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Force-Dependent Fluxes of Adiabatic Classical and Quantum Rocking Ratchets

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We consider the adiabatic mode of Brownian particle motion in a periodic potential under the action of symmetric dichotomous fluctuations of an external force F with zero mean value (rocking ratchet), in which the fluctuation frequency is much less than the inverse relaxation time of the particle in each of the states of the dichotomous process. Expressions are given for force-dependent fluxes of an adiabatic classical rocking ratchet. In the absence of thermal fluctuations, within the semiclassical approximation, analytical expressions are obtained for the rocking-ratchet tunneling flux in a sawtooth periodic potential of arbitrary asymmetry and in the potential of two sinusoids. It is shown that the tunneling flux has a linear asymptotics in modulus of small F due to (i) the absence of reverse tunneling fluxes with respect to the direction F and (ii) the root dependence of the integrand of the Gamow formula on the potential energy. The main parameters of the model are the energy barrier V_0 and the period L of the potentials, as well as the rocking force F and the asymmetry parameter $\xi = l/L$ of the sawtooth potential with the width of one of its teeth equal to l. It is shown that the direction of quantum ratchet motion is opposite to the motion direction of the corresponding classical ratchet in a limited range of values of the rocking force $|F|\, L/V_0 < \alpha_c$, where the parameter α_c changes from the value $\alpha_1 = (\sqrt{5}-1)/2 \approx 0.618$ for the extremely asymmetric sawtooth potential ($\xi = 1$) to the value $\alpha_2 = 2/3 \approx 0.667$ for the symmetric potential ($\xi = 1/2$). In the range of values $\alpha_1 < \alpha_c < \alpha_2$, the sign of the tunneling flux changes with the change in the asymmetry parameter ξ. Numerical calculations for the potential of two sinusoids corresponding to the effective value $\xi \approx 0.655$ of the asymmetry lead to similar results with $\alpha_c \approx 0.81$.

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1. Introduction

Among various models of controlled nanotransport, the most common models are the so-called rocking ratchets, in which, spatial and/or temporal symmetry is being broken, unbiased nonequilibrium fluctuations of an external force can lead to directed motion of nanoparticles [1–4]. The main factor determining the direction of Brownian ratchet motion is the asymmetry of the potential relief [3, 5–8]. In addition to the asymmetry and features of the potential relief, the motion direction is also

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significantly affected by dynamic effects [9]. More possibilities for controlling nanotransport arise for nanoparticles of sufficiently large or small mass, for which inertial or quantum effects are to be taken into account [2, 3]. Models that use classical description in the framework of diffusion dynamics based on the Fokker-Planck equation [10] are the most studied. Accounting for laws of quantum mechanics in describing the ratchet functioning leads to fundamentally new effects that were impossible to discover within the classical description [2]. In this case, the critical factor is the temperature of the process. At certain temperature, the mechanism of the process is known to be transformed from the low-temperature tunnel mechanism to the classical over-barrier one (see, for example, Refs. [11, 12] and references therein). In other words, there is a criterion, depending on the parameters of the barrier and the particle mass, for the temperature at which the mechanism of particle transfer alters. In particular, this applies to the quantum rocking ratchet, which was first described in the pioneering work [13]. The model of the quantum ratchet system considered in [13] included but a two-sinusoidal potential and fixed values of the external force; it was demonstrated that, at sufficiently high temperatures, the ratchet effect led to particle motion in the direction coinciding with the classical case, while at low temperatures, when the tunnel motion prevails, the particle moved in opposite direction. This theoretical result was confirmed experimentally in Ref. [14]. In this paper, we consider the adiabatic mode of Brownian particle motion in a periodic potential relief under the action of symmetric dichotomous fluctuations of an external force F with zero mean value (rocking ratchet). We analyze the dependencies of the particle flux on the magnitude of the fluctuating force in both the classical and quantum rocking systems. Section 2 presents the main results of the theory of classical adiabatic rocking ratchets with arbitrary periodic potentials and temperature values; the results are then concretized to potentials of a sawtooth shape with the energy barrier V_0 and low temperatures ($V_0/k_BT >> 1$,k_B is the Boltzmann constant, T is the absolute temperature) when the kinetic description can be used. The main result of this section is that the dependence of the flux on the rocking force F in the interval of values $FL < V_0$ (L is the period of the potential) is a monotonically increasing function, quadratic in F if F is small. In Section 3, within the framework of the semiclassical approximation, analytical expressions are obtained for the tunneling flux of an adiabatic rocking ratchet in a sawtooth periodic potential of arbitrary asymmetry at zero absolute temperature. It follows from these expressions that, for a quantum rocking ratchet, the dependence of the flux on F in the same range of its values, $FL < V_0$, is a nonmonotonic function having a section of linear dependence at small F, where the sign of the flux is opposite to the analogous contribution for the classical rocking ratchet. The presence of this linear contribution is due to the absence of the reverse tunneling flux at fixed F. For certain values of F from the same interval, $FL < V_0$, the direction of the quantum ratchet motion reverses and becomes the same as that of the classical ratchet. A similar result is obtained for the motion of a rocking ratchet when choosing the two-sinusoidal form of the potential, analogous to considered in Ref. [13]. The discussion and conclusions are presented in the final Section 4.

