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On One Approach to Mathematical Modeling of Socio-Economic Development of Regions

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In this paper we use various methods of modeling of random processes to research the stochastic modification of the model of the controlled sustainable development of the region. Numerical experiments were carried out for the proposed model. The experimental results indicate a significant influence of randomness on the behavior of the system.

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1. Introduction

At present, almost all problems arising in industry and industry go through the stage of mathematical modeling. This allows us to predict the results of activities and reduce the costs of achieving our goals.

Usually deterministic models are used for modeling, but now it is not enough to use models of this kind in such industries as finance, planning, and modeling various kinds of diffusion processes. It is necessary to propose models which take into account the random nature of ongoing processes. Moreover, problems arise not only with regard to the influence of randomness in the construction of a certain model, but also with the choice of methods for solving an already posed problem. The authors propose an approach based on the introduction of a random term into the existing deterministic model of the controlled sustainable development of the region and using numerical methods to approximate for the resulting system.

In this article, we will focus on specific theoretical approaches to taking into account the random nature of the simulated process, so we present some preliminary data used below.

Let us define stochastic or random process as a family of random variables

$$\{X_t \equiv X(t) \equiv X(\omega, t),\ \omega \in \Omega,\ t \in [0,T]\}$$

which we consider on the probability space $(\ \Omega,\ A, \{\mathcal{F}_t\},\ P\ )$, where Ω is a sample space, A is an event space, P is a probability function, $\{\mathcal{F}_t\}$ is a filtration, i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq A,\ s < t$.

There are a large number of random processes that correspond to this definition and have various properties. So processes can be continuous, discrete, stationary, with independent increments, etc. But in this paper we will focus on the consideration of random processes based on a mathematical abstraction of Brownian motion, which is also called the Wiener process.

Let us define the Wiener process $W_t,\ t \in [0,T]$ as follows:

1. W_0=0 almost surely,
2. W_t has independent increments,

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3. W_t is continuous on t,

4. The increments are normally distributed, i.e. $W_t - W_s \equiv \mathcal{N}(0, t-s)\,, \quad \forall t, s : t > s$, here $\mathcal{N}$ is normal random value with mean 0 and variance $t - s$

The Wiener process is stationary and Markovian. The last property means that the conditional probability distribution of future states depends only on the current state (and not on past states). But the random processes observed in reality depend on their history, so using a Wiener-like process is not enough to formally describe observed processes. Should be considered dependence on the entire process trajectory. To take into account such a dependence, we can use the concept of a stochastic integral. Currently, there are many types of stochastic integrals, so in our research we will consider stochastic Ito integral, which was proposed in [13].

Formally, the integral is introduced according to a scheme similar to the Riemann-Stieltjes integral, but this is only an external similarity, and the integral has a number of properties that greatly complicate calculations when working with it. For example, it should be noted that

$$\int_0^t W_t dW_t = \frac{1}{2}\left(W_t^2 - t\right).$$

This small example shows us, that we can not use well-known rules of integrating of deterministic functions. In his work [14] Itô developed the foundations of the theory of stochastic differential equations (SDE). So stochastic differential equation in Ito sense this is the equation of the form:

$$dX_t = \alpha(X_t, t)dt + \beta(X_t, t)dW_t, \qquad (1)$$

but this is rather a formal description. In research the corresponding integral representation is more often used:

$$X_t = X_0 + \int_0^t \alpha(X_s, s)ds + \int_0^t \beta(X_{s-}, s)dW_s.$$

Properties of the Itô integral and SDE are described in detail in [4, 21].

One of the main questions is the question of the existence of a solution. The conditions of existence of strong solution are following:

$$\|\alpha(x,t) - \alpha(y,t)\|^2 + \|\beta(x,t) - \beta(y,t)\|^2 \le K_1 \|x - y\|^2,$$

$$\|\alpha(x,t)\|^2 + \|\beta(x,t)\|^2 \le K_2\left(1 + \|x\|^2\right)$$

where x, y are elements of some space $\mathcal{X}$, $\|\cdot\|$ is a norm defined over space $\mathcal{X}$, $K_1 < \infty$, $K_2 < \infty$. Here we need to remind, that existence of strong solution means, that exists some process which satisfies to equation (1). In what follows, we will consider only those equations that satisfy the above conditions.

