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Linear Skorokhod SDE: Evaluation of Expectations of Functionals

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This paper considers a linear stochastic differential equation (SDE) containing the Skorokhod integral. A formula for the approximate calculation of functionals of solutions of this equation is constructed, which is approximately exact for polynomials of the third order.

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1. Introduction

Known functional quadrature formulas for calculating the mathematical expectations of nonlinear functionals from the trajectories of random processes, as a rule, require the fulfillment of the condition of their accuracy for polynomial functionals of a given order. The most frequently used formulas are exact for functional polynomials of the third order, which are used to obtain a fast initial approximation, as well as in combination with other approximations [1-5]. However, in the case of processes specified as complex functions of other random processes, the constructed functional quadrature formulas are difficult for numerical implementation. In this paper, we solve the problem of constructing approximate formulas in which the requirement of their accuracy for approximations of polynomials is imposed instead of accuracy for the polynomials themselves. Since the mathematical expectations of functional polynomials from a random process depend linearly on the moments of the process, in fact, when constructing approximate formulas, approximations of the moments are considered. The solution of this problem is considered on the example of a linear stochastic Skorokhod equation with a Wiener leading process and an initial condition in the form of a function of a linear functional of the Wiener process. When constructing approximate formulas, the moments of solving the equation up to the third order inclusive and their approximations obtained by expanding the functional that specifies the initial condition in a Taylor series are used. The main goal of the paper is an approximate formula for the mathematical expectation of nonlinear functionals from the solution of the linear Skorokhod equation, which has a third-order error in time.

2. Approximate formula

We consider only such linear Skorokhod SDEs for which the moment of the first order from the solution is equal to zero [6]. Moments of the second and third orders in the general case cannot be represented in a multiplicative form. However, their approximations can often be represented as finite sums with terms that have a multiplicative

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form. Let some approximation of the moment $M_3(t_1, t_2, t_3)$ of the third order of the random process $X(t)$, $t \in [0, T]$, be represented as

$$\hat{M}_3(t_1, t_2, t_3) = \sum_{k=1}^{4} a_{3,k} M_{3k}(t_1, t_2, t_3). \qquad (2.1)$$

Here we have chosen sum of only four components to facilitate the construction of approximations, $a_{3,k}$ are constants, which are the mathematical expectations of the functionals given by the conditions of the problem, which can be written out exactly or with a sufficiently high degree of accuracy. The symbol ˆ was introduced to stress the approximateness of the formula (2.1). M_{3k} has the general multiplicative form:

$$M_{3k}(t_1, t_2, t_3) = \sum_{j=1}^{3} x_{k,j,1}(\overline{t_1}) x_{k,j,2}(\overline{t_2}) x_{k,j,3}(\overline{t_3}). \qquad (2.2)$$

Here again the sum of three component us chosen with above mentioned reasons. where $x_{k,j,l}(t)$, $l = 1, 2, 3$ are real functions, $\overline{t_1}, \overline{t_2}, \overline{t_3}$ is ordered ascending numbers t_1, t_2, t_3 and some of the factors in (2.2) may be equal to 1.

Similarly, the approximation of the moment $M_2(t_1, t_2)$ of the second order of the random process $X(t)$, $t \in [0, T]$, is represented as

$$\hat{M}_2(t_1, t_2, t_3) = \sum_{k=1}^{4} a_{2,k} M_{2k}(t_1, t_2), \qquad (2.3)$$

where $a_{2,k}$ are constants, which are also the mathematical expectations of the functionals specified by the conditions of the problem, which can be written out exactly, or with a sufficiently high degree of accuracy, and M_{2k} may not depend on some of the variables t_1, t_2; while $M_{2k}(t_1, t_2)$ have a multiplicative form:

$$M_{2k}(t_1, t_2) = \sum_{j=1}^{3} x_{k,j,1}(\overline{t_1}) x_{k,j,2}(\overline{t_2}), \qquad (2.4)$$

where $x_{k,j,l}(t)$, $l = 1, 2, 3$, are real functions, $\overline{t_1}, \overline{t_2}$ are ordered ascending number t_1, t_2, and a factor $x_{k,j,2}(\overline{t_2})$ may be equal to 1.

