

ON GENUS OF DIVISION ALGEBRAS

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ABSTRACT. The genus $\mathbf{gen}(\mathcal{D})$ of a finite-dimensional central division algebra \mathcal{D} over a field F is defined as the collection of classes $[\mathcal{D}'] \in \mathrm{Br}(F)$, where \mathcal{D}' is a central division F -algebra having the same maximal subfields as \mathcal{D} . We show that the fact that quaternion division algebras \mathcal{D} and \mathcal{D}' have the same maximal subfields does not imply that the matrix algebras $M_l(\mathcal{D})$ and $M_l(\mathcal{D}')$ have the same maximal subfields for $l > 1$. Moreover, for any odd $n > 1$, we construct a field L such that there are two quaternion division L -algebras \mathcal{D} and \mathcal{D}' and a central division L -algebra \mathcal{C} of degree and exponent n such that $\mathbf{gen}(\mathcal{D}) = \mathbf{gen}(\mathcal{D}')$ but $\mathbf{gen}(\mathcal{D} \otimes \mathcal{C}) \neq \mathbf{gen}(\mathcal{D}' \otimes \mathcal{C})$.

The genus $\mathbf{gen}(\mathcal{D})$ of a finite-dimensional central division algebra \mathcal{D} over a field F is defined as the collection of classes $[\mathcal{D}'] \in \mathrm{Br}(F)$, where \mathcal{D}' is a central division F -algebra having the same maximal subfields as \mathcal{D} . This means that \mathcal{D} and \mathcal{D}' have the same degree n , and a field extension K/F of degree n admits an F -embedding $K \hookrightarrow \mathcal{D}$ if and only if it admits an F -embedding $K \hookrightarrow \mathcal{D}'$. Different variations of the notion of the genus are mentioned in [1].

The following questions were formulated in [2, footnote 1 and Remark 2.2]:

Does the fact that division algebras \mathcal{D} and \mathcal{D}' have the same maximal subfields imply that the matrix algebras $M_l(\mathcal{D})$ and $M_l(\mathcal{D}')$ have the same maximal subfields / étale subalgebras for any (or even some) $l > 1$?

Let n_1 and n_2 be relatively prime positive integers. Let also \mathcal{D}_i and \mathcal{D}'_i be central division algebras of degree n_i over a field F for $i = 1, 2$. Is it true that if $\mathbf{gen}(\mathcal{D}_i) = \mathbf{gen}(\mathcal{D}'_i)$ for $i = 1, 2$, then $\mathbf{gen}(\mathcal{D}_1 \otimes \mathcal{D}_2) = \mathbf{gen}(\mathcal{D}'_1 \otimes \mathcal{D}'_2)$?

Negative answers to these questions are given in Theorem 5 and Corollary 6 below.

We use the following notation. For a field F , F^* denotes the multiplicative group of F . F^{*2} denotes the subgroup of squares in F^* . For a field extension K/F and a central simple F -algebra \mathcal{A} , \mathcal{A}_K denotes the tensor product $\mathcal{A} \otimes_F K$ and $\mathrm{res}_{K/F} : \mathrm{Br}(F) \rightarrow \mathrm{Br}(K)$ denotes the restriction homomorphism.

Let F be a field, $\mathrm{char}(F) \neq 2$. Assume that there are two non-isomorphic quaternion division F -algebras \mathcal{A} and \mathcal{B} . We will be interested in finite separable field extensions K/F that satisfy the following three conditions:

- (A) There is $d \in K^* \setminus K^{*2}$ such that there is no $a \in F^*$ such that $da \in K^{*2}$;
- (B) K does not split \mathcal{B} ;
- (C) $K(\sqrt{d})$ splits \mathcal{B} but does not split \mathcal{A} .