There are few of stochastic equations can be solved exactly. One of them is linear equation:

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad r, \sigma \in \mathbb{R}.$$

The solution of this equation has the form:

$$X_t = X_0 \exp\left\{rt - \frac{1}{2}\sigma^2 t + \sigma W_t\right\}.$$

The equations of this type have found wide application in various fields, for example, in the construction of various models in economics and finance (see, e.g. [1–3, 7]). There are many models for describing the behavior of the financial market, e.g. the Vasicek model, the Cox–Ingersoll–Ross model, and so on (e.g. [17, 18]) but separately we need to mention the works of Black, Scholes and Merton [5, 19], who proposed the asset fluctuation model and estimated the price of a European option using this model. In the 1997 Merton and Scholes received Nobel Memorial Prize in Economic Sciences for this result.

This work shows us that we are interested in the values of the functionals on the trajectories of solutions, and not in the trajectories themselves, for example, functionals of the form $E\left[F\left(X_{(\cdot)}, t\right)\right]$. The most commonly used examples of such functionals are the average value of the process, its volatility, asymmetry, etc.

2. Numerical methods for SDE

Since only a small class of SDEs can be solved exactly, the use of numerical methods is of particular importance, among which it is necessary to distinguish two main classes of methods: strong approximations and weak approximations.

Strong approximations are simulation methods. They are based on building a process, that is in some sense close to solution of SDE. Examples are the Euler–Maruyama method, the Milstein method, the Runge–Kutta stochastic method, etc. (see [12, 15, 16, 22]). One of the most commonly used methods is the Milstein method [20].

If it is necessary to calculate the value of the functional on the solution, then it is necessary to simulate the set of trajectories. In this case, the discreteness in time and the size of the set depend on the required accuracy of estimating the value of the functional. Thus, the use of such methods requires significant computing power and large amount of memory.

Weak approximations methods are based on the construction of approximations of some parameters of random processes, for example, moments, distributions, characteristic functionals of solution and so on. In our studies, we will use approximations for the first three moments of the solution of equation (1). The ideas behind this method are described in [10, 11].

In the works [23, 24] was constructed the following formula, which is approximatelly exact for the first three moments of of the equation (1) in the sentence about the existence of a strong solution:

$$E\left[F\left[X_{(\cdot)}\right]\right] \approx J\left[F,Y\right] = \frac{1}{2}\sum_{j=1}^{2} A_j \times \int_0^1 \int_0^1 \int_{-1}^1 F[Y_j(\cdot,\tau_1,\tau_2,v)]d\tau_1 d\tau_2 dv \qquad (2)$$

where $A_1 + A_2 = 1$, $a_{1,1} = \frac{1}{2}\left(1-\sqrt{-\frac{A_2}{A_1}}\right)$, $a_{1,2} = \frac{1}{2}\left(1+\sqrt{-\frac{A_2}{A_1}}\right)$, $a_{2,1} = \frac{1}{2}\left(1-\sqrt{-\frac{A_1}{A_2}}\right)$, $a_{2,2} = \frac{1}{2}\left(1+\sqrt{-\frac{A_1}{A_2}}\right)$, $\rho_{j,k}\left(t,\tau_k\right) = a_{j,k}1_{[\tau_k,1]}\left(t\right)$, $k = 1,2$, $\rho\left(t,v\right) = \mathrm{sign}(v)\,1_{[|v|,1]}(t)$, and

$$\begin{aligned}
Y_j\left(t,\tau_1,\tau_2,v\right) = X\left(0\right) + \alpha\Big(& X(0) + \alpha\left(X(0) + \beta\left(X\left(0\right)\right)\ \mathrm{sign}\left(v\right) 1_{(|v|,1]}\left(\tau_2\right)\right)\ a_{j,2}1_{(\tau_2,1]}\left(\tau_1\right) \\
& + \beta\left(X\left(0\right) + \alpha\left(X\left(0\right)\right) a_{j,2}1_{(\tau_2,1]}\left(|v|\right)\right) \mathrm{sign}\left(v\right) 1_{(|v|,1]}\left(\tau_1\right) \Big)\rho_{j,1}(t,\tau_1) \\
& + \alpha\Big(X\left(0\right) + \alpha\left(X\left(0\right) + \beta\left(X(0)\right)\mathrm{sign}(v)\, 1_{(|v|,1]}\left(\tau_1\right)\right) a_{j,1}1_{(\tau_1,1]}\left(\tau_2\right) \\
& + \beta\left(X\left(0\right) + \alpha\left(X(0)\right) a_{j,1}1_{(\tau_1,1]}\left(|v|\right)\right) \mathrm{sign}(v)\, 1_{(|v|,1]}\left(\tau_2\right)\Big)\rho_{j,2}(t,\tau_2) \\
& + \beta\Big(X\left(0\right) + \alpha\left(X\left(0\right) + \alpha\left(X(0)\right)\ a_{j,2}1_{(\tau_2,1]}\left(\tau_1\right)\right)\ a_{j,1}1_{(\tau_1,1]}\left(|v|\right) \\
& + \alpha\left(X\left(0\right) + \alpha\left(X\left(0\right)\right) a_{j,1}1_{(\tau_1,1]}\left(\tau_2\right)\right)\ a_{j,2}1_{(\tau_2,1]}\left(|v|\right)\Big)\rho(t,v).
\end{aligned}$$

