

Article

Fractional Calculus in Russia at the End of XIX Century

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Abstract: In this survey paper, we analyze the development of Fractional Calculus in Russia at the end of the XIX century, in particular, the results by A. V. Letnikov, N. Ya. Sonine, and P. A. Nekrasov. Some of the discussed results are either unknown or inaccessible.

Keywords: fractional integrals and derivatives; Grünwald-Letnikov approach; Sonine kernel; Nekrasov fractional derivative

MSC: primary 26A33; secondary 34A08; 34K37; 35R11; 39A70



Citation: Rogosin, S.; Dubatovskaya, M. Fractional Calculus in Russia at the End of XIX Century. *Mathematics* **2021**, *9*, 1736. <https://doi.org/10.3390/math9151736>

Academic Editor: Clemente Cesarano

Received: 28 June 2021

Accepted: 19 July 2021

Published: 22 July 2021

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1. Introduction

The year of the birth of Fractional Calculus is considered 1695, when Leibniz discussed the possibility of introducing the derivative of an arbitrary order in his letters to Wallis and Bernoulli. Several attempts were made to give a precise meaning to this new notion. A comprehensive detailed analysis of the history of Fractional Calculus is given in Reference [1]. One of most productive periods in this history was the middle-end of the XIX century. Here, we can mention works by Legendre, Fourier, Peacock, Kelland, Tardi, Roberts, and others. The most advanced approach to the determination of the fractional derivative of an arbitrary order was proposed by Liouville. A deep analysis of the results on this subject was given in the article [2] by Letnikov. In particular, he recognized the basic role of Liouville's approach. Letnikov said ([2], p. 92): "... we give a survey of the results by Liouville whom we ought to consider as the first scientist paid a serious attention to clarifying the question on the derivative of an arbitrary order. In 1832, he started to publish a series of articles devoted to the foundation and the application of his theory of general differentiation which is the first complete discussion of this topic. Before his work, only few very important but not completely clear remarks were made on this subject".

It should be noted that the works by A. V. Letnikov constitute the first rigorous and comprehensive construction of the theory of the fractional integro-differentiation. An extended description of the results by Letnikov is presented in the articles of References [3,4] and in the book of Reference [5], written in Russian.

In the middle-end of the XIX century, an interest to Fractional Calculus in Russia grew significantly [6–10]. One of the reasons for it was a high standard in the research in Real and Complex Analysis in Russia in this period. Russian Universities took care of the level of the education of young scientists. Many applicants for a professorship had been given the opportunity to spend 1–2 years at the leading research centers and to attend the lectures of known mathematicians.

Letnikov's results attracted people to this branch of the Science, at least in Russia. Nevertheless, these works remained unknown abroad and, for a long time, were inaccessible. After contribution by Letnikov, the serious works on Fractional Calculus in Russia in the second part of the XIX century were published by N. Ya. Sonine and P. A. Nekrasov. They introduced the complex-analytic technique into the study and application of derivatives and integrals of an arbitrary order. It should be noted that Complex Analysis was

traditionally highly developed discipline starting from Leonard Euler, who worked for a long period in Russia (1726–1741 and 1776–1783). This part of Mathematical Analysis was essentially developed in the XIX century by M. V. Ostrogradsky, V. Ya. Bunyakovsky, P. L. Chebyshev, A. M. Lyapunov, and many others. In particular, Sonine and Nekrasov found a fractional analog of the classical Cauchy integral formula for analytic functions.

In our article, we describe the contribution of Alexey Vasil'evich Letnikov (1837–1888), Nikolai Yakovlevich Sonine (1849–1915), and Pavel Alekseevich Nekrasov (1853–1924) to Fractional Calculus and the role of these results in the modern Fractional Calculus and its Applications.