It is important to note that the sets of functions $x_{k,j,l}$ for the moments of the third order and the second order are different.

For the component $x_{k,j,1}(\overline{t_1}) x_{k,j,2}(\overline{t_2}) x_{k,j,3}(\overline{t_3})$ from (2.2) we construct an approximate formula in a way to give precisely zero for a constant functional and for any first-order and second-order monomial, whereas for an above written third order monomial the formula should give a precise value.

We denote the approximant as $J_{3,k,j}(F(X_{(\cdot)}))$ and write it down in the form

$$J_{3,k,j}(F(X_{(\cdot)})) = -\int_0^T \int_0^T \frac{\partial}{\partial s}\left(\frac{x_{k,j,1}(s)}{x_{k,j,2}(s)+h}\right) \frac{\partial}{\partial \tau}\left(\frac{x_{k,j,3}(\tau)}{x_{k,j,2}(\tau)+h}\right) \qquad (2.5)$$

$$\times \Lambda F((x_{k,j,2}(\cdot)+h)1_{[0,\cdot]}(s)1_{[\cdot,T]}(\tau))dsd\tau$$

$$+\frac{x_{k,j,3}(T)}{x_{k,j,2}(T)+h} \int_0^T \frac{\partial}{\partial s}\left(\frac{x_{k,j,1}(s)}{x_{k,j,2}(s)+h}\right) \Lambda F((x_{k,j,2}(\cdot)+h)1_{[0,\cdot]}(s))ds$$

$$-\frac{x_{k,j,1}(0)}{x_{k,j,2}(0)+h} \int_0^T \frac{\partial}{\partial \tau}\left(\frac{x_{k,j,3}(\tau)}{x_{k,j,2}(\tau)+h}\right) \Lambda F((x_{k,j,2}(\cdot)+h)1_{[\cdot,T]}(\tau))d\tau$$

$$+\frac{x_{k,j,1}(0)}{x_{k,j,2}(0)+h}\frac{x_{k,j,3}(T)}{x_{k,j,2}(T)+h}\Lambda F((x_{k,j,2}(\cdot)+h))$$

$$+h\int_0^T\int_0^T\frac{\partial}{\partial s}(x_{k,j,1}(s))\frac{\partial}{\partial\tau}(x_{k,j,3}(\tau))\Lambda F(1_{[0,\cdot]}(s)1_{[\cdot,T]}(\tau))dsd\tau$$

$$-hx_{k,j,3}(T)\int_0^T\frac{\partial}{\partial s}(x_{k,j,1}(s))\Lambda F(1_{[0,\cdot]}(s))ds$$

$$+hx_{k,j,1}(0)\int_0^T\frac{\partial}{\partial\tau}(x_{k,j,3}(\tau))\Lambda F(1_{[\cdot,T]}(s))d\tau-hx_{k,j,1}(0)x_{k,j,3}(T)\Lambda F(1),$$

where $\Lambda F(x)=\frac{1}{2}(F(x)-F(-x))$, $1_{[0,t]}(s)$ is the indicator function of $[0,t]$. The symbol "$\cdot$" is used to denote the value of functional in a current point. Note that this formula is also valid for calculating the contribution in the case $x_{k,j,3}(\overline{t_3})=1$.

The constant h is chosen in such a way that in the fractions presented in the formula (2.5) the denominator does not turn to 0 for any values of the argument of the function $x_{k,j,2}(t)$ used in the course of calculations. It should be noted that the formula (2.5) is constructed in such a way that the formula does not depend on h in cases when F is a monomial of the first, second, third order or a constant.