$$Y_j(t,\tau_1,\tau_2,v) = X(0) + \alpha\Big(X(0) + \alpha\left(X(0) + \beta\left(X(0)\right)\ \mathrm{sign}\left(v\right) 1_{(|v|,1]}\left(\tau_2\right)\right)\ a_{j,2}1_{(\tau_2,1]}\left(\tau_1\right)$$
$$+\beta\left(X(0) + \alpha\left(X(0)\right) a_{j,2}1_{(\tau_2,1]}\left(|v|\right)\right)\mathrm{sign}\left(v\right) 1_{(|v|,1]}\left(\tau_1\right)\Big)\rho_{j,1}(t,\tau_1)$$
$$+\alpha\Big(X(0) + \alpha\left(X(0) + \beta\left(X(0)\right)\mathrm{sign}(v)\, 1_{(|v|,1]}\left(\tau_1\right)\right) a_{j,1}1_{(\tau_1,1]}\left(\tau_2\right)$$
$$+\beta\left(X(0) + \alpha\left(X(0)\right) a_{j,1}1_{(\tau_1,1]}\left(|v|\right)\right)\mathrm{sign}(v)\, 1_{(|v|,1]}\left(\tau_2\right)\Big)\rho_{j,2}(t,\tau_2)$$
$$+\beta\Big(X(0) + \alpha\left(X(0) + \alpha\left(X(0)\right)\ a_{j,2}1_{(\tau_2,1]}\left(\tau_1\right)\right)\ a_{j,1}1_{(\tau_1,1]}\left(|v|\right)$$
$$+\alpha\left(X(0) + \alpha\left(X(0)\right) a_{j,1}1_{(\tau_1,1]}\left(\tau_2\right)\right)\ a_{j,2}1_{(\tau_2,1]}\left(|v|\right)\Big)\rho(t,v).$$

3. A mathematical model of the controlled sustainable development of the region

The formula (2) can be applied to study the behavior of economic models containing a chaotic component. Let us consider the the model of the controlled sustainable development of the region. The model was proposed in [8] and has the form

$$\begin{aligned} ds_1(t) &= (u_1 s_1(t) - u_2 s_1(t) s_2(t) \\ &\quad - u_3 s_1(t) s_3(t))\, dt, \\ ds_2(t) &= (-u_4 s_2(t) + u_5 s_1(t) s_2(t) \\ &\quad + u_6 s_2(t) s_3(t))\, dt, \\ ds_3(t) &= (u_7 s_3(t) - u_8 s_1(t) s_3(t) \\ &\quad - u_9 s_2(t) s_3(t))\, dt \end{aligned} \qquad (3)$$

where s_1 is the regional population, s_2 is number of jobs in the real sector of the regional economy, s_3 is the indicator of the regional energy supply, with following parameters:

- u_1 is the demographic activity coefficient,
- u_2 is the coefficient of people's anti-motivation to childbearing,
- u_3 is the energy supply coefficient,
- u_4 is the coefficient of people's interest in economic development,
- u_5 is the coefficient of the real sector economic development,
- u_6 is the coefficient of energy supply per workplace,
- u_7 is the energy supply coefficient of the region,
- u_8 is the conformity ratio of the population with the energy supply,
- u_9 is the conformity ratio of the economic development with the energy supply.

Of particular interest are the values of the parameters at which a stable behavior of the solution of the system is observed, so in work [9] this model was examined for the presence of stability points (or close to them). One of these points is given by the parameters $u_1 = 0.087$, $u_2 = 0.087$, $u_3 = 0.087$, $u_4 = 0.049$, $u_5 = 1.02$, $u_6 = 1.02$, $u_7 = 7.7$, $u_8 = 0.095$, $u_9 = 3.9$.

