown quantum phases. One of them is related to the motion of an electric dipole in an electric field,

$$\delta_{pE} = -\frac{1}{\hbar c^2}\int (\boldsymbol{p}\cdot\boldsymbol{E})\boldsymbol{v}\cdot d\boldsymbol{s}, \quad (13)$$

and the second one is related to the motion of a magnetic dipole in a magnetic field

$$\delta_{mB} = -\frac{1}{\hbar c^2}\int (\boldsymbol{m}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{s}. \quad (14)$$

The next problem is to disclose the physical meaning of the four quantum phases δ_{mE}, δ_{pB}, δ_{pE}, δ_{mB} for moving dipoles. In the solution of this problem, we applied for the first time the "superposition principle" for quantum phase effects, where the observed phase for a dipole represents a superposition of corresponding phases for each of its charge [9,10].

Solving this problem, we have to express the phases for the dipoles through the scalar φ and vector $\boldsymbol{A}$ potentials by analogy with the quantum phases for point-like charges.

We start our analysis with the HMW and A-C phases, which have already been tested experimentally and thus, the results obtained with these phases will be especially convincing.

First, addressing to the HMW phase, we derive:

$$\delta_{pB} = -\frac{1}{\hbar c}\int (\boldsymbol{p}\times\boldsymbol{B})\cdot d\boldsymbol{s} =$$

$$-\frac{1}{\hbar c}\int_V\int_L (\boldsymbol{P}\times\boldsymbol{B})\cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c}\int_V\int_L (\boldsymbol{P}\times(\nabla\times\boldsymbol{A}))\cdot d\boldsymbol{s}dV = \frac{1}{\hbar c}\int_V\int_L \rho\boldsymbol{A}\cdot d\boldsymbol{s}dV. \quad (15)$$

Here, we used the equalities $\boldsymbol{p} = \int_V \boldsymbol{P}dV$ *(where **P** is the polarization, and V the volume of the dipole),* $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, and the vector identity [17]

$$\int_S (\boldsymbol{A}\cdot\boldsymbol{P})d\boldsymbol{S} - \int_S \boldsymbol{P}(\boldsymbol{A}\cdot d\boldsymbol{S}) - \int_S \boldsymbol{A}(\boldsymbol{P}\cdot d\boldsymbol{S}) = \int_V \boldsymbol{A}\times(\nabla\times\boldsymbol{P})dV + \int_V \boldsymbol{P}\times(\nabla\times\boldsymbol{A})dV -$$

$$\int_V \boldsymbol{A}(\nabla\cdot\boldsymbol{P})dV - \int_V \boldsymbol{P}(\nabla\cdot\boldsymbol{A})dV;$$

further on, we have taken into account that the polarization $\boldsymbol{P}$ is vanishing on the boundary surface S of the dipole, so that all integrals on lhs of this identity are equal to zero. Finally, we have used the equalities $\rho = -\nabla\cdot\boldsymbol{P}$ (where ρ is the charge density), $\nabla\times\mathbf{P} = 0$, as well as the Coulomb gauge ($\nabla\cdot\mathbf{A} = 0$).

We clarify the physical meaning of (15) for the simplest model of an electric dipole – two point-like charges $+e$ and $-e$ located at opposite ends of a rigid rod of length d. For such a dipole, equation (15) yields:

$$\delta_{pB} = \frac{e}{\hbar c}\left(\oint_{L_+} \boldsymbol{A}(\boldsymbol{r}+\boldsymbol{d})\cdot d\boldsymbol{s} - \oint_{L_-} \boldsymbol{A}(\boldsymbol{r})\cdot d\boldsymbol{s}\right). \quad (16)$$

where L_+ (L_-) stands for the path of the positive (negative) charge of dipole, and $\boldsymbol{r}$ being the radial coordinate.

Equation (16) signifies that the HMW phase represents the algebraic sum of the magnetic A-B phases (2) for each charge constituting the dipole.

This comfortable result demonstrates the significance of the "superposition principle" for the quantum phases of moving dipoles in elucidating their physical meaning, and the fact of experimental confirmation of the HMW phase mentioned above [8], along with observations of magnetic A-B phase (see, e.g., [18]) does validate the entire approach based on the superposition of quantum phases of point-like charges to compose the observed phases for the dipoles.