For the component $x_{k,j,1}(\overline{t_1})x_{k,j,2}(\overline{t_2})$, from (2.4) we construct an approximate formula $J_{2,k,j}(F(X_{(\cdot)}))$ and write it down in the form

$$J_{2,k,j}(F(X_{(\cdot)}))=\int_0^T\frac{\partial}{\partial s}\left(\frac{x_{k,j,1}(s)}{x_{k,j,2}(s)+h}\right)\Delta F((x_{k,j,2}(\cdot)+h)1_{[0,\cdot]}(s))ds$$

$$-h\int_0^T\frac{\partial}{\partial s}x_{k,j,1}(s)\Delta F(1_{[0,\cdot]}(s))ds+\frac{x_{k,j,1}(0)}{x_{k,j,2}(0)+h}\Delta F(x_{k,j,2}(\cdot)+h)-hx_{k,j,1}(0)\Delta F(1),$$

where $\Delta F(x)=\frac{1}{2}(F(x)+F(-x))$. In the case of second-order moments, the contribution equal to the term $x_{k,j,1}(\overline{t_1})$, is given by the same formula in which $h=0$ and $x_{k,j,2}(\overline{t_2})=1$. This formula gives a zero value for monomials of the first and third orders in $X_{(\cdot)}$.

Now, let us sum up all above introduced approximants $J_{2,k,j}, J_{3,k,j}$, $k=1,4,$, $j=1,3$ together and obtain the following theorem:

Theorem 1. *The approximate formula*

$$E[F(X_{(\cdot)})] \approx J(F(X_{(\cdot)})) \equiv F(0) + \sum_{k=1}^{4} a_{2,k} \sum_{j=1}^{3} \sum_{r=1}^{2} B_r J_{2,k,j}(F(q_r X_{(\cdot)}))$$

$$+ \sum_{k=1}^{4} a_{3,k} \sum_{j=1}^{3} \sum_{r=1}^{2} A_r J_{3,k,j}(F(c_r X_{(\cdot)})), \qquad (2.6)$$

where $A_1 = -\frac{1}{3}$, $A_2 = \frac{1}{6}$, $c_1 = 1$, $c_2 = 2$, $B_1 = 2$, $B_2 = -2$, $q_1 = 1$, $q_2 = \frac{1}{\sqrt{2}}$, *is approximately exact for functional polynomials of the third order.*

Let us consider the application of the proposed approach to the calculation of the mathematical expectation of functionals from the solution of the Skorokhod equation:

$$X_t = X_0 + \int_0^t \sigma(s) X_s \delta W_s, \qquad (2.7)$$

where $X_0 = g\Big(\int_0^T a(\tau)dW_\tau\Big)$; W_t, $t \in [0,T]$, is Wiener process, defined on a probability space $\Omega = C_0([0,T])$, $W_t(\omega) = \omega(t)$; $\sigma(s)$, $g(u)$, $a(\tau)$ are deterministic functions, $\int_0^T \sigma^2(s)ds < \infty$; $g(u)$ is differentiable the required number of times, $a(\tau) \in L_2([0,T])$, $\int_0^T a(\tau)dW_\tau$ is a stochastic integral in Ito sense.

The integral on the right side (2.7) is the Skorokhod integral, because X_0 is not adapted to the filtration generated by the Wiener process.

The solution of this equation can be fount in explicit form (see [6-8])

$$X_t = g\Big(\int_0^T a(\tau)dW_\tau - \int_0^t a(\tau)\sigma(\tau)d\tau\Big)$$
$$\times \exp\Big\{\int_0^t \sigma(s)dW_s - \frac{1}{2}\int_0^t \sigma^2(s)ds\Big\},$$

and first three moments has the following form:

$$E[X_t] = E\Big[g\Big(\int_0^T a(\tau)dW_\tau\Big)\Big],$$

$$E[X_{t_1}X_{t_2}] = \exp\Big\{\int_0^{t_1\wedge t_2} \sigma^2(\tau)d\tau\Big\} E\Big[g\Big(\int_0^T a(\tau)dW_\tau + \int_0^{t_1} a(\tau)\sigma(\tau)d\tau\Big)$$
$$\times g\Big(\int_0^T a(\tau)dW_\tau + \int_0^{t_2} a(\tau)\sigma(\tau)d\tau\Big)\Big] \equiv M_2(t_1,t_2),$$

$$E\Big[\prod_{k=1}^{3} X_{t_k}\Big] = \prod_{\{t_i,t_j\}} \exp\Big\{ \int_0^{t_i\wedge t_j} \sigma^2(\tau)d\tau\Big\}$$

$$\times E\Big[\prod_{\{t_i,t_j\}} g_1\Big(\int_0^T a(\tau)dW_\tau; t_i, t_j\Big)\Big] \equiv M_3(t_1,t_2,t_3),$$

where $t_i \wedge t_j = \min(t_i,t_j)$,

$$g_1\Big(\int_0^T a(\tau)dW_\tau; t_i, t_j\Big)$$

$$= g\Big(\int_0^T a(\tau)dW_\tau + \int_0^{t_i} a(\tau)\sigma(\tau)d\tau + \int_0^{t_j} a(\tau)\sigma(\tau)d\tau\Big),$$

and a pair $\{t_i,t_j\}$ takes values $\{t_1,t_2\},\{t_1,t_3\},\{t_2,t_3\}$.

We obtain a third-order moment approximation in $t \in T$ using the Taylor series expansion of the function $g_1(u)$ and the assumption that $\sup_{\tau\in[0,T]} |a(\tau)\sigma(\tau)| < 1$:

$$g_1(\langle a,W\rangle, t_i, t_j) = g(\langle a,W\rangle) + g'(\langle a,W\rangle)(p(t_i)+p(t_j)) + \frac{1}{2}g''(\langle a,W\rangle)(p(t_i)+p(t_j))^2$$

$$+\frac{1}{3!}g^{(3)}(\langle a,W\rangle + \theta(p(t_i)+p(t_j)))(p(t_i)+p(t_j))^3.$$

Let us denote $\langle a,W\rangle = \int_0^T a(\tau)dW_\tau$, $p(t) = \int_0^t a(\tau)\sigma(\tau)d\tau$, $\phi(t) = \int_0^t \sigma^2(\tau)d\tau$. Then one gets M_3 in the following explicit form:

$$M_3(t_1,t_2,t_3) = \Bigg[E[g^3(\langle a,W\rangle)] + 2E[g^2(\langle a,W\rangle)g'(\langle a,W\rangle)]\sum_{i=1}^{3} p(t_i)$$

$$+ (E[g^2(\langle a,W\rangle)g''(\langle a,W\rangle)] + 3E[g(\langle a,W\rangle)(g'(\langle a,W\rangle))^2])(p(t_1)p(t_2) + p(t_1)p(t_3) + p(t_2)p(t_3))$$

$$+ (E[g^2(\langle a,W\rangle)g''(\langle a,W\rangle)] + E[g(\langle a,W\rangle)(g'(\langle a,W\rangle))^2])\sum_{i=1}^{3} p^2(t_i)\Bigg] M_{31}(t_1,t_2,t_3) + O(\max(t_1,t_2,t_3))$$

$$= \widehat{M}_3(t_1,t_2,t_3) + O(\max(t_1,t_2,t_3)). \quad (2.8)$$

where $M_{31}(t_1,t_2,t_3) = \exp\{p(t_1 \wedge t_2) + p(t_1 \wedge t_3) + p(t_2 \wedge t_3)\}$. Now, taking into account the fact that