Example 1. Let M be a field containing a primitive n th root of 1, $\mathrm{char}(M) \nmid 2n$. Let also $F := M(x, y, z, w)$ be a purely transcendental extension of M of transcendence degree 4 and $\mathcal{A} := (x, y)$, $\mathcal{B} := (w, y)$ quaternion F -algebras. Then $K := F(\sqrt[n]{z})$ is a cyclic extension of F of degree n and K does not split \mathcal{B} . Finally, let $d := w + y(\sqrt[n]{z} + 1)^2$.

Since $K(\sqrt{d})$ is a purely transcendental extension of $M(x, y)$, then $K(\sqrt{d})$ does not split \mathcal{A} . On the other side, $K(\sqrt{d})$ splits \mathcal{B} since d is represented over K by the quadratic form $\langle w, y, -yw \rangle$. Finally, note that there is no $a \in F^*$ such that $da \in K^{*2}$. Thus the extension K/F satisfies conditions (A) - (C). Note also that the symbol F -algebra $(x, z)_n$ of degree and exponent n is split by K . Moreover, if $\text{char}(M) = 0$ and M contains all roots of 1, then for any $n > 1$, the field F has an extension of degree n satisfying conditions (A) - (C).

In the notation above, we have the following

Proposition 2. *Let $c \in F^* \setminus F^{*2}$. Then there exists a regular field extension F_c/F such that*

- (1) *the homomorphism $\text{res}_{F_c/F} : \text{Br}(F) \rightarrow \text{Br}(F_c)$ is injective;*
 - (2) *the field $F_c(\sqrt{c})$ splits the algebras \mathcal{A}_{F_c} and \mathcal{B}_{F_c} .*
- Moreover, if a field extension K/F satisfies conditions (A) - (C) and $c \notin K^{*2}$, then*
- (3) *there is no $a \in F_c$ such that $da \in F_c K^{*2}$;*
 - (4) *the composite $F_c K$ does not split \mathcal{B}_{F_c} ;*
 - (5) *the composite $F_c K(\sqrt{d})$ splits \mathcal{B}_{F_c} but does not split \mathcal{A}_{F_c} .*

Proof. Let $F(x)$ be a purely transcendental extension of F of transcendence degree 1. Let also

$$\mathcal{C} := \mathcal{A}_{F(x)} \otimes (c, x)$$

be a biquaternion $F(x)$ -algebra and F_1 the function field of the Severi-Brauer variety of the algebra \mathcal{C} .

Now let $F_1(y)$ be a purely transcendental extension of F_1 of transcendence degree 1 and

$$\mathcal{D} := \mathcal{B}_{F_1(y)} \otimes (c, y)$$

a biquaternion $F_1(y)$ -algebra. Let also F_c be the function field of the Severi-Brauer variety of the algebra \mathcal{D} .

Since the kernel of the restriction homomorphism $\text{res}_{F_1/F(x)}$ is generated by the class of the algebra \mathcal{C} , then the homomorphism $\text{res}_{F_1/F}$ is injective. Indeed, \mathcal{C} ramifies at the discrete valuation (trivial on F) of $F(x)$ defined by the polynomial x . Hence $[\mathcal{C}] \neq [\mathcal{Q}_{F(x)}]$ for any central simple F -algebra \mathcal{Q} .

Note that F_1 splits \mathcal{C} . Then $\mathcal{A}_{F_1} \cong (c, x)_{F_1}$ and $F_1(\sqrt{c})$ splits \mathcal{A}_{F_1} .

Let K/F be a field extension satisfying conditions (A) - (C). Since F_1/F is a regular field extension, then there is no $a \in F_1$ such that $da \in F_1 K^{*2}$. In particular, this means that $dc \notin F_1 K^{*2}$. Moreover, $F_1 K/K$ is a regular extension of K . Thus, if $c \notin K^{*2}$, then $c \notin F_1 K^{*2}$.