We add that equation (16) has already been derived in [19], but the authors were not aimed by then, to disclose its actual physical meaning.

Thus, in a view of the obtained physical interpretation of (16), it becomes important to clarify the meaning of other quantum phases for the dipoles entering into equation (12).

Next, we first address the A-C phase, which, like the HMW phase, has been confirmed experimentally [6,7]. Therefore, the clarification of its physical meaning via the superposition principle for quantum phases is indeed intriguing.

Analyzing equation (3) for the A-C phase, we assume for simplicity that the external electric field does not contain an inductive component and is defined through the gradient of the scalar potential, i.e., $\boldsymbol{E} = -\nabla\varphi$. Substituting this equality into (3), and using the definition $\boldsymbol{m} = \int \boldsymbol{M} dV$ (where $\boldsymbol{M}$ denotes the magnetization), we obtain:

$$\delta_{mE} = -\frac{1}{\hbar c}\int_V\int \left(\boldsymbol{M}\times\nabla\varphi\right)\cdot d\boldsymbol{s}dV =$$

$$\frac{1}{\hbar c}\int_V\int \nabla\times\left(\boldsymbol{M}\varphi\right)\cdot d\boldsymbol{s}dV - \frac{1}{\hbar c}\int_V\int \varphi\left(\nabla\times\boldsymbol{M}\right)\cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c}\int_V\oint \varphi\left(\nabla\times\boldsymbol{M}\right)\cdot d\boldsymbol{s}dV. \quad (17)$$

In the derivation of (17), we used the vector identity $\nabla\times\left(\boldsymbol{M}\varphi\right) = \varphi\nabla\times\boldsymbol{M} - \boldsymbol{M}\times\nabla\varphi$, and took into account that the volume integral $\int_V\oint \nabla\times\left(\boldsymbol{M}\varphi\right)\cdot d\boldsymbol{s}dV$ can be transformed into a surface integral, where the magnetization $\boldsymbol{M}$ is vanishing.

In order to present (17) in a more convenient form, we introduce the current density

$$\boldsymbol{j} = \nabla\times\boldsymbol{M} + \partial\boldsymbol{P}/\partial t, \quad (18)$$

and adopt a stationary polarization, where $\partial\boldsymbol{P}/\partial t = -\left(\boldsymbol{v}\cdot\nabla\right)\boldsymbol{P}$. Thus, we obtain:

$$\delta_{mE} = -\frac{1}{\hbar c^2}\int_V\oint \varphi\left(\boldsymbol{j} + \left(\boldsymbol{v}\cdot\nabla\right)\boldsymbol{P}\right)\cdot d\boldsymbol{s}dV. \quad (19)$$

We clarify the physical meaning of (19) with the simplest model of a magnetic dipole, presented by an electrically neutral current loop with a steady current. For such a dipole, its proper polarization is equal to zero, while the polarization for a moving dipole $\boldsymbol{P} = \boldsymbol{v}\times\boldsymbol{M}_0/c$ is always orthogonal to the vector $d\boldsymbol{s}=\boldsymbol{v}dt$. Hence, equation (19) yields

$$\delta_{mE} = -\frac{1}{\hbar c^2}\int_V\oint \varphi\boldsymbol{j}\cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c^2}\int_V\oint \varphi\rho\boldsymbol{u}\cdot d\boldsymbol{s}dV, \quad (20)$$

where we have used the equality $\boldsymbol{j}=\rho\boldsymbol{u}$.

Equation (20) expresses the A-C phase through the scalar potential, and for determination of its physical meaning we use the model of magnetic dipole adopted above. For this model, the positive charges of the frame of the loop are immovable in the proper frame of the dipole, and their phase is defined as

$$\left(\delta_{mE}\right)_{positive} = -\frac{1}{\hbar c^2}\int_V \oint \varphi\rho_+ \boldsymbol{v}\cdot d\boldsymbol{s}dV . \tag{21a}$$

Considering the contribution of the negative charged (the carries of current), we designate their flow velocity as $\boldsymbol{u}$, and obtain

$$\left(\delta_{mE}\right)_{negative} = -\frac{1}{\hbar c^2}\int_V \oint \varphi\rho_- \left(\boldsymbol{u}+\boldsymbol{v}\right)\cdot d\boldsymbol{s}dV . \tag{21b}$$

Summing equations (21a) and (21b) for an electrically neutral magnetic dipole *(where $\rho_+ = -\rho_-$),* we arrive at phase (20).