$$M_{31}(t_1,t_2,t_3) = e^{2\phi(\bar{t}_1)}e^{\phi(\bar{t}_2)}, \quad (2.9)$$
$$M_{32}(t_1,t_2,t_3) = p(\bar{t}_1)e^{2\phi(\bar{t}_1)}e^{\phi(\bar{t}_2)} + e^{2\phi(\bar{t}_1)}p(\bar{t}_2)e^{\phi(\bar{t}_2)} + e^{2\phi(\bar{t}_1)}e^{\phi(\bar{t}_2)}p(\bar{t}_3), \quad (2.10)$$
$$M_{33}(t_1,t_2,t_3) = p(\bar{t}_1)e^{2\phi(\bar{t}_1)}p(\bar{t}_2)e^{\phi(\bar{t}_2)} + p(\bar{t}_1)e^{2\phi(\bar{t}_1)}e^{\phi(\bar{t}_2)}p(\bar{t}_3) + e^{2\phi(\bar{t}_1)}p(\bar{t}_2)e^{\phi(\bar{t}_2)}p(\bar{t}_3), \quad (2.11)$$
$$M_{34}(t_1,t_2,t_3) = p^2(\bar{t}_1)e^{2\phi(\bar{t}_1)}e^{\phi(\bar{t}_2)} + e^{2\phi(\bar{t}_1)}p^2(\bar{t}_2)e^{\phi(\bar{t}_2)} + e^{2\phi(\bar{t}_1)}e^{\phi(\bar{t}_2)}p^2(\bar{t}_3), \quad (2.12)$$

$$a_{3,1} = E[g^3(\langle a,W\rangle)],$$
$$a_{3,2} = 2E[g^2(\langle a,W\rangle)g'(\langle a,W\rangle)],$$
$$a_{3,3} = E[g^2(\langle a,W\rangle)g''(\langle a,W\rangle)] + 3E[g(\langle a,W\rangle)(g'(\langle a,W\rangle))^2],$$
$$a_{3,4} = E[g^2(\langle a,W\rangle)g''(\langle a,W\rangle)] + E[g(\langle a,W\rangle)(g'(\langle a,W\rangle))^2]$$

the expression (2.8) can be rewritten in the form

$$M_3(t_1,t_2,t_3) = \sum_{k=1}^{4} a_{3,k}M_{3k}(t_1,t_2,t_3).$$

It is clear that $M_{3k}(t_1,t_2,t_3)$ is determined by the expression (2.2). Let us present the form of the corresponding $x_{i,j,k}$ for various M_{3k} as

for M_{31}

$$x_{1,1,1}(t) = e^{2\phi(t)}, \quad x_{1,1,2}(t) = e^{\phi(t)}, \quad x_{1,1,3}(t) = 1,$$
$$x_{1,2,1}(t) = x_{1,2,2}(t) = x_{1,2,3}(t) = x_{1,3,1}(t) = x_{1,3,2}(t) = x_{1,3,3}(t) = 0;$$

for M_{32}

$$x_{2,1,1}(t) = p(t)e^{2\phi(t)}, \quad x_{2,1,2}(t) = e^{\phi(t)}, \quad x_{2,1,3}(t) = 1,$$
$$x_{2,2,1}(t) = e^{2\phi(t)}, \quad x_{2,2,2}(t) = p(t)e^{\phi(t)}, \quad x_{2,2,3}(t) = 1,$$
$$x_{2,3,1}(t) = e^{2\phi(t)}, \quad x_{2,3,2}(t) = e^{\phi(t)}, \quad x_{2,3,3}(t) = p(t);$$

for M_{33}

$$x_{3,1,1}(t) = p(t)e^{2\phi(t)}, \quad x_{3,1,2}(t) = p(t)e^{\phi(t)}, \quad x_{3,1,3}(t) = 1,$$
$$x_{3,2,1}(t) = p(t)e^{2\phi(t)}, \quad x_{3,2,2}(t) = e^{\phi(t)}, \quad x_{3,2,3}(t) = p(t),$$
$$x_{3,3,1}(t) = e^{2\phi(t)}, \quad x_{3,3,2}(t) = p(t)e^{\phi(t)}, \quad x_{3,3,3}(t) = p(t);$$