2. Classical rocking ratchet

An analytical expression for the classical-rocking-ratchet flux in the adiabatic regime of the ratchet motion can be derived from the exact solution of the problem of diffusion of a Brownian particle in a stationary spatially periodic potential $V(x)$, of a period L, under the action of a uniform and stationary force F [10, 15, 16]:

$$J_{cl} = D\frac{\left(1-e^{-\beta FL}\right)}{\int_0^L e^{-\beta U(x)}dx\int_0^L e^{\beta U(x)}dx-\left(1-e^{-\beta FL}\right)\int_0^L dx e^{-\beta U(x)}\int_0^x dx' e^{\beta U(x')}}, \quad (1)$$

where $U(x) = V(x) - Fx$, $\beta = (k_BT)^{-1}$, D is the diffusing coefficient. The average velocity of the directed particle motion, v, is then expressed through J_{cl} as

$$v = J_{cl}L$$

, so that, in linear in F approximation (in the limit of small forces $\beta FL \ll 1$), it will be

$$v = \mu_{\rm eff}F$$

, where $\mu_{\rm eff}$ is the effective coefficient of particle mobility in the periodic potential [10, 15, 16]:

$$\mu_{\rm eff} = \zeta^{-1}L^2Z_+^{-1}Z_-^{-1}, \quad (2)$$
$$Z_\pm = \int_0^L dx \exp\left[\pm\beta V(x)\right].$$

This formula was first obtained by Lifson and Jackson [17] when another problem (namely, determining the diffusion coefficient of a particle in a periodic potential) was solved using the Einstein relation $D_{eff} = \mu_{\rm eff}k_BT$. Eq. (2) shows that as $V(x) \to 0$, and the presence of the potential always reduces the particle mobility. The integrals in Eqs. (1) and (2) are readily evaluated if we choose $V(x)$ in the form of a sawtooth potential with the potential barrier V_0, given in its major region by the following expression:

$$V(x) = V_0 \times \begin{cases} x/l, & 0 \le x \le l, \\ (L-x)/(L-l), & l \le x \le L. \end{cases} \quad (3)$$

The slopes of the linear sections $0 \le x < l$ and $l \le x < L$ become $f_+ = V_0/l - F$ and $f_- = -V_0/(L-l) - F$, respectively, and we then obtain [18, 19]:

$$\beta D J_{\rm cl}^{-1} = -\frac{l}{f_+} - \frac{L-l}{f_-} + \beta^{-1}\left\{\frac{e^{\beta f_-(L-l)}-1}{f_-^2} - \frac{e^{-\beta f_+ l}-1}{f_+^2} + \frac{1}{e^{-\beta f_+ l} - e^{\beta f_-(L-l)}}\left[\frac{e^{\beta f_-(L-l)}-1}{f_-} - \frac{e^{-\beta f_+ l}-1}{f_+}\right]^2\right\},$$
$$\mu_{\rm eff} = \zeta^{-1}w^2/\sinh^2 w, \quad w = \beta V_0/2. \quad (4)$$

Thus, for the sawtooth potential, the effective mobility does not depend on its asymmetry. Consider a dichotomous process of alterning two states, denoted by $''+''$ and $''-''$, which are characterized by durations $\tau_\pm$ and potential profiles $U_\pm(x) = V(x) \mp Fx$ (Fig. 1).