2. Liouville's Approach and Its Analysis by Letnikov

As it was already said, A. V. Letnikov considered (see, e.g., Reference [2,6,7]) that the Liouville's theory constitutes the only complete treatment of differentiation of an arbitrary order. Realizing the great importance of this theory, Letnikov had seen that its certain parts did not receive a proper justification and led to some misunderstanding in the works of Liouville's followers.

Let us present here Letnikov's description of the elements of the Liouville's theory following Reference [2]. Letnikov started his analysis with the definitions given by Liouville.

Definition 1. Let the function $y(x)$ be represented in the form of the following series of exponents:

$$y(x) = A_1 e^{m_1 x} + A_2 e^{m_2 x} + \dots, \quad (1)$$

which is denoted for shortness as $\sum A_m e^{m x}$.

Fractional derivative of the order p is defined by multiplying each term of the series by p -th power of the index m :

$$\frac{d^p y}{dx^p} = \sum A_m m^p e^{m x}. \quad (2)$$

If p is negative, then Formula (2) determined the fractional integral of order $-p$.

Fractional integral of order $-p$ is denoted by Liouville as $\int^{-p} y dx^{-p}$. Liouville considered this definition as the only possible way to generalize the usual derivative. Evaluating its role, Letnikov stressed that Definition 1 contains a key ideas to establish a deep analogy with differences and powers and, thus, could lead to a more simple construction.

Nevertheless, the above definition had a very important restriction. It cannot be applied to an arbitrary function since not all of them possess representations in series of exponents. Liouville himself understood this difficulty. He proposed a way to overcome it. By performing the change of variable $z = e^x$ for the function $y = F(x)$, one can expand the composite function $y = F(\ln z)$ (with $x = \ln z$) in a converging power series:

$$F(\ln z) = \sum A_m z^m. \quad (3)$$

Thus, the initial function $y = F(x)$ admits representation via series of exponents

$$F(x) = \sum A_m e^{m x}. \quad (4)$$

But the possibility to represent $y = F(\ln z)$ in form (3) met several restrictions. For instance, if we suppose to get representation of $y = F(\ln z)$ in a form of series in positive powers of z , then all derivatives of $y = F(x)$ at $x = \infty$ should be equal to zero since

$$(F(\ln z))'_z = \frac{F'(\ln z)}{z}, (F(\ln z))''_z = \frac{F''(\ln z) - F'(\ln z)}{z^2}, \dots$$

Similar restriction appears if we suppose to represent $y = F(\ln z)$ in a form of series in negative powers of z since we deal in this case with the function $y = F(-\ln z)$. Such conditions look fairly strong. Moreover, they are neither necessary nor sufficient for the representation of the type (4).

Liouville met such a restriction trying to calculate the derivative of a fractional order of the power function. He started with the Euler formula

$$\frac{1}{x^m} = \frac{\int_0^{\infty} e^{-xz} z^{m-1} dz}{\Gamma(m)}.$$

Liouville supposed that the above integral can be represented in a form of the exponential sum $\sum A_n e^{-nx}$. Here, all coefficients in such representation should be infinitely small. Then, using his main definition, Liouville arrived at the formula of the derivative of this function:

$$\frac{d^p}{dx^p} \frac{1}{x^m} = \frac{\int_0^{\infty} e^{-xz} (-z)^p z^{m-1} dz}{\Gamma(m)}. \quad (5)$$

Thus, by definition of Γ -function, we get, after substituting $xz = t$, the following formula:

$$\frac{d^p}{dx^p} \frac{1}{x^m} = \frac{(-1)^p \Gamma(m+p)}{\Gamma(m) x^{m+p}}. \quad (6)$$

In his first articles, Liouville used the only definition of the Γ -function of positive variable (later, he noted that he was not familiar with the general definition of the Γ -function by Legendre and Gauss). Therefore, he supposed that, due to assumptions $m > 0$, $m + p > 0$, one needs to use in the above definition so-called auxiliary functions (more detailed discussion of the role of auxiliary function is presented below in Section 4; in fact, such notion appeared in the works by Liouville since he used indefinite integral for fractional integration). Being very important, the use of auxiliary functions did not lead to a general definition of the fractional derivative. Liouville showed that, if one supposed an existence of auxiliary functions, then these necessarily had to be entire functions.