for M_{34}

$$x_{4,1,1}(t) = p^2(t)e^{2\phi(t)}, \quad x_{4,1,2}(t) = e^{\phi(t)}, \quad x_{4,1,3}(t) = 1,$$
$$x_{4,2,1}(t) = e^{2\phi(t)}, \quad x_{4,2,2}(t) = p^2(t)e^{\phi(t)}, \quad x_{4,2,3}(t) = 1,$$
$$x_{4,3,1}(t) = e^{2\phi(t)}, \quad x_{4,3,2}(t) = e^{\phi(t)}, \quad x_{4,3,3}(t) = p^2(t).$$

Similarly, for the moment of the second order, we obtain

$$M_2(t_1,t_2)=\sum_{k=1}^{4}a_{2,k}M_{2k}(t_1,t_2)+O(\max(t_1,t_2)),$$

where

$$\begin{aligned}M_{21}(t_1,t_2) &= e^{\phi(\bar t_1)},\\ M_{22}(t_1,t_2) &= p(\bar t_1)e^{\phi(\bar t_1)}+e^{\phi(\bar t_1)}p(\bar t_2),\\ M_{23}(t_1,t_2) &= p^2(\bar t_1)e^{\phi(\bar t_1)}+e^{\phi(\bar t_1)}p^2(\bar t_2),\\ M_{24}(t_1,t_2) &= p(\bar t_1)e^{\phi(\bar t_1)}p(\bar t_2),\end{aligned}$$

$$\begin{aligned}a_{2,1}&=E[g^2(\langle a,W\rangle)],\\ a_{2,2}&=E[g(\langle a,W\rangle)g'(\langle a,W\rangle)],\\ a_{2,3}&=E[g(\langle a,W\rangle)g''(\langle a,W\rangle)],\\ a_{2,4}&=E[(g'(\langle a,W\rangle))^2],\end{aligned}$$

where for M_{21}

$$x_{1,1,1}(t)=e^{\phi(t)},\quad x_{1,1,2}(t)=1,$$
$$x_{1,2,1}(t)=x_{1,2,2}(t)=x_{1,3,1}(t)=x_{1,3,2}(t)=0,$$

for M_{22}

$$x_{2,1,1}(t)=p(t)e^{\phi(t)},\qquad x_{2,1,2}(t)=1,$$
$$x_{2,2,1}(t)=e^{\phi(t)},\qquad x_{2,2,2}(t)=p(t),$$
$$x_{2,3,1}(t)=x_{2,3,2}(t)=0,$$

for M_{23}

$$x_{3,1,1}(t)=p^2(t)e^{\phi(t)},\qquad x_{3,1,2}(t)=1,$$
$$x_{3,2,1}(t)=e^{\phi(t)},\qquad x_{3,2,2}(t)=p^2(t),$$
$$x_{3,3,1}(t)=x_{3,3,2}(t)=0,$$

for M_{24}

$$x_{4,1,1}(t)=p(t)e^{\phi(t)},\quad x_{4,1,2}(t)=p(t),$$
$$x_{4,2,1}(t)=x_{4,2,2}(t)=x_{4,3,1}(t)=x_{4,3,2}(t)=0.$$

In the paper, the approximate formulas are constructed for the class of functionals that can be represented in the form

$$F(X_{(\cdot)})=F(0)+\sum_{k=1}^{3}\frac{1}{k!}\int_0^T\overset{(k)}{\cdots}\int_0^T A_k(t_1,\dots,t_k)\prod_{l=1}^{k}X_{t_l}dt_1\cdots dt_k$$
$$+\frac{1}{3!}\int_0^1(1-\tau)^3\int_0^T\overset{(4)}{\cdots}\int_0^T G(\tau X_{(\cdot)},t_1,\dots,t_k)\prod_{l=1}^{4}X_{t_l}dt_1\cdots dt_4d\tau,\qquad(2.13)$$

where $T<1$, $A_k(t_1,\dots,t_k)$ are deterministic functions, (k) denotes an integral multiplicity.