Fig. 1 shows the trajectory of the solution of the system under the specified initial conditions $(s_1(0), s_2(0), s_3(0)) = (1, 1, 1)$. At the figure we denotes s_1, s_2, s_3 as x, y, z correspondingly.

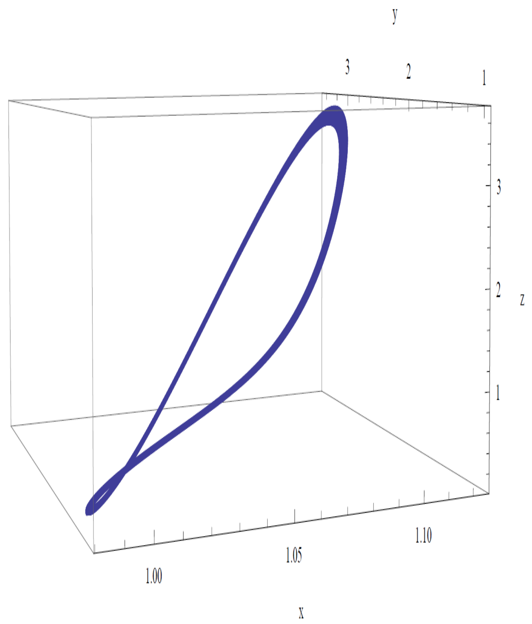

FIG. 1: The solution of the model (3).

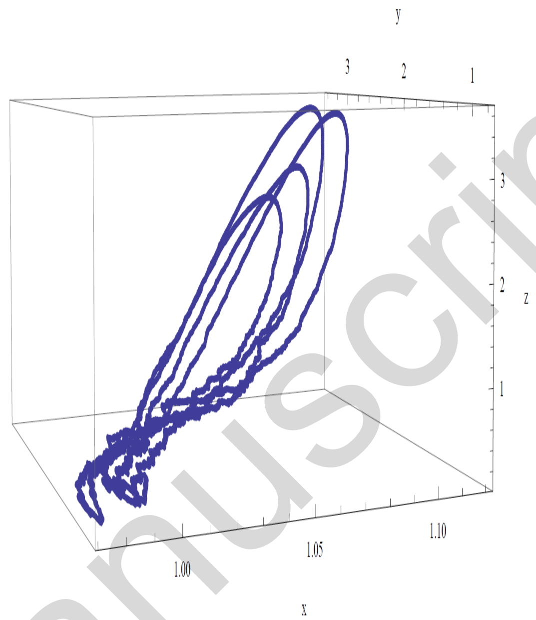

FIG. 2: Single trajectory of SDE solution.

The proposed deterministic model takes into account the change in the population of the region and the indicator of energy supply per number of jobs, but it is obvious that such an impact has a random nature. To assess the impact of randomness on behavior, the following changes were made to the model [6]:

$$ds_1(t) = (u_1 s_1(t) - u_2 s_1(t) s_2(t) - u_3 s_1(t) s_3(t))\, dt,$$

$$ds_2(t) = (-u_4 s_2(t) + u_5 s_1(t) s_2(t) + u_6 s_2(t) s_3(t))\, dt + \left(\frac{a}{b + c s_1^2(t)}\right)\left(\frac{d}{e + f s_3^2(t)}\right) dW(t),$$

$$ds_3(t) = (u_7 s_3(t) - u_8 s_1(t) s_3(t) - u_9 s_2(t) s_3(t))\, dt,$$

where a, b, c ,d, e, f are calibration coefficients that can be obtained from statistical data processing later. Since the purpose of this work is to evaluate the influence of the stochastic term on the behavior of the model as a whole, we added randomness to only one equation.

Fig. 2 show us one single trajectory of the solution of the equation. Here we use Milstein method. Parameters $a = b = c = d = f = 1$,

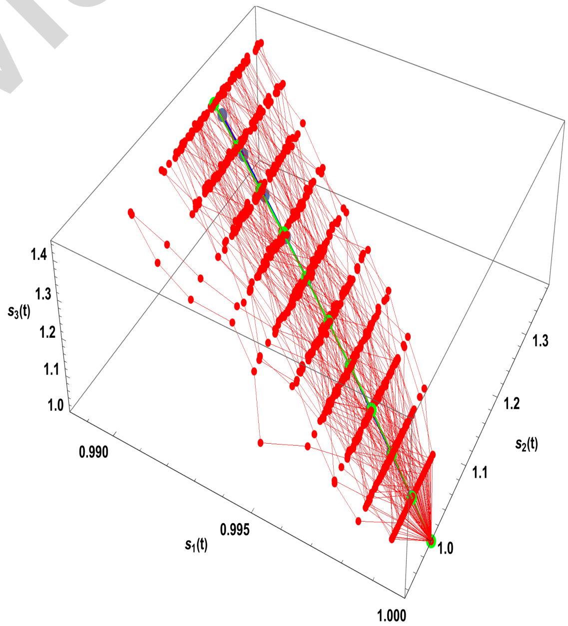

FIG. 3. The result of Monte Carlo simulations vs proposed method.