The applied forces make the potential profiles tilted (tilted periodic potentials, washboard potentials); in each of the profiles a stationary particle flux $J_\pm$ occurs. In the adiabatic approximation, the values $\tau_\pm$ significantly exceed the characteristic relaxation times (the times during which the initial conditions are forgotten and the fluxes become stationary); and, for a symmetric dichotomous process ($\tau_+ = \tau_-$), the average flux is then the arithmetic mean of the fluxes $J_\pm$:

$$\langle J\rangle = (J_+ + J_-)/2. \tag{5}$$

It is this average flux that is the rocking-ratchet flux in the adiabatic approximation.

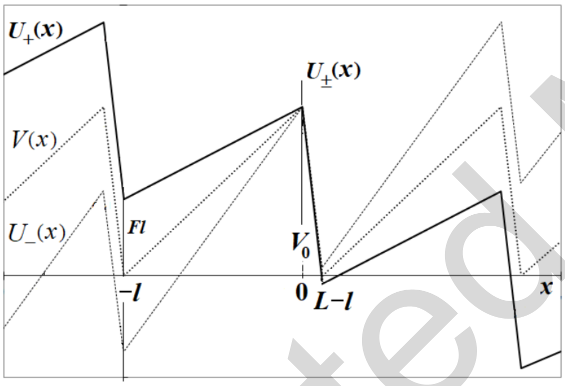

FIG. 1. The mechanism of occurrence of directed motion in a rocking ratchet, induced by fluctuations of the applied force F, due to which the potential profiles $U_\pm(x) = V(x) \mp Fx$ ($V(x)$ periodic potential, which is shown as a sawtooth in the figure) become tilted, and in each of these profiles occurs either thermoactivated flux of particles (at high temperatures) or tunneling flux (at low temperatures).

The features of the dependence $\langle J_{\rm cl}\rangle$ on F for the sawtooth potential (3) was illustrated in [18] (see Fig. 2 therein). At either $Fl < V_0$ or $F(L - l) < V_0$ (that is when the particle motion along the F direction is inevitably accompanied by thermally activated overcoming the barrier), the magnitude of the flux is small, and the function describing its dependence on F monotonically increases from zero at $\xi > 1/2$, while, at $\xi < 1/2$, it monotonically decreases. In the absence of thermally activated overcoming the barriers, the absolute value of the flux increases a lot when $FL > V_0$, and then, for $FL \gg V_0$, decreases to zero, since there is no ratchet effect in the absence of an asymmetric potential profile. We will not be interested in the section of nonmonotonicity corresponding large values of the rocking force ($FL > V_0$), since we intend to compare the classical thermally activated barrier overcoming with quantum tunneling in the context of their manifestation in ratchet characteristics. In the expansion of the expressions (5) and (1) in small βFL, the linear in F contributions are compensated, and the average flux begins with the quadratic, in F, terms [20, 21], that is

$$\begin{aligned}
&\langle J_{\rm cl}\rangle = -\beta F^2 \mu_{\rm eff}\,\Phi + O[(\beta FL)^4],\\
&\Phi = \int_0^L dx\left[\rho_+(x) - L^{-1}\right]\int_0^x dy\left[\rho_-(y) - L^{-1}\right],\\
&\rho_\pm(x) = Z_\pm^{-1}\exp\left[\pm\beta V(x)\right],
\end{aligned} \tag{6}$$

where the quantity Φ determines the average flux for an adiabatically driven on-off flashing ratchet, $\langle J_{\rm cl}\rangle_{\rm flash} = \tau^{-1}\Phi$, $\tau = \tau_+ + \tau_-$ is the period of the dichotomous process ($\tau_+ = \tau_-$), $\mu_{\rm eff}$ is determined by the relation (2), and $O(z)$ denotes an infinitesimal quantity of the order of z. For the sawtooth potential (3), the relations (6) are reduced, up to the contributions of the order of F^2, to the following expression:

$$\langle J_{\rm cl}\rangle \approx (2\xi - 1)\,\frac{F^2}{2\zeta V_0}\frac{w^3}{\sinh^2 w}\left(\coth w + \frac{w}{\sinh^2 w} - \frac{2}{w}\right) \approx (2\xi - 1)\,\frac{\beta^3 V_0^2 F^2}{\zeta} \times \begin{cases} w/360, & w \ll 1,\\ e^{-\beta V_0}, & w \gg 1.\end{cases} \tag{7}$$

where $w = \beta V_0/2, \quad \xi = l/L$.