Such a representation has, for example, some functionals that have functional derivatives up to the 4th order, inclusive, in accordance with the Taylor functional formula

$$F(X_{(\cdot)}) = F(0) + \sum_{k=1}^{3} \frac{1}{k!} \int_0^T \overset{(k)}{\cdots} \int_0^T F^{(k)}(0, t_1, \ldots, t_k) \prod_{l=1}^{k} X_{t_l} dt_1 \cdots dt_k$$

$$+ \frac{1}{3!} \int_0^1 (1-\tau)^3 \int_0^T \overset{(4)}{\cdots} \int_0^T F^{(4)}(\tau X_{(\cdot)}, t_1, \ldots, t_4) \prod_{l=1}^{4} X_{t_l} dt_1 \cdots dt_4 d\tau$$

$$\equiv P_3(F(X_{(\cdot)})) + r(F(X_{(\cdot)})),$$

$F^{(4)}(\tau X_{(\cdot)}, t_1, \ldots, t_k)$ functional derivative of F.

The following theorem gives an estimate for the remainder $r(F(X_{(\cdot)}))$:

Theorem 2. *Let functional $F(X_{(\cdot)})$ can be represented in the form (2.13) and the conditions*

$$\sup_{\tau \in [0,1], t_1, \ldots, t_4 \in T} |E[G(\tau X_{(\cdot)}, t_1, \ldots, t_4) \prod_{l=1}^{4} X_{t_l}]| \leq b_1 \equiv \text{const},$$

$$\sup_{\tau \in [0,1], t_1, \ldots, t_4 \in T} |G(\tau c_r(x_{k,j,l}(\cdot) + h) 1_{[0,\cdot]}(s) 1_{[\cdot,T]}(\tau), t_1, \ldots, t_4)| \leq b_2 \equiv \text{const},$$

and the set of functions $x_{k,j,1}$, $x_{k,j,2}$ for the moments of the second order are satisfies to the condition

$$\sup_{\bar{t}_l \in [0,T], j=1,2} \left(\left| x_{k,j,1}(\bar{t}_1) \prod_{l=2}^{4} x_{k,j,2}(\bar{t}_l) \right| + h |x_{k,j,1}(\bar{t}_1)| \right) \leq 1,$$

where $\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4$ is ordered ascending numbers t_1, t_2, t_3, t_4.

Then

$$|R| = |E[F(X_{(\cdot)})] - J(F(X_{(\cdot)}))| \leq |P_3(X_{(\cdot)}) - \widehat{P}_3(X_{(\cdot)})| + T^4 \frac{1}{4!}(b_1 + b_2).$$

Here $\hat{P}$ is the polynomial P, where expectations $E[\sum_l X_{t_l}]$ are replaced according to proposed approximations.

Proof

Let us estimate the error $R = E[F(X_{(\cdot)})] - J(F(X_{(\cdot)}))$:

$$|R| \leq |E[P_3(X_{(\cdot)})] - J(P(X_{(\cdot)}))| + |E[r(F(X_{(\cdot)})] - J(r(F(X_{(\cdot)})))|$$

$$\leq |P_3(X_{(\cdot)}) - \widehat{P}_3(X_{(\cdot)})|$$

$$+\frac{1}{3!}\int\limits_0^1 (1-\tau)^3 \int\limits_0^T \overset{(4)}{\cdots} \int\limits_0^T \Big[\Big|E\Big[G(\tau X_{(\cdot)}, t_1, \ldots, t_4)\prod_{l=1}^4 X_{t_l}\Big]\Big| + \Big|J\Big(G(\tau X_{(\cdot)}, t_1, \ldots, t_k)\prod_{l=1}^4 X_{t_l}\Big)\Big|\Big] dt_1 \cdots dt_k d\tau$$

