Let $f$ be an invertible function. A parastrophe $\mathcal{f}$ of $f$ is defined by

$$\mathcal{f}(x_1; x_2) = x_{3\sigma} \iff f(x_1; x_2) = x_3$$

for any $\sigma \in S_3$, where $S_3 := \{e, r, s, r s, r s s\}$ and $s := (12)$, $e := (31)$. Let $f_1, f_2, f_3, f_4$ be binary invertible functions defined on a set $Q$. Then a quadruple $(f_1; \ldots; f_4)$ is a solution of

$$F_1(y; x) = F_4(x; F_3(x; f_4(x; x)))$$

iff there exists a substitution $\alpha$ and an element $a \in Q$ such that

$$f_1(y; y) = a, \quad f_2(x; \alpha x) = a, \quad f_4(x; x) = \mathcal{f}_3(x; \alpha x).$$

References


SYSTEMS OF LINEAR CONGRUENCES, BALANCED MODULAR LABELLINGS OF GRAPHS AND CHROMATIC TOTIENTS

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The present research is devoted to some number-theoretic consequences of certain notions and results of algebraic graph theory. Given a finite simple connected graph $G = (V, E)$, an orientation of its edges and a natural number $k$, we consider edge $k$-labellings $f : E \to \mathbb{Z}_k^*$ satisfying Kirchhoff’s circuit law, where $\mathbb{Z}_k^*$ is the set of invertible elements of the ring $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. In terms of variables $x_e = f(e), e \in E$, we consider the system of homogeneous linear congruences modulo $k$ which correspond to the (simple, independent) cycles of $G$, have coefficients $0$ and $\pm 1$ (moreover, their matrix is unimodular) and are subject to the 'side condition' that all values of their variables are coprime with $k$. The solutions with the latter property are called invertible. The choices of edge orientations and independent cycles do not influence the resulting system of congruences up to equivalence. Let $R(G, k)$ be the number of invertible solutions of such a system.

**Theorem.** For any finite connected graph $G$, $R(G, k)$ is the multiplicative arithmetic function of $k$ that is determined by the formula

$$R(G, p^\alpha) = \chi(G, p)p^{(a-1)(n-1)-1}$$  \hspace{1cm} (1)

for every prime $p$ and integer $\alpha \geq 1$, where $\chi(G, z)$ is the chromatic polynomial of $G$ and $n = |V|$ is the number of vertices.

This basic equation shows that $R(G, k)$ is a kind of totient functions [1], which we call a chromatic totient. In particular, $R(K_2, k) = \phi(k)$, Euler’s totient function, where $K_2 = \bullet \rightarrow \bullet$
is the graph consisting of two vertices and one edge. When $G$ is a cycle, the general formula reduces to the well-known formula of Rademacher—Brauer [2, Ch. 3] for the number of invertible solutions of the congruence $x_1 + \cdots + x_n \equiv 0 \pmod{k}$ (cf. also the concluding remark in [3]).

**Corollary.** The system of congruences corresponding to the cycles of $G$ has an invertible solution modulo $k$ if and only if $p \geq \lambda(G)$ for all prime $p$ dividing $k$, where $\lambda(G)$ is the chromatic number of $G$.

We refer to [4] for useful details concerning the chromatic polynomials and numbers of graphs. There are three main ingredients of the proof:

- the existence of a bijection between proper vertex $p$-colorings of a rooted connected graph and nowhere-zero $\mathbb{Z}_p$-labellings of its edges that satisfy Kirchhoff’s second law;
- the equivalence of the restrictions $\gcd(p, i) = 1$ and $i \neq 0 \pmod{p}$ for any prime $p$;
- the familiar fact (see, e.g., [5]) that in a fundamental cycle base $\mathcal{B}$ of $G$, every cycle $C$ contains an exclusive edge (such as, e.g., the edge $e \in G - T$ by which $C$ is determined as the unique cycle of the subgraph $e \cup T$, where $T = T_{\mathcal{B}}$ is the corresponding spanning tree of $G$).

Formula (1) extends easily to disconnected graphs. A generalization to non-homogeneous systems of linear congruences where some variables get prescribed values holds with an appropriate “partial” chromatic polynomial instead of $\chi(G, z)$.

**References**


**GROUPS WITH PRESCRIBED PROPERTIES OF FINITE SUBGROUPS GENERATED BY COUPLES OF 2-ELEMENTS**

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We discuss results of the research started in [1] and continued in [2–4].

**Theorem.** Suppose that in a group $G$ the order of the product of every two involutions is finite. If every finite subgroup of $G$ generated by a couple of 2-elements is either nilpotent of class at most 2 or has an exponent dividing 4, then all 2-elements of $G$ form a normal subgroup which is either nilpotent of class at most 2 or has an exponent dividing 4.

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