Next, we observe that obtained equations (21a-b) are fulfilled only in the case, where the motion of a point-like charge in the field of a scalar potential φ yields the phase

$$\delta_{c\varphi} = \frac{e}{\hbar c^2}\int \varphi\boldsymbol{v}\cdot d\boldsymbol{s} , \tag{22}$$

which we named in references [11,12] as a complementary electric A-B phase, supplying it with the subscript "*c*".

By such a way, we disclose the physical meaning of the A-C phase for a moving magnetic dipole as a superposition of complementary electric A-B phases (22) for all charges of the dipole.

Further on, we consider the remaining phases (13) and (14) for a moving dipole, and again clarify their physical meaning using the superposition principle for quantum phases.

Addressing the phase δ_{pE} (13), we assume as before that the external electric field $\boldsymbol{E}$ does not contain an inductive component (i.e., $\partial\boldsymbol{A}/\partial t = 0$). Hence, using equality $\boldsymbol{E} = -\nabla\varphi$, definition $\boldsymbol{p} = \int_V \boldsymbol{P}dV$, as well as equality $\rho = -\nabla\cdot\boldsymbol{P}$, we derive

$$\delta_{pE} = -\frac{1}{\hbar c^2}\int \left(\boldsymbol{p}\cdot\boldsymbol{E}\right)\boldsymbol{v}\cdot d\boldsymbol{s} = \frac{1}{\hbar c^2}\int_V \oint \left(\boldsymbol{P}\cdot\nabla\varphi\right)\boldsymbol{v}\cdot d\boldsymbol{s}dV =$$

$$\frac{1}{\hbar c^2}\int_V \oint \nabla\cdot\left(\boldsymbol{P}\varphi\right)\boldsymbol{v}\cdot d\boldsymbol{s}dV - \frac{1}{\hbar c^2}\int_V \oint \rho\varphi\boldsymbol{v}\cdot d\boldsymbol{s}dV = -\frac{1}{\hbar c^2}\int_V \oint \rho\varphi\boldsymbol{v}\cdot d\boldsymbol{s}dV . \tag{23}$$

Here, we have used the vector identity $\nabla\cdot\left(\boldsymbol{P}\varphi\right) = \varphi\nabla\cdot\boldsymbol{P} + \boldsymbol{P}\cdot\nabla\varphi$, and taken into account that the volume integral $\int_V \oint \nabla\cdot\left(\boldsymbol{P}\varphi\right)\boldsymbol{v}\cdot d\boldsymbol{s}dV$ can be transformed into a surface integral by the Gauss theorem, where the polarization $\boldsymbol{P}$ vanishes.

In order to clarify the physical meaning of (23), we again use the simplest model of an electric dipole – two point-like particles with an opposite electric charge, which are separated by a small distance $\boldsymbol{d}$. For such a dipole, equation (23) takes the form:

$$\delta_{pE} = -\frac{1}{\hbar c^2}\int_V \int_L \rho\varphi\boldsymbol{v}\cdot d\boldsymbol{s}dV = -\frac{e}{\hbar c^2}\int_{L_+} \varphi\left(\boldsymbol{r}+\boldsymbol{d}\right)\boldsymbol{v}\cdot d\boldsymbol{s} + \frac{e}{\hbar c^2}\int_{L_+} \varphi\left(\boldsymbol{r}\right)\boldsymbol{v}\cdot d\boldsymbol{s} . \tag{24}$$

This equation indicates that the observed quantum phase δ_{pE} for an electric dipole represents a superposition of complementary electric A-B phases $\delta_{c\varphi}$ (22), and once again confirms the necessity of introducing this phase with regards to the explanation of quantum phase effects for moving dipoles.