Letnikov claimed and proved that it follows from his analysis that Liouville's formulas were so general that they had no need of any auxiliary function. Later on, several attempts to correct Liouville's approach were made. In particular, Letnikov analyzed in Reference [2] the works by Kelland, Tardi, and Roberts. But the really rigorous approach which transformed Liouville's formulas to the general definition of the fractional derivative was proposed by Letnikov. We have to note that Letnikov used a definite integral in his construction (see Section 3.1). For such construction, the notion of auxiliary function becomes needless.

3. Letnikov's Contribution to Fractional Calculus

3.1. Letnikov or Grünwald-Letnikov Derivative

Starting his work on determination of the derivative of an arbitrary order, Letnikov posed this problem [6,7] as interpolation in form of the elements of two sequences consisting of successive derivatives of the function $f(x)$

$$(a) \quad f(x), f'(x), f''(x), \dots, f^{(n)}(x), \dots \quad (7)$$

and of successive n -fold integrals of this function:

$$(b) \quad f(x), \int f(x) dx, \int^2 f(x) dx^2, \dots, \int^n f(x) dx^n, \dots \quad (8)$$

In other words, he tried to find such a formula of the derivative of an arbitrary order α which, for nonnegative integer, $\alpha = 0, 1, 2, \dots$ coincides with the corresponding elements of the sequence (a), and, for nonpositive integer, $\alpha = 0, -1, -2, \dots$ coincides with the corresponding elements of the sequence (b). Denoting this formula by

$$D^\alpha f(x) \quad \text{or} \quad \frac{d^\alpha f(x)}{dx^\alpha},$$

he expected to get this new object to have (whenever it is possible) that same properties as elements of sequence (a) or (b) when α is an integer.

The next idea by Letnikov was to restrict the generality of the above question and to consider, instead of the sequence (b) (of indefinite n -fold integrals), the sequence of definite integrals, supposing that $f(x)$ is continuous on certain interval $[a, x]$, i.e., to interpolate in form elements of the double sequence

$$(A) \quad \dots \int_a^x \int_a^x f(x) dx^2, \int_a^x f(x) dx, f'(x), f''(x), \dots, \quad (9)$$

in which any element is the derivative of the previous one.

The corresponding interpolating object he denoted as

$$[D^\alpha f(x)]_a^x.$$

In order to get such interpolation, Letnikov proposed to examine the following formula:

$$\frac{\sum_{k=0}^n (-1)^k \binom{\alpha}{k} y(x - kh)}{h^\alpha}, \quad (10)$$

where $h = \frac{x-a}{n}$, and $\binom{\alpha}{k}$ denotes the binomial coefficient. This approach was independently used by Grünwald [11] and by Letnikov [6]. When Letnikov found the paper by Grünwald, he decided to decline publication of his work, but later changed his mind. Letnikov developed in Reference [6] more rigorously than in Reference [11] the theory of the derivative of an arbitrary order and found its relationship with many results known in this area.

Elementary algebra yields that, for $\alpha = m$ being positive integer number, the derivatives of the corresponding order can be defined as a limit of the above expression

$$f^{(m)}(x) = \lim_{\delta \rightarrow 0} \frac{f(x) - \binom{m}{1} f(x-\delta) + \binom{m}{2} f(x-2\delta) - \dots + (-1)^n \binom{m}{n} f(x-n\delta)}{\delta^m}. \quad (11)$$

Here, $\delta \rightarrow 0$ is equivalent to $n \rightarrow \infty$, but the sum in the numerator remains finite since all binomial coefficients with $n > m$ vanishing. Thus, Formula (11) can be taken as the definition of the derivative of order $m \in \mathbb{N}$.