The sign of the flux is determined by the sign of $(2\xi - 1)$, that is, by the asymmetry of the sawtooth potential. For the symmetric potential $(2\xi - 1 = 0)$ the flux is always zero. For the stated purposes, it is sufficient to confine the analysis to the approximation of large, compared to the thermal energy $k_B T$, potential barriers (this low-temperature regime allows for using kinetic approach), when the particle flux in each state $U_\pm(x)$ can be calculated using the Arrhenius law. If we choose the coordinate origin coinciding with one of the barriers V_0 of the sawtooth potential (Fig. 2), then

$$J_\pm^{(\mathrm{cl})} = C\left[e^{-\beta(V_0 - U_\pm(-l))} - e^{-\beta(V_0 - U_\pm(L-l))}\right] = Ce^{-\beta V_0}\left[e^{\pm\beta F l} - e^{\mp\beta F(L-l)}\right], \quad (8)$$

where C is a pre-exponential factor describing the frequency with which particles hit the barrier, and the quantities $U_\pm(-l)$ and $U_\pm(L-l)$ specify the potential energies at the potential energy minima closest to the chosen origin. Note that the signs of the expressions for the fluxes in "+" and "−" states, in which the rocking force acts to the right and to the left, are obtained automatically: $J_+^{(\mathrm{cl})} > 0$, $J_-^{(\mathrm{cl})} < 0$.

Averaging Eqs. (8) using (5) gives:

$$\langle J_{\mathrm{cl}}\rangle = 2Ce^{-\beta V_0}\sinh(\beta F L/2)\sinh[\beta F(2l - L)/2]. \quad (9)$$

At $\beta F L \ll 1$, formula (5) yields $\langle J_{\mathrm{cl}}\rangle \approx (C/2)(2\xi - 1)(\beta F L)^2 e^{-\beta V_0}$. Comparing this result with the approximate expression for the flux (7) with $w \gg 1$ gives the value of the pre-exponential factor: $C = 2\beta V_0^2/(\zeta L^2)$. The dependence of $\langle J_{\mathrm{cl}}\rangle$ on the dimensionless force $\beta F L/2$ is shown in the inset to Fig. 3. This dependence on the interval $FL < V_0$ is indeed a monotonically increasing function when $\xi > 1/2$.

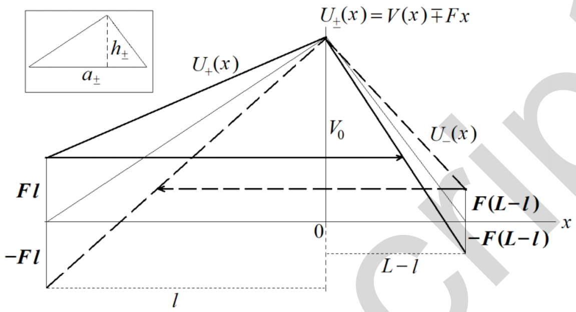

FIG. 2. A fragment of a sawtooth potential illustrating the tunneling of a quantum particle in the direction of the applied force F. At $F = 0$, the initial triangle with the base L equal to the period of the potential and the height V_0 (depicted by thin lines) undergoes distortions caused by the force fluctuations. The result is two distorted triangles, depicted in thick solid and dashed lines, with bases $a_\pm$ and heights $h_\pm$ (see the inset, upper left), which can be found from the similarity of right triangles with vertical legs along the ordinate axis.