The composite $F_1 K(\sqrt{d})$ is the function field of the Severi-Brauer variety of the $K(\sqrt{d})(x)$ -algebra $\mathcal{C}_{K(\sqrt{d})(x)}$. Hence the kernel of the restriction homomorphism $\text{res}_{F_1 K(\sqrt{d})/K(\sqrt{d})(x)}$ is generated by the class of $\mathcal{C}_{K(\sqrt{d})(x)}$. Since $dc \notin K^{*2}$, then $c \notin K(\sqrt{d})^{*2}$ and $\mathcal{C}_{K(\sqrt{d})(x)}$ ramifies at the discrete valuation (trivial on $K(\sqrt{d})$) of $K(\sqrt{d})(x)$ defined by the polynomial x , but $\mathcal{A}_{K(\sqrt{d})(x)}$ is unramified at this valuation, hence $[\mathcal{A}_{K(\sqrt{d})(x)}] \neq [\mathcal{C}_{K(\sqrt{d})(x)}]$ and $F_1 K(\sqrt{d})$ does not split \mathcal{A}_{F_1} . Analogously, the composite $F_1 K$ does not split \mathcal{B}_{F_1} .

Thus the extension F_1K/F_1 satisfies conditions (A)-(C) with respect to the algebras \mathcal{A}_{F_1} and \mathcal{B}_{F_1} .

The field F_c satisfies conditions (1)-(5) of the proposition by the same arguments as for the field F_1 . We just replace the ground field F by F_1 and the extension K/F by F_1K/F_1 . \square

Remark 3. Conditions (1) and (3)-(5) of Proposition 2 say that if K/F is a field extension satisfying conditions (A)-(C), then the extension F_cK/F_c satisfies conditions (A)-(C) with respect to the algebras \mathcal{A}_{F_c} and \mathcal{B}_{F_c} .

In the notation above, we also have the following

Proposition 4. *Let $U := \{c \in F^* \setminus F^{*2} \mid F(\sqrt{c}) \text{ splits } \mathcal{A} \text{ or } \mathcal{B}\}$. There exists a regular field extension $E(F)/F$ such that*

- (1) *the homomorphism $\text{res}_{E(F)/F} : \text{Br}(F) \rightarrow \text{Br}(E(F))$ is injective;*
 - (2) *the field $E(F)(\sqrt{c})$ splits the algebras $\mathcal{A}_{E(F)}$ and $\mathcal{B}_{E(F)}$ for any $c \in U$.*
- Moreover, if a field extension K/F satisfies conditions (A) - (C), then*
- (3) *there is no $a \in E(F)$ such that $da \in E(F)K^{*2}$;*
 - (4) *the composite $E(F)K$ does not split $\mathcal{B}_{E(F)}$;*
 - (5) *the composite $E(F)K(\sqrt{d})$ splits $\mathcal{B}_{E(F)}$ but does not split $\mathcal{A}_{E(F)}$.*

Proof. Note that for any field extension K/F satisfying conditions (A) - (C), $U \cap K^{*2} = \emptyset$ since K does not split \mathcal{A} and \mathcal{B} .

Let $<$ be a well-ordering on the set U and let c_0 denote its least element. Set $E^{c_0} := F_{c_0}$, where the field F_{c_0} is constructed in Proposition 2.

For $c \in U$, $c \neq c_0$, set

$$E^{<c} := \bigcup_{c' < c} E^{c'} \text{ and } E^c := E_c^{<c},$$

where the field E^c is obtained by applying Proposition 2 to the field $E^{<c}$ and the element $c \in E^{<c}$ and the algebras $\mathcal{A}_{E^{<c}}$ and $\mathcal{B}_{E^{<c}}$. Define also $E(F) := \bigcup_{c \in U} E^c$.