$$\leq |P_3(X_{(\cdot)}) - \widehat{P}_3(X_{(\cdot)})| + \frac{1}{4!}T^4(b_1+b_2)J\left(\Big|\prod_{l=1}^4 X_{t_l}\Big|\right) \leq |P_3(X_{(\cdot)}) - \widehat{P}_3(X_{(\cdot)})| + \frac{1}{4!}T^4(b_1+b_2)J_2\left(\Big|\prod_{l=1}^4 X_{t_l}\Big|\right),$$

Here we use the fact, that $\Big|\prod_{l=1}^4 X_{t_l}\Big|$ is even functional, and the property of the operator Λ. Let us denote $J_{2,k}(F(X_{(\cdot)})) = \sum_{j=1}^2 J_{2,k,j}(F(X_{(\cdot)}))$. So

$$J_{2,k}\left(\Big|\prod_{l=1}^4 X_{t_l}\Big|\right) = \sum_{j=1}^2 J_{2,k,j}\left(\Big|\prod_{l=1}^4 X_{t_l}\Big|\right) = \sum_{j=1}^2 \int\limits_0^T \frac{\partial}{\partial s}\left(\frac{x_{k,j,1}(s)}{x_{k,j,2}(s)+h}\right)\prod_{l=1}^4 (x_{k,j,2}(t_l)+h)1_{[0,t_l]}(s)ds$$

$$-h\sum_{j=1}^2 \int\limits_0^T \frac{\partial}{\partial s}(x_{k,j,1}(s))\prod_{l=1}^4 1_{[0,t_l]}(s)ds + \sum_{j=1}^2 \frac{x_{k,j,1}(0)}{x_{k,j,2}(0)+h}\prod_{l=1}^4 (x_{k,j,2}(t_l)+h)1_{[0,t_l]}(s) - h\sum_{j=1}^2 x_{k,j,2}^2(0)$$

$$\leq |x_{k,j,1}(\bar{t}_1)\prod_{l=2}^4 x_{k,j,2}(\bar{t}_l)| + h|x_{k,j,1}(\bar{t}_1)|.$$

The assertion of the theorem follows from the obtained estimates for. ∎

3. Numeric experiment

Let us consider the application of proposed formula.

Let $T = 1$, $a(x) = x$, $\sigma(x) = \sqrt{x}$ and $g(x) = x$ and $F(X) = X^3$. The results of calculation of expectation for various t are shown in the table 1.

Let $T = 1$, $a(x) = x$, $\sigma(x) = \sqrt{x}$, $g(x) = x^3$ and $F(X) = \sin X$. The results of calculation of expectation are represented in the table 2.

4. Conclusion

In this paper, an approximate exact formula has been constructed for functional polynomials from the solution of a stochastic differential equation in the sense of Skorokhod. For the proposed formula, an estimate of the accuracy of the approximate values calculated with its help was obtained. An example of applying the formula

Table 1. Estimations and errors for the functional $E[X_t^3]$.

t	Approximate	Exact	Error
0.1	0.00256806	0.00256807	0.00000001
0.2	0.01519577	0.01519888	0.00000311
0.3	0.04513598	0.04520618	0.00007020
0.4	0.10291310	0.10358754	0.00067445
0.5	0.20576686	0.20988220	0.00411534

Table 2. Estimations and errors for the functional $E[\sin X_t]$.

t	Approximate	Exact	Error
0.1	-0.00016877	-0.00359667	0.00342790
0.2	-0.00085016	-0.01073725	0.00988709
0.3	-0.00173725	-0.02061174	0.01887449
0.4	-0.00127556	-0.03287304	0.03159748
0.5	0.00482462	-0.04724638	0.05207100

is given.

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