3. Quantum rocking ratchet

At zero temperature, there are no thermoactivated contributions to the flux, and the only mechanism for the motion occurrence is the effect of quantum tunneling. By the semiclassical approximation, the tunneling flux is proportional to the rate constant of overcoming the potential barrier and is determined by the expression

$$J_\pm^{(\mathrm{qm})} = \pm A e^{-S_\pm/\hbar}, \qquad S_\pm = 2\left|\int_{x_{0,\pm}}^{x_{1,\pm}} dx\sqrt{2m\left[U_\pm(x) - U(x_{0,\pm})\right]}\right|, \quad (10)$$

where A is the pre-exponential factor, $\hbar$ is the Planck constant; the action $S_\pm$ is represented by the Gamow formula, in which m is the particle mass and the integration is carried out over the sub-barrier region of the potential profile $U_\pm(x)$ with the entry and exit points $x_{0,\pm}$ and $x_{1,\pm}$, respectively, and $U_\pm(x_{0,\pm}) = U_\pm(x_{1,\pm})$ (when $x_{0,\pm} > x_{1,\pm}$, modulus is required). Unlike the

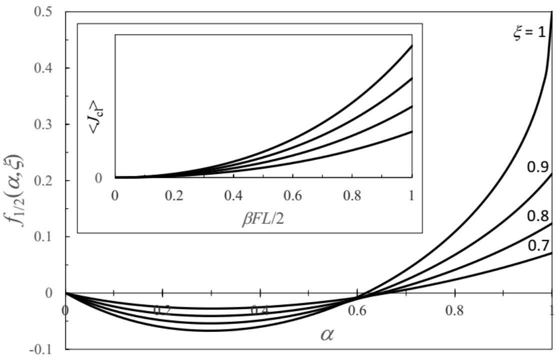

FIG. 3. A family of dependencies of the function $f_{1/2}(\alpha,\xi)$ on α, describing force-dependent tunnel fluxes of the adiabatic quantum rocking ratchet for various values of the sawtooth-potential asymmetry parameter ξ (indicated near the curves). The inset shows schematically the thermoactivated fluxes of the corresponding classical rocking ratchet for the same values of the asymmetry parameter, in the order from top to bottom.

expression (8) for the classical thermoactivated flux, for which its sign is obtained automatically from the sign of the applied force, the sign of the expression for the tunneling flux (10) is not automatically processed and must be specified explicitly. Represent the expression for the average quantum flux (5) in a form convenient for analyzing its sign:

$$\langle J_{\rm qm}\rangle = Ae^{-S_+/\hbar}\{1-\exp\left[-(S_- - S_+)/\hbar\right]\} \tag{11}$$

so the sign of $J^{(\rm qu)}$ is the same as the sign of the difference $S_- - S_+$. For a sawtooth potential, the sub-barrier region is a triangle, so that, in Eq. (10), the integral over its region can be taken analytically. To understand the features of the integration result, we consider a more general case of the integrand: The piecewise linear function $y(x)$, describing the sides of the triangle, enters the integrand to an arbitrary power ν. The result of the integration can readily be written as:

$$I_\nu = \int\limits_{x_0}^{x_1} y^\nu(x)dx = \frac{1}{\nu+1}h^\nu a, \tag{12}$$

where $a = |x_1 - x_0|$ and h are the base and height of the triangle, respectively. The values a and h are different in the states $''+''$ and $''-''$, so we will use the notations $a_\pm$ and $h_\pm$ for them (see the inset in Fig. 2). For the potential $U_+(x)$, the subbarrier region is defined as follows. Tunneling can occur from a potential minimum only to the right (see the triangle in Fig. 2 depicted by the solid thick lines). Then from the geometric constructions in Fig. 2 it follows that $h_+ = V_0 - Fl$ and $a_+ = V_0L/[V_0 + F(L-l)]$. Similarly, for the potential $U_-(x)$, tunneling can only occur to the left (see the triangle in Fig. 2 depicted by the dashed thick lines), so that $h_- = V_0 - F(L-l)$ and $a_- = V_0L/(V_0 + Fl)$. Let us introduce the quantities convenient for further analysis

$$\alpha \equiv FL/V_0\,, \quad \xi \equiv l/L. \tag{13}$$

They characterize the relative magnitude of the rocking force ($0 \le \alpha < 1$) and the asymmetry coefficient of the sawtooth potential. Then the expressions for the quantities $I_{\nu,\pm}$ defined by Eq. (12) will take on the form:

$$I_{\nu,\pm}(\alpha,\xi) = \frac{V_0^\nu L}{\nu+1}\times\begin{cases}\varphi_\nu(\alpha,\xi),\\ \varphi_\nu(\alpha,1-\xi),\end{cases}$$

$$\varphi_\nu(\alpha,\xi) \equiv \frac{(1-\alpha\xi)^\nu}{1+\alpha(1-\xi)}. \tag{14}$$