By Proposition 2 and transfinite induction, the field $E(F)$ satisfies conditions (1)-(5) of the proposition. \square

Theorem 5. *Let F be a field such that there are two non-isomorphic quaternion F -algebras \mathcal{A} and \mathcal{B} . There exists a regular field extension L/F with the following properties:*

- (1) *\mathcal{A}_L and \mathcal{B}_L are division algebras and $\mathbf{gen}(\mathcal{A}_L) = \mathbf{gen}(\mathcal{B}_L)$;*
- (2) *If K/F is a field extension of degree n satisfying properties (A) - (C) with respect to the algebras \mathcal{A} and \mathcal{B} , then the matrix algebras $M_n(\mathcal{A}_L)$ and $M_n(\mathcal{B}_L)$ do not have the same maximal subfields;*
- (3) *If K is a field from the previous item and C is a central division F -algebra of exponent n which is split by K , then C_L is a division algebra of exponent n and the algebras $\mathcal{A}_L \otimes C_L$ and $\mathcal{B}_L \otimes C_L$ do not have the same maximal subfields.*

Proof. Let $K_0 := F$. We recursively define K_i , $i \in \mathbb{Z}_{>0}$, to be the field $E(K_{i-1})$ constructed by applying Proposition 4 to the field K_{i-1} and the algebras $\mathcal{A}_{K_{i-1}}$ and $\mathcal{B}_{K_{i-1}}$. Let also $L := \bigcup_{i \geq 0} K_i$.

By induction and Proposition 4, the homomorphism $res_{L/F} : \text{Br}(F) \rightarrow \text{Br}(L)$ is injective. Hence \mathcal{A}_L and \mathcal{B}_L are non-isomorphic division algebras.

Assume that M is a maximal subfield of \mathcal{A}_L . Then there exists $i \geq 0$ such that $M = LM'$, where M' is a quadratic extension of K_i that splits \mathcal{A}_{K_i} . By the construction of K_{i+1} , the field $K_{i+1}M'$ splits the algebra $\mathcal{B}_{K_{i+1}}$. Hence M splits \mathcal{B}_L . Analogously, every maximal subfield of \mathcal{B}_L splits \mathcal{A}_L . Thus the algebras \mathcal{A}_L and \mathcal{B}_L have the same family of maximal subfields, i.e., $\mathbf{gen}(\mathcal{A}_L) = \mathbf{gen}(\mathcal{B}_L)$.

Assume that K/F is a field extension of degree n satisfying conditions (A) - (C) with respect to the algebras \mathcal{A} and \mathcal{B} . By induction and Proposition 4, the composite $LK(\sqrt{d})$ splits \mathcal{B}_L but does not split \mathcal{A}_L . Then $LK(\sqrt{d})$ embeds in $M_n(\mathcal{B}_L)$ but does not embed in $M_n(\mathcal{A}_L)$. Hence $M_n(\mathcal{A}_L)$ and $M_n(\mathcal{B}_L)$ do not have the same maximal subfields.

Finally, let \mathcal{C} be a central division F -algebra of exponent n which is split by K . Since the homomorphism $res_{L/F}$ is injective, then the exponent of \mathcal{C}_L is n . Since the exponent of \mathcal{C}_L divides its index, then \mathcal{C}_L is a division algebra. The composite $LK(\sqrt{d})$ splits $\mathcal{B}_L \otimes \mathcal{C}_L$ but does not split $\mathcal{A}_L \otimes \mathcal{C}_L$. This means that $\mathcal{A}_L \otimes \mathcal{C}_L$ and $\mathcal{B}_L \otimes \mathcal{C}_L$ do not have the same maximal subfields. \square

Corollary 6. *There exists a field L such that there are two quaternion division L -algebras \mathcal{D} and \mathcal{D}' such that $\mathbf{gen}(\mathcal{D}) = \mathbf{gen}(\mathcal{D}')$, but for any $n > 1$ the matrix algebras $M_1(\mathcal{D})$ and $M_1(\mathcal{D}')$ do not have the same maximal subfields.*

Proof. Let F be a field such that there are two non-isomorphic quaternion F -algebras \mathcal{A} and \mathcal{B} and for any $n > 1$, the field F has an extension of degree n satisfying conditions (A) - (C) with respect to the algebras \mathcal{A} and \mathcal{B} . Let L be the field constructed in Theorem 5. By Theorem 5, the algebras $\mathcal{D} := \mathcal{A}_L$ and $\mathcal{D}' := \mathcal{B}_L$ have the required properties. \square

REFERENCES

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