Finally, we consider the phase δ_{mB} represented by equation (14) and express it through the vector potential $\boldsymbol{A}$, using the equality $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ along with definition $\boldsymbol{m} = \int \boldsymbol{M} dV$ and vector identity $\nabla \cdot (\boldsymbol{M} \times \boldsymbol{A}) = \boldsymbol{M} \cdot (\nabla \times \boldsymbol{A}) + \boldsymbol{A} \cdot (\nabla \times \boldsymbol{M})$. Hence, we derive from (14):

$$\delta_{mB} = -\frac{1}{\hbar c^2}\int_V \oint (\boldsymbol{M} \cdot (\nabla \times \boldsymbol{A}))\boldsymbol{v} \cdot d\boldsymbol{s} dV = \frac{1}{\hbar c^2}\int_V \oint (\nabla \cdot (\boldsymbol{M} \times \boldsymbol{A}))\boldsymbol{v} \cdot d\boldsymbol{s} dV - \frac{1}{\hbar c^2}\int_V \oint (\boldsymbol{A} \cdot (\nabla \times \boldsymbol{M}))\boldsymbol{v} \cdot d\boldsymbol{s} dV = -\frac{1}{\hbar c^2}\int_V \oint (\boldsymbol{A} \cdot (\nabla \times \boldsymbol{M}))\boldsymbol{v} \cdot d\boldsymbol{s} dV, \tag{25}$$

where we have taken into account that the volume integral

$$\int_V \oint (\nabla \cdot (\boldsymbol{M} \times \boldsymbol{A}))\boldsymbol{v} \cdot d\boldsymbol{s} dV$$

is vanishing by the Gauss theorem. For evaluation of the remaining integral (25), we involve equality (18) and again assume that the polarization is stationary, i.e. $d\boldsymbol{P}/dt = \partial \boldsymbol{P}/\partial t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{P} = 0$, and $\partial \boldsymbol{P}/\partial t = -(\boldsymbol{v} \cdot \nabla)\boldsymbol{P}$. Hence, combining equations (18) and (25), one gets:

$$\delta_{mB} = -\frac{1}{\hbar c^3}\int_V \int (\boldsymbol{A} \cdot \boldsymbol{j} + \boldsymbol{A} \cdot (\boldsymbol{v} \cdot \nabla)\boldsymbol{P})\boldsymbol{v} \cdot d\boldsymbol{s} dV. \tag{26}$$

The second integral in (26) can be presented in the form:

$$\int_V \int (\boldsymbol{A} \cdot (\boldsymbol{v} \cdot \nabla)\boldsymbol{P})\boldsymbol{v} \cdot d\boldsymbol{s} dV = \int_V \int (\boldsymbol{A} \cdot (\boldsymbol{v} \cdot \nabla)\boldsymbol{P}) v^2 dt dV = v^2 \int_V \int \frac{d}{dt}(\boldsymbol{A} \cdot \boldsymbol{P}) dt dV - v^2 \int \left(\boldsymbol{p} \cdot \frac{d\boldsymbol{A}}{dt} \right) dt .$$

Therefore, it is vanishing under the natural conditions $\boldsymbol{A}(t=0)=\boldsymbol{A}(t=\infty)=0$, and equation (26) yields

$$\delta_{mB} = -\frac{1}{\hbar c^3}\int_V \int (\boldsymbol{j} \cdot \boldsymbol{A})\boldsymbol{v} \cdot d\boldsymbol{s} dV = -\frac{1}{\hbar c^3}\int_V \int (\rho \boldsymbol{u} \cdot \boldsymbol{A})\boldsymbol{v} \cdot d\boldsymbol{s} dV, \tag{27}$$

where we have used, once again, the equality $\boldsymbol{j}=\rho\boldsymbol{u}$.