Vice versa, for $\alpha = -m$ being negative integer, the expression under the limit sign in the right-hand side of (11) equals to

$$\frac{f(x) + \binom{m}{1} f(x-\delta) + \binom{m}{2} f(x-2\delta) + \dots + \binom{m}{n} f(x-n\delta)}{\delta^{-m}}. \quad (12)$$

Letnikov showed [6] (pp. 5–12) that the limit of this expression as $\delta \rightarrow 0$, or equivalently as $n \rightarrow \infty$, is equal to the multiple integral, i.e., (in his notation)

$$\begin{aligned} [D^{-m} f(x)]_a^x &= \lim_{\delta \rightarrow 0} \frac{f(x) + \binom{m}{1} f(x-\delta) + \binom{m}{2} f(x-2\delta) + \dots + \binom{m}{n} f(x-n\delta)}{\delta^{-m}} \\ &= \int_a^x dx_1 \int_a^{x_1} dx_2 \dots \int_a^{x_{m-1}} f(x_m) dx_m. \end{aligned} \quad (13)$$

This magnitude $[D^{-m} f(x)]_a^x$ satisfies certain properties. First of all, if we apply to it similar operation of order $-p$, $p > 0$, then we will have

$$[D^{-p}D^{-m}f(x)]_a^x = [D^{-m-p}f(x)]_a^x.$$

Next, if we take the derivative $\frac{d^p}{dx^p}$ of order $p > 0$, then we will have

$$\frac{d^p}{dx^p} [D^{-m}f(x)]_a^x = [D^{-m+p}f(x)]_a^x, \quad \text{if } m > p,$$

and

$$\frac{d^p}{dx^p} [D^{-m}f(x)]_a^x = \frac{d^{p-m}f(x)}{dx^{p-m}}, \quad \text{if } m < p.$$

Thus, in particular, the symbol $[D^{-m}f(x)]_a^x$ means m -times differentiable function whose all derivatives up to m -th order are vanishing at $x = a$.

Formulas (13) and (11) coincide with the corresponding elements of the double sequence (A). Therefore, it led Letnikov to the conclusion that the limit

$$[D^\alpha f(x)]_a^x := \lim_{\delta \rightarrow 0} \frac{f(x) - \binom{\alpha}{1} f(x-\delta) + \binom{\alpha}{2} f(x-2\delta) - \dots + (-1)^n \binom{\alpha}{n} f(x-n\delta)}{\delta^\alpha} \quad (14)$$

is a good candidate to solve the interpolation problem for the sequence (A), i.e., to be the derivative of arbitrary order.

The relations of this new object to known formulas of the fractional derivatives (integrals) were described by Letnikov [6] (p. 15) using the following elementary:

Lemma 1. Let (α_k) be a sequence of (real or complex) numbers such that

$$\lim_{k \rightarrow \infty} \alpha_k = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} (\alpha_1 + \alpha_2 + \dots + \alpha_k) = C,$$

and let (β_k) be a sequence of (real or complex) numbers such that

$$\lim_{k \rightarrow \infty} \beta_k = 1.$$

Then, the sequence of their products has the limits equal to C , i.e.,

$$\lim_{k \rightarrow \infty} (\alpha_1 \beta_1 + \alpha_2 \beta_2 + \dots + \alpha_k \beta_k) = \lim_{k \rightarrow \infty} (\alpha_1 + \alpha_2 + \dots + \alpha_k) = C.$$

The above formula is valid for $[D^\alpha f(x)]_a^x$ with $\alpha < 0$ (i.e., for fractional integral of order $-\alpha$ in modern language).