other parameters and initial conditions are the same as for model 3.

The main conclusion of the simulation results, that the random component significantly changes the behavior of the trajectory. And this

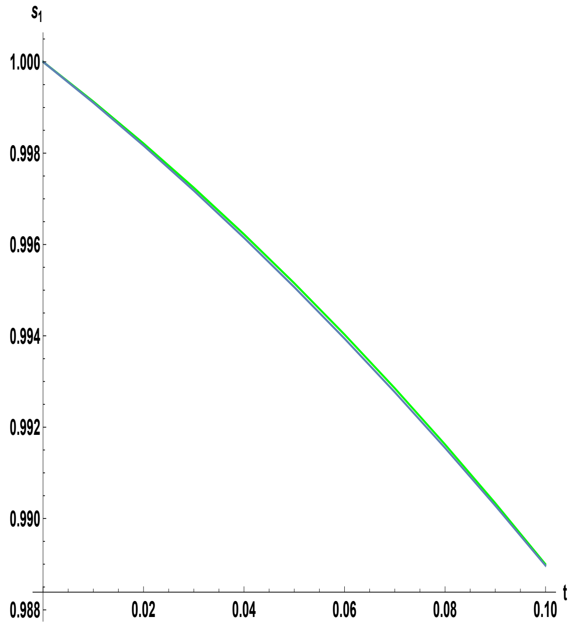

FIG. 4. The result of Monte Carlo simulations vs proposed method for s_1.

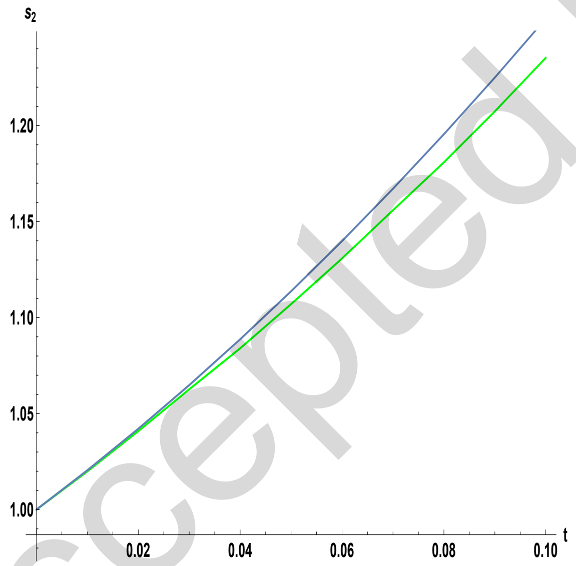

FIG. 5. The result of Monte Carlo simulations vs proposed method for s_2.

is only one trajectory, and the calculation of the creation of the average value will require modeling a significant number of trajectories.

Using the formula (2) will significantly reduce the calculation time. First, we write the

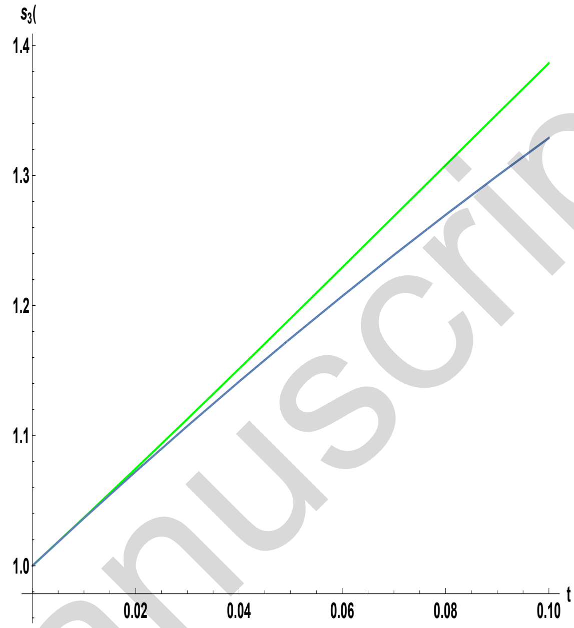

FIG. 6. The result of Monte Carlo simulations vs proposed method for s_3.