Applying equation (27) to the adopted model of a magnetic dipole as an electrically neutral current loop with a steady current, we obtain by analogy with (21a-b):

$$(\delta_{mB})_{positive} = -\frac{1}{\hbar c^2}\int_V \oint \rho_+ (\boldsymbol{v} \cdot \boldsymbol{A})\boldsymbol{v} \cdot d\boldsymbol{s} dV ,$$

$$(\delta_{mB})_{negative} = -\frac{1}{\hbar c^2}\int_V \oint \varphi \rho_- ((\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{A})(\boldsymbol{u} + \boldsymbol{v}) \cdot d\boldsymbol{s} dV .$$

These equations indicate that the phase δ_{mB} for a moving magnetic dipole in the presence of a magnetic field can be presented through a superposition of new quantum phases for point-like charges of the dipole

$$\delta_{cA} = -\frac{e}{\hbar c^3}\int (\boldsymbol{v} \cdot \boldsymbol{A})\boldsymbol{v} \cdot d\boldsymbol{s} . \tag{28}$$

In references [11,12], we have entitled the phase (28) as complementary magnetic A-B phase.

Thus, the application of "superposition principle" for the quantum phases of dipoles and charges allowed us to conclude that the quantum phases δ_{mE}, δ_{pB}, δ_{pE}, δ_{mB} for an electric/magnetic dipole, given by equation (12), originate from three fundamental phases for the moving point-like charges:

- magnetic A-B phase δ_A (2);
- complementary electric A-B phase $\delta_{c\varphi}$ (22);
- complementary magnetic A-B phase δ_{cA} (28).

3. Total quantum phase for moving charges and generalized de Broglie relationship

The obtained results indicate that the total velocity-dependent component of the quantum phase for a moving charge in an EM field is given by the sum of equations (2), (22) and (28):

$$\delta_{EM}(\boldsymbol{v}) = \frac{e}{\hbar c}\int \boldsymbol{A}\cdot d\boldsymbol{s} + \frac{e}{\hbar c^2}\int \varphi\boldsymbol{v}\cdot d\boldsymbol{s} - \frac{e}{\hbar c^3}\int \boldsymbol{v}(\boldsymbol{A}\cdot\boldsymbol{v})\cdot d\boldsymbol{s}. \tag{29}$$

In references [12,20], we revealed that this phase is directly linked with the interactional EM field momentum of the system "charged particle in an EM field":

$$\boldsymbol{P}_{EM} = \frac{1}{4\pi c}\int_V (\boldsymbol{E}\times\boldsymbol{B}_e)dV + \frac{1}{4\pi c}\int_V (\boldsymbol{E}_e\times\boldsymbol{B})dV = \frac{e\boldsymbol{A}}{c} + \frac{\boldsymbol{v}e\varphi}{c^2} - \frac{e\boldsymbol{v}(\boldsymbol{A}\cdot\boldsymbol{v})}{c^3}, \tag{30}$$

where $\boldsymbol{E}_e$, $\boldsymbol{B}_e$ are respectively the electric and magnetic field of the moving charge.

Thus, comparing equations (29) and (30), we obtain the equality

$$\delta_{EM}(\boldsymbol{v}) = \frac{1}{\hbar}\int \boldsymbol{P}_{EM}\cdot d\boldsymbol{s}. \tag{31}$$

This equation, first obtained in [20], opens the key for a new understanding of quantum phase effects for moving charges in the presence of an EM field, which could not be revealed before when only the magnetic A-B phase (2) was associated with the moving charge.

Now, with the disclosure of complementary A-B phases, we can see that the observed quantum phase for moving charges in an EM field is directly defined by the interactional EM field momentum for the system "charged particle in an EM field".

What is more, introducing the wave vector $\boldsymbol{k}_{EM}$, associated with the quantum phase $\delta_{EM}(\boldsymbol{v})$, we can write

$$\delta_{EM}(\boldsymbol{v}) = \int \boldsymbol{k}_{EM}\cdot d\boldsymbol{s}. \tag{32}$$