The desired expressions for the action $S_\pm$ (10) and the difference $S_- - S_+$, which specifies the sign of the average tunneling flux (11), are expressed in terms of functions $\varphi_\nu(\alpha,\xi)$ as follows:

$$S_\pm/\hbar = \frac{4L\sqrt{2mV_0}}{3\hbar}\times\begin{cases}\varphi_{1/2}(\alpha,\xi),\\ \varphi_{1/2}(\alpha,1-\xi),\end{cases}$$

$$(S_- - S_+)/\hbar = \frac{4L\sqrt{2mV_0}}{3\hbar}f_{1/2}(\alpha,\xi)\,, \tag{15}$$

where we introduced the auxiliary function

$$f_\nu(\alpha,\xi) \equiv \varphi_\nu(\alpha,1-\xi) - \varphi_\nu(\alpha,\xi). \quad (16)$$

The expansion of functions $\varphi_\nu(\alpha,\xi)$ and $f_\nu(\alpha,\xi)$ in small parameter α, the values ν being arbitrary, can be written as:

$$\varphi_\nu(\alpha,\xi) \approx 1 - \alpha[1+(\nu-1)\xi] + \alpha^2[1+(\nu-2)\xi+(\nu-2)(\nu-1)\xi^2/2],$$
$$f_\nu(\alpha,\xi) \approx (2\xi-1)\alpha\,[\nu-1+(\nu-2)(\nu+1)\alpha/2]. \quad (17)$$

For the value of interest $\nu = 1/2$, we obtain the important result

$$f_{1/2}(\alpha,\xi) \approx -(2\xi-1)(\alpha/2)\,(1-9\alpha/4), \quad (18)$$

from which it follows that the contribution to the average flux, which is linear in the force modulus (α), is not only nonzero, but also opposite in sign to the analogous contribution to the flux for the classical rocking ratchet (compare with the expression (7)). The nonzero value of the linear contribution is a consequence of the absence of reverse tunneling fluxes, with respect to the direction of the applied force, as well as of the root dependence of the integrand in the Gamow formula (10). The root dependence, in turn, is a consequence of the quadratic dependence of the kinetic energy of particles on their velocities. Note that at $\nu = 1$, the linear term of the expansion of $f_\nu(\alpha,\xi)$ in small α (polarizability) is absent, while at $\nu > 1$, it becomes positive. The result of the integration in formula (12) says that, at $\nu < 1$, the dependence $a(\alpha)$ will predominate over the dependence $h(\alpha)$. Since, at $\xi > 1/2$, the bases of the triangles in Fig. 2 in the state $''+''$ greater than those in the state $''-''$ ($a_+ > a_-$), then the flux in the state $''+''$ is less than that in the state $''-''$; hence, the resulting flux will be in the direction opposite to the direction of the classical flux with the same asymmetry. At $\xi < 1/2$, the motion directions of both the classical and quantum ratchets reverse, but the resulting flux will again be in the direction opposite to the direction of the classical flux. The family of dependencies of the function $f_{1/2}(\alpha,\xi)$ on α corresponding different values $\xi > 1/2$ illustrates this result (Fig. 3). While the approximate Eq. (18) predicts the flux reversal at $\alpha = 4/9 \approx 0.44$, the exact values of the parameter α, at which the flux reverses, depend on the asymmetry parameter ξ and lie in the region of values exceeding 0.6. The family of functions $f_{1/2}(\alpha,\xi)$ for various α is shown in Fig. 4. For small α values, the function $f_{1/2}(\alpha,\xi)$ is negative, while for large α values, it is positive. The smallest α value, at which, for $\xi \neq 1/2$, $f_{1/2}(\alpha,\xi) = 0$, is achieved for the extremely asymmetric sawtooth potential ($\xi = 1$). The inequality $f_{1/2}(\alpha,1) = 0$ takes place at $\alpha^2 + \alpha - 1 = 0$; hence, the root of the last equation $\alpha_1 = (\sqrt{5}-1)/2 \approx 0.618$ is precisely the desired minimum value. Note that the boundary value $F \approx 0.618\,V_0/L$ was given in the review [1] under the semiclassical consideration of the rocking ratchet in the extremely asymmetric potential case. The largest α value, at which, for $\xi \neq 1/2$, it will be obtained $f_{1/2}(\alpha,\xi) = 0$, is determined from the condition of the zero first derivative of the function $f_{1/2}(\alpha,\xi)$ at the point $\xi = 1/2$ ($f'_\alpha(1/2) = \alpha(1-3\alpha/2)(1-\alpha/2)^{-5/2} = 0$), which yields $\alpha_2 = 2/3 \approx 0.667$. Therefore, in the range of values $\alpha_1 < \alpha < \alpha_2$, the sign of $f_{1/2}(\alpha,\xi)$ depends on the asymmetry of the sawtooth potential (parameter ξ).