equation in appropriate integral form as follows

$$X(t) = X(0) + \int_0^t \alpha(X(s))ds + \int_0^t \beta\left(X(s-)\right) dW(s) \tag{4}$$

where $W(t)$ is the Wiener process, $X(t) = (s_1(t), s_2(t), s_3(t))$,

$$\alpha(X(t))$$

$$= \begin{pmatrix} u_1 s_1(t) - u_2 s_1(t) s_2(t) - u_3 s_1(t) s_3(t) \\ -u_4 s_2(t) + u_5 s_1(t) s_2(t) + u_6 s_2(t) s_3(t) \\ u_7 s_3(t) - u_8 s_1(t) s_3(t) - u_9 s_2(t) s_3(t) \end{pmatrix},$$

$$\beta\left(X\left(t\right)\right) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \left(\frac{a}{b+cs_1^2(t)}\right)\left(\frac{d}{e+fs_3^2(t)}\right) \\ 0 \end{pmatrix}.$$

We suppose that the integrals $\int_0^t \beta_k\left(X\left(s-\right)\right) \mathrm{dW}(s)$, $k = 1, 2, 3$ are Itô's integrals.

Now we can apply the proposed formula to the equation (4). We use parameters same

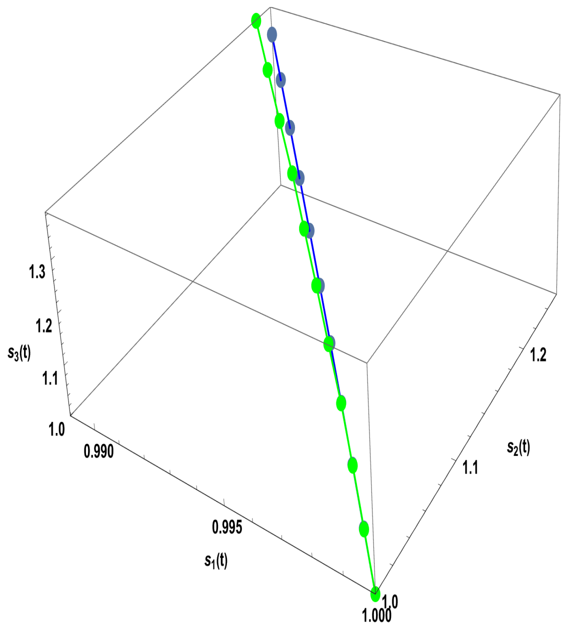

FIG. 7. The result of Monte Carlo simulations vs proposed method in 3D.

as for previous calculations. Calculated fuctional is average of trajectories, i.e. $E[X(t)] = (E[s_1(t)], E[s_2(t)], E[s_3(t)])$. The approximation has the form:

$$E[s_1(t)] \approx 1 - 0.087\,t - 0.233213\,t^2 + 3.58739 \times 10^{-6}\,t^3,$$
$$E[s_2(t)] \approx 1 + 1.991\,t + 5.6641\,t^2 - 0.0845331\,t^3,$$
$$E[s_3(t)] \approx 1 + 3.705\,t - 4.19939\,t^2 + 0.315994\,t^3.$$

The calculation performed at the interval $[0, 0.05]$ with step of discretization $\Delta t = 0.01$.

Fig. 3 demonstrate results of calculations. Here and below the blue line denotes the value, calculated by composition Monte Carlo method and Milstein scheme, red lines are simulated trajectories, which were used in Monte Carlo method. At this figure we show only 100 trajectories, but in calculations 1000 of trajectories were used.

Figures 4, 5, 6 show the dynamics of coordinates $E[s_1(t)], E[s_2(t)], E[s_3(t)]$, and Fig. 7 shows the relative position of the averaged trajectory in 3D.

W need to note, that the values, calculated by both methods are close, but weak method is much faster.

4. Conclusion

This paper shows that the addition of a random component can lead to a significant change in the behavior of the model used. In some cases, when researchers are interested not so much in the shape of the trajectories of individual solutions as in the values of some given functionals, weak methods can be used.

Weak methods provide the specified accuracy and minimize the requirements for computing systems used in modeling. To calculate the moments of random processes at relatively large values, the approach proposed in [25].

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