Thus, comparing equations (31) and (32), we obtain the equality

$$\boldsymbol{k}_{EM} = \frac{\boldsymbol{P}_{EM}}{\hbar}. \tag{33}$$

Next, we remind that for a freely moving charge, its wave vector is defined through the mechanical momentum $\boldsymbol{P}_M$ as

$$\boldsymbol{k}_M = \frac{\boldsymbol{P}_M}{\hbar}. \tag{34}$$

Summing up equations (33) and (34), we arrive at a new expression for the total wave vector k of a charged particle, moving in an EM field:

$$\boldsymbol{k} = \boldsymbol{k}_M + \boldsymbol{k}_{EM} = \frac{\boldsymbol{P}_M + \boldsymbol{P}_{EM}}{\hbar}, \tag{35}$$

which allows to straightforwardly generalize the expression for the de Broglie wavelength of a charged particle, moving in an EM field:

$$\lambda = \frac{h}{\left|\boldsymbol{P}_M + \boldsymbol{P}_{EM}\right|}. \tag{36}$$

Equation (36) indicates that the de Broglie wavelength for such a particle is defined through the modulus of the vector sum of its mechanical momentum $\boldsymbol{P}_M$ and the interactional EM momentum $\boldsymbol{P}_{EM}$. This means, in particular, that at constant $\boldsymbol{P}_M$, when no net force acts on the charge, its de Broglie wavelength may nevertheless change due to the variation of the interactional EM field momentum $\boldsymbol{P}_{EM}$. This result, we believe, sheds light on the actual physical meaning of quantum phase effects.

It is interesting to notice that equation (36) is also fulfilled for a bound electron in hydrogenlike atoms with application of the appropriate expression for the interactional EM field momentum for this system [21,22], which allows assuming that equation (36) has the general character and reflects the deepest laws of quantum mechanics.

4. New form of the momentum operator

The results we obtained above indicate that a moving point-like charge in the presence of an EM field is characterized by three quantum phases: i) the previously known magnetic A-B phase (2), ii) as well as the new complementary electric (21), and iii) the new complementary magnetic (29) A-B phases. The disclosure of the full set of quantum phase effects for charged particles allowed us to determine the direct relationship between the velocity dependent quantum phase $\delta_{EM}(\boldsymbol{v})$ and the interactional EM momentum for the system "charged particle and external EM field". This finding straightforwardly led to the generalized de Broglie relationship (36), which makes understandable the origin of quantum phase effects for electrically charged particles.

Derivation of these results makes especially topical the problem to understand the failure of the Schrödinger equation with the standard Hamiltonian (6) for deriving complementary electric (22) and magnetic (28) A-B phases. What is more, the fundamental equations of relativistic quantum mechanics are also failed to obtain complementary A-B phases [12,13].

The way to solution of this puzzle can be clarified by comparing equations (31) and (2), which show that replacing the term $e\boldsymbol{A}/c$ in the integrand of equation (2) by the interactional EM field momentum $\boldsymbol{P}_{EM}$ transforms it into equation (31), defining the total velocity-dependent phase. One can easy realize that solution (31) instead of (2) can be obtained via replacing the term $e\boldsymbol{A}/c$ by $\boldsymbol{P}_{EM}$ in Hamiltonian (6), which yields

$$\hat{H} = \frac{\left(-i\hbar\nabla - \boldsymbol{P}_{EM}\right)^2}{2m} + e\varphi. \tag{37}$$

In turn, one can see that the transition from Hamiltonian (6) to its new form (37) corresponds to the redefinition of the momentum operator (7), where it is associated with the sum of mechanical $\boldsymbol{P}_M$ and electromagnetic $\boldsymbol{P}_{EM}$ momenta for the system "charged particle in an external EM field", instead of the canonical momentum $\boldsymbol{P}_M + e\boldsymbol{A}/c$ in the customary definition (5).

Returning to the explicit expression for $\boldsymbol{P}_{EM}$ (equation (30)), we see that the new definition of the momentum operator (7) is more general than the old one (5), and is equally applicable to both non-relativistic and relativistic quantum mechanics [12].

One more argumentation against the old definition of the momentum operator is related to the well-known fact that the canonical momentum of a charged particle associated with this operator in the definition (5) does not have a real physical meaning, and emerges in classical electrodynamics only as a formal variable (see, e.g. [23]). What is more, using the explicit presentation of the interactional EM field momentum $\boldsymbol{P}_{EM}$ (30), we can see that the sum of mechanical $\boldsymbol{P}_M$ and electromagnetic $\boldsymbol{P}_{EM}$ momenta reduces to the canonical momentum $\boldsymbol{P}_M + e\boldsymbol{A}/c$ only in the limit $\boldsymbol{v}\to 0$, whereas a non-zero mechanical momentum implies the inequality $v\neq 0$. This makes the canonical momentum actually inconsistent from a physical viewpoint.