The corresponding justification of the formula of $[D^\alpha f(x)]_a^x$ with $\alpha > 0$ (i.e., representation of fractional derivative) Letnikov supposed additionally that the function $f(x)$ is $(n+1)$ -times continuously differentiable on the interval (a, x) , where n is a largest integer smaller than α , i.e., $n < \alpha < n+1$. Then, using quite cumbersome transformation of the binomial coefficients [6] (pp. 21–26), he has got that the limit in (14) is equal:

$$[D^\alpha f(x)]_a^x = \frac{f(a)(x-a)^{-\alpha}}{\Gamma(-\alpha+1)} + \frac{f'(a)(x-a)^{-\alpha+1}}{\Gamma(-\alpha+2)} + \dots + \frac{f^{(n)}(a)(x-a)^{-\alpha+n}}{\Gamma(-\alpha+n+1)} + \quad (15)$$

$$+ \frac{1}{\Gamma(-\alpha+n+1)} \int_a^x (x-\tau)^{-\alpha+n} f^{(n+1)}(\tau) d\tau.$$

Note that the same result is true if $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$. Integration by parts showed that (15) can be taken as the definition of fractional derivative of an arbitrary order $\alpha > 0$. A slightly more general form can be written for any $s \in \mathbb{Z}$, $s \geq n$, $n < \operatorname{Re} \alpha < n+1$ (of course, under additional smoothness conditions):

$$[D^\alpha f(x)]_a^x = \sum_{k=0}^s \frac{f^{(k)}(a)(x-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(-\alpha+s+1)} \int_a^x (x-\tau)^{-\alpha+s} f^{(s+1)}(\tau) d\tau. \quad (16)$$

In Reference [6], Letnikov paid attention to relationship of his formulas with known constructions. In particular, he showed that, if the function $f(x)$ is defined, infinitely differentiable on $[x, \infty)$, and vanishes together with any derivative when x is tending to ∞ , then the following formula hold for any $\alpha, \operatorname{Re} \alpha < 0$:

$$[D^\alpha f(x)]_{+\infty}^x = \frac{1}{\Gamma(-\alpha)} \int_{+\infty}^x (x-\tau)^{-\alpha-1} f(\tau) d\tau = \frac{1}{(-1)^\alpha \Gamma(-\alpha)} \int_0^{+\infty} z^{-\alpha-1} f(x+z) dz,$$

i.e., coincides with the corresponding integral defined by Liouville. Similarly, for any $\alpha, 0 \leq n < \operatorname{Re} \alpha < n+1$ Letnikov discovered that

$$[D^\alpha f(x)]_{+\infty}^x = \frac{1}{\Gamma(-\alpha+n+1)} \int_{+\infty}^x \frac{f^{(n+1)}(\tau) d\tau}{(x-\tau)^{\alpha-n}} = \frac{1}{(-1)^{\alpha-n-1} \Gamma(-\alpha+n+1)} \int_0^{+\infty} \frac{f^{(n+1)}(x+z) dz}{z^{\alpha-n}}. \quad (17)$$

He also noted that the considered class of functions is not empty, it contains, in particular, all functions of the form $x^m e^{-x}$.

In Reference [6], Letnikov also presented a series of formulas for the values of his derivative of an arbitrary order of elementary functions, such as power function $(x-a)^\beta$, exponential function e^{mx} , logarithmic function $\log x$, exponential-trigonometric functions $e^{\beta x} \sin \gamma x$, $e^{\beta x} \cos \gamma x$, and rational functions $\frac{P(x)}{Q(x)}$. These formulas coincide with nowadays known formulas (see, e.g., Reference [1,12]). Composition formulas for fractional derivatives and integrals were found in Reference [6], too. The last result, presented in Reference [6], was the so-called Leibniz rule for the fractional derivative/integral of the product of functions. Note, that after the death of A. V. Letnikov, it was created a committee examined some of his manuscripts [13]. A few results were then published, but not all were found. In particular, the members of the committee reported that they did not find any results on Abel integrals, as it was expected by some researchers.