Note that the dependence of the motion direction of a quantum rocking ratchet on the applied force is an attribute not only of the sawtooth shape of the potential. In Ref. [13], the periodic potential profile was chosen as the sum of two sinusoids

$$V(x) = \tilde{V}_0[\sin(2\pi x/L) - 0.22\sin(2\pi x/L)], \quad (19)$$

and the only force value F corresponded to the modified parameter $\tilde{\alpha} \approx FL/\tilde{V}_0 = 0.4\pi$. It can be readily shown that the best approximation of the function (19) by a sawtooth potential is realized at $V_0 \approx 2.50\,\tilde{V}_0$ and $\xi \approx 0.655$ (see the inset in Fig. 5). Since $\alpha \equiv FL/V_0 \approx 0.4\,\tilde{\alpha}$,

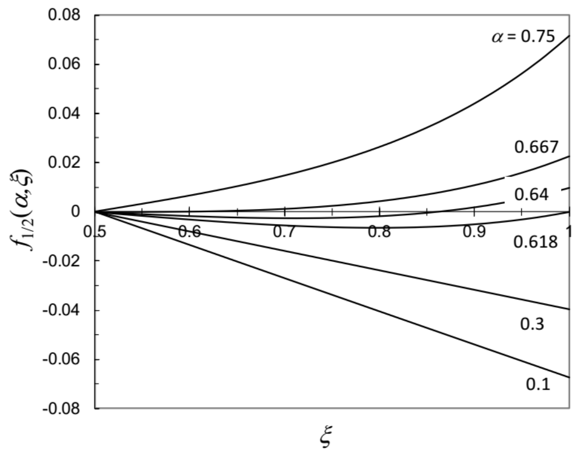

FIG. 4. A family of dependencies of the function $f_{1/2}(\alpha, \xi)$ on the asymmetry parameter ξ for various values of the force parameter α.

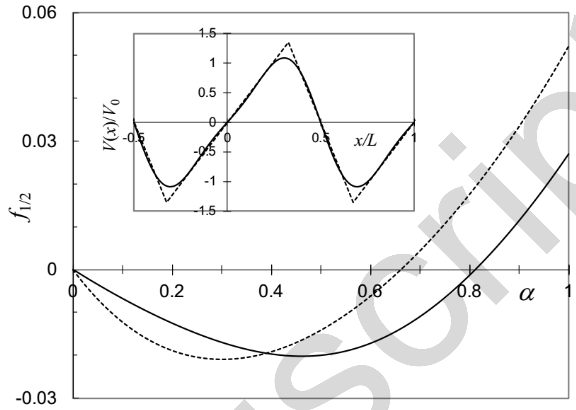

FIG. 5. The dependence of the function $f_{1/2}$, describing the tunneling flux for the potential profiles (19) (solid curve) and (3) (dashed curve), on the parameter α. In the inset, the corresponding curves show the potential of two sinusoids (19), which is approximated, using the least squares method, by the sawtooth potential (3) with parameters $V_0 \approx 2.50\,\tilde{V}_0$ and $\xi \approx 0.655$.