In contrast, in the proposed redefinition of the momentum operator (7), both components of the momenta $\boldsymbol{P}_M$ and $\boldsymbol{P}_{EM}$ are defined at the same velocity of a charged particle, which is quite logical from a physical viewpoint.

5. Discussion

Having disclosed the complete set of quantum phases for electric/magnetic dipoles (12) (see [9, 10]), and applying the superposition principle for quantum phases, we have found a complete set of quantum phases for electrically charged particles (29), which contains two previously known electric and magnetic A-B phases, as well as two new phases (22) and (28), which we have named as complementary electric and magnetic A-B phases, correspondingly [11,12].

The disclosure of all quantum phase effects for point-like charges allowed us to reveal a direct relationship between the total velocity-dependent quantum phase for a charged particle $\delta_{EM}(\boldsymbol{v})$ and the interactional EM momentum $\boldsymbol{P}_{EM}$ for the system "charged particle in external EM field", given by equation (31). This relationship straightforwardly leads to a generalization of the de Broglie relationship for a moving charged particle, where its wavelength depends on the modulus of the vector sum $\left|\boldsymbol{P}_M + \boldsymbol{P}_{EM}\right|$ (see (36)). The latter equation gives a key to understanding the physical origin of quantum phase effects for moving charges, which predicts the variation of their de Broglie wavelength with scalar and vector potentials even at constant $\boldsymbol{P}_M$ in the absence of a classical force.

As we have shown above, these findings prove the need to re-define the momentum operator (5) in a new form (7), where the canonical momentum of a charged particle in an EM field is replaced by the sum $\boldsymbol{P}_M + \boldsymbol{P}_{EM}$.

The unification of the laws of nature requires introducing a re-definition of the momentum operator (5) into basic equations of relativistic quantum mechanics, too [12,13]. One should emphasize that such a redefinition of the momentum operator does not require an additional redefinition of the energy operator in the presence of an EM field

$$\hat{E} \to \hat{E} - e\varphi,$$

where the term $e\varphi$, as is known, can be considered as the interaction electric energy for the system "charged particle in an EM field". This is explained by the known fact that the magnetic interactional energy for this system anyway does not enter into the Hamiltonian. From a physical viewpoint, this is explained by the known fact in classical electrodynamics that the magnetic force does not make work (see, e.g. [23]).

An analysis of physical implications, resulting from the application of the momentum operator (7) to relativistic quantum mechanical falls outside the scope of the present contribution. Now we will emphasize only some important consequences of (7) for free and electrically bound charges:

1. For a freely moving charge, the Klein-Gordon and Dirac equations, like the Schrödinger equation for a charged particle in an EM field, yield a complete set of velocity-dependent quantum phases (29) using the momentum operator (7). It is important to notice here that the Klein-Gordon equation with momentum operator (7) no longer contains solutions with negative probability density and thus becomes well applicable to spinless charged particles.

2. For bound charged particles, the expression for the interactional EM field momentum (30) becomes, in general, inapplicable and can take different forms for different problems. In particular, for hydrogenlike atoms, the re-definition of the momentum operator (7) with the corresponding estimations of the interactional EM field momentum for the one-body and two-body problems, leads to an

appropriate modification of the basic equations of atomic physics. In fact, such modifications have already been introduced by us into the Breit equation without external field, albeit with a different physical argumentation [24]. We named this approach as the Purely Bound Field Theory, which actually achieved undoubted successes in eliminating the available subtle deviations between the QED calculations and the experimental results [25,26].

Last, but not the least, one should add that the results obtained earlier by Yarman [21,27] and later advanced by our team [22,28] also indicate the need to redefine the momentum operator in the form (7). These and further consequences of the proposed redefinition of the momentum operator (7), eventually in full conformity with the law of energy conservation, encompassing the mass and energy equivalence of the special theory of relativity, will be considered in more details elsewhere.

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