3.2. Solution to Certain Differential Equations

In Reference [14], Liouville made a background for further development of Fractional Calculus. In order to show an importance of the new branch of Science, he solved in Reference [15] a number of problems (mainly from geometry, classical mechanics, and mathematical physics) by using his constructions of integral and derivatives of an arbitrary order. Later, in Reference [16], he also discussed the tautochrone problem and usage of fractional derivatives to its solution.

In his master thesis, Letnikov carefully examined these results by Liouville and came to the conclusion that Liouville's solutions of the problems can be obtained by using more traditional methods, too. He also remarked that incorrect usage of Liouville construction by his followers led to certain misunderstandings, and even mistakes. Note that the master thesis by Letnikov was reprinted in Russian recently in Reference [4,5].

Nevertheless, Letnikov believed that newly created technique could find proper applications. One of these applications was presented in his article, Reference [17], devoted to use of the fractional derivative to the solution of the differential equation

$$(a_n + b_n x) \frac{d^n y}{dx^n} + (a_{n-1} + b_{n-1} x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + (a_0 + b_0 x) y = 0. \quad (18)$$

These results were lectured by Letnikov at the meeting of Mathematical Society on 16 April 1876, and at the meeting of the Warsaw Congress of naturalists on 3 September 1876. They were reprinted by P. A. Nekrasov, who parsed the Letnikov's archive after his death.

Denoting

$$\varphi(\rho) := a_n \rho^n + a_{n-1} \rho^{n-1} + \dots + a_1 \rho + a_0, \quad \psi(\rho) := b_n \rho^n + b_{n-1} \rho^{n-1} + \dots + b_1 \rho + b_0,$$

Equation (18) can be rewritten in the following symbolic form:

$$\varphi\left(\frac{d}{dx}\right)y + x\psi\left(\frac{d}{dx}\right)y = 0. \quad (19)$$

Suppose that equation

$$\psi(\lambda) = 0 \quad (20)$$

has different zeroes $\lambda_1, \lambda_2, \dots, \lambda_n$. Denoting for each $j = 1, 2, \dots, n$

$$y := e^{\lambda_j x} Y, \quad (21)$$

one can rewrite (19)

$$\varphi\left(\lambda_j + \frac{d}{dx}\right)Y + x\psi\left(\lambda_j + \frac{d}{dx}\right)Y = 0. \quad (22)$$

A crucial idea by Letnikov was to look for the solution to Equation (22) in the form:

$$Y = [D^p y_j]_a^x, \quad (23)$$

where y_j is a new unknown function, and $[D^p]_a^x$ is an inter-limit (Letnikov-type) derivative whose order p is to be defined later.

Let there exist the function y_j , vanishing at $x = a$, together with all derivatives up to order $n - 2$, satisfying the following equation:

$$\varphi_j\left(\frac{d}{dx}\right)y_j + x\psi_j\left(\frac{d}{dx}\right)y_j = 0, \quad (24)$$

where

$$\psi_j(\rho) = \frac{\psi(\lambda_j + \rho)}{\rho}, \quad p + 1 = \frac{\varphi(\lambda_j)}{\psi_j(0)} = \frac{\varphi(\lambda_j)}{\psi'(\lambda_j)} = A_j, \quad \varphi_j(\rho) = \frac{\varphi(\lambda_j + \rho) - A_j \psi_j(\rho)}{\rho}.$$

Then, there exists the solution to Equation (19), which is represented in the form:

$$y = e^{\lambda_j x} [D^{A_j-1} y_j]_a^x. \quad (25)$$

Thus, by this transformation, we reduce Equation (18) of order n to Equation (24) of order $n - 1$. By applying this method, one can reduce the order of the equation up to 1 and get the possible solution via successive application of the inter-limit derivative.