then the value $\tilde{\alpha} \approx 0.4\pi$ from [13] corresponds to $\alpha \approx 0.503 < \alpha_1$, so that $f_{1/2}(\alpha, \xi) < 0$ and the direction of the quantum flux is indeed opposite to the direction of the classical one. For the average quantum flux defined by the relations (11) and (15), the dependence $f_{1/2}$ on α with the potential profile (19) is calculated by numerical evaluating the integrals in (10). Comparison of the numerical result with the analytical result based on the sawtooth approximation to the potential (the solid and dashed curves in Fig. 5) shows that both dependencies behave in a similar way, and the boundary value α at which $f_{1/2}$ changes its sign is approximately equal to 0.81 for the potential of two sinusoids and 0.66 for the sawtooth one.

4. Discussion and conclusions

The reversal of the direction of the rocking-ratchet motion with decreasing temperature when the assumption of classical character of the motion is replaced by the quantum one lies in different dependencies of fluxes on the barrier parameters and the probability of particle transfer in the thermally activated and tunneling cases. In the first case, only the value of the barrier to be overcome becomes important; in the second case, the value of the tunneling path in the sub-barrier region also does. The use of a sawtooth potential as a model for explaining the occurrence of the motion in a certain direction turned out to be very useful, since it allowed to obtain analytical dependencies of the fluxes on the applied fluctuating force, to visualize and explain under which conditions a decrease in the tunneling path can dominate over a decrease in the barrier value. The result of competition of these two factors is determined by the symmetry of the periodic potential profile and the magnitude of the applied fluctuating force which perturbs this profile. The analysis of the obtained expressions showed that, in the region of small values of the fluctuating applied force F, the quantum rocking ratchet moves in the direction opposite to the classical one (in full accordance with the results of Ref. [13]). A new result of our work is that this behavior changes in the region of large F: The quantum ratchet moves in the same direction as the classical one. This result can be understood if we take into account that, at large F, one of the potential barriers becomes small;

this increases not only the thermally activated flux, but also the tunneling flux in the same direction. To analyze the results being obtained, a more general case of the integrand in the Gamow formula was considered, in which the potential energy enters to an arbitrary power ν (instead of the root dependence with $\nu = 1/2$). It has been shown that the contribution to the flux, which is linear in absolute value of the rocking force, occurs from the absence of tunneling fluxes, which are inverse to the direction of the applied force, and occurs only at $\nu < 1$, that is, just in the case of the root dependence of the integrand on the potential energy in the Gamow formula. In this case, the main contribution to the flux comes from the force dependence of the tunneling-path length compared to the force dependence of the barrier height. The analysis of the lengths of tunneling paths in sawtooth potentials of various asymmetries, when the rocking force is applied in the forward and reverse directions, does lead to the conclusion that the quantum rocking ratchet moves in the direction opposite to the classical one.

It has been found that for the interval of values of the asymmetry parameter $\xi \equiv l/L$ of the sawtooth potential (l is the width of one of its linear sections), corresponding to the positive thermoactivation flux ($1/2 < \xi \leq 1$), there exists such an interval of values of $\alpha \equiv FL/V_0$ (V_0 is the energy barrier of the periodic potential with the period L), $0.618 < \alpha < 0.667$, in which the sign of the tunneling flux changes with changes in the asymmetry parameter ξ. The tunneling flux is also numerically calculated for the potential of two sinusoids, which can be approximated with a high accuracy by a sawtooth potential. The results obtained for the potential of two sinusoids also confirm the conclusion that the effect of reversal of the motion of the quantum rocking ratchet with respect to the classical one (motion reversal with decreasing temperature) takes place only for small α values, $\alpha < 0.81$. This allows one to state that the regularities established in this paper do not depend on the specific type of an asymmetric periodic profile perturbed by an external fluctuating force, and are of a general nature. This conclusion can be confirmed by the experimentally observed dependence of the tunneling flux on the applied force (see Fig. 2 (B) in [14]): At sufficiently low temperatures, the nonmonotonic function $J(F)$ changes its sign at a certain value of F, while at higher temperatures, the function $J(F)$ becomes monotonic and of constant sign.

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