4. Sonine-Letnikov Discussion

In 1868, A. V. Letnikov published the main part of his master thesis as an article in Mathematical Sbornik [6], supplemented by the historical survey on the development of the theory of differentiation of an arbitrary order [2]. This article, and Grünwald's article [11], was criticized by N. Ya. Sonine in Reference [9], who also presented in Reference [9] his own approach to determine the derivatives of an arbitrary order. Sonine's article started with the discussion of Liouville's definition of the derivative of an arbitrary order (not necessarily positive). The latter definition is based on the derivative of an arbitrary order $p \in \mathbb{R}$ of exponential function

$$\frac{d^p}{dx^p} e^{mx} = m^p e^{mx}$$

and on the possibility to expand a differentiable function into exponential series (the Dirichlet series in modern language):

$$f(x) = \sum_{k=-\infty}^{+\infty} A_k e^{m_k x}.$$

Sonine made two important remarks concerning Liouville's definition. First, he showed that the derivative of negative order (i.e., the fractional integral) cannot be con-

sidered as an inverse to the fractional derivative. The second remark by Sonine is related to the problem discovered by Liouville himself. If one applies Liouville's definition of the derivative of arbitrary order to power function, then it leads immediately to a kind of contradiction. Liouville founded this phenomena using the following representation of x :

$$x = \lim_{\beta \rightarrow 0} \frac{e^{\beta x} - e^{-\beta x}}{2\beta}.$$

If we suppose that the limit and the fractional derivative are interchangeable, then the half-derivative of x becomes infinite. Moreover, the derivative of an infinitesimal quantity could be finite. From these facts, Liouville concluded the existence of additional functions that are the derivative of zero function and coincide with an entire function with arbitrary coefficients. Contrariwise, Sonine has shown that such contradiction follows from a not completely rigorous way of expansion of the function into the series in exponents. He also remarked that Liouville's proof of existence of additional functions does not have a proper rigor.

In the second part of his article [9], Sonine criticized an approach by Grünwald and Letnikov, in which the fractional derivative is defined by the following limit:

$$D^\alpha [f(x)]_{x=a}^{x=x} := \lim_{\delta \rightarrow 0} \frac{f(x) - \binom{\alpha}{1} f(x-h) + \binom{\alpha}{2} f(x-2h) - \dots + (-1)^n \binom{\alpha}{n} f(x-nh)}{h^\alpha}, \quad (26)$$

where $nh = x - a$.

Sonine had two main objections. First, he noted that the series in the numerator of (26) is converging. Hence, the fraction should be infinite. The second remark by Sonine was that, if we apply to the fractional integral $D^{-\alpha}$ the fractional derivative D^β ($\alpha, \beta > 0$), then, by Leibnitz rule, the result should coincide with $D^{\beta-\alpha}$. It leads to contradiction, even for the function $f(x) = 1$, since it should exist for any β , but it is so only for $\text{Re } \alpha > [\text{Re } \beta]$, where $[\cdot]$ means the integer part of the number.

Concerning the first remark, Letnikov noted in Reference [7] that, in the case $\text{Re } \alpha > 0$, Formula (26) determines the fractional integral (coinciding with m -times repeated integral if $\alpha = m \in \mathbb{N}$) if the series in the numerator of (26) is converging, and its sum is equal to zero. Moreover, Letnikov gave sufficient conditions for existence of the limit in (26).

The second question by Sonine appeared due to his incorrect application of the Leibnitz rule. Letnikov noted that fractional integral and fractional derivative are defined by different formulas:

$$D^\alpha [f(x)]_{x=a}^{x=x} = \frac{1}{\Gamma(-\alpha)} \int_a^x (x-t)^{-\alpha-1} f(t) dt, \quad \alpha < 0, \quad (27)$$

$$D^\alpha [f(x)]_{x=a}^{x=x} = \sum_{k=0}^m \frac{f^{(k)}(a)(x-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(-\alpha+m+1)} \int_a^x (x-t)^{-\alpha+m} f^{(m+1)}(t) dt, \quad (28)$$

$$\alpha > 0, m = [\alpha].$$

Both formulas are applied under certain conditions. Thus, successive application of these two formulas can lead to certain contradiction if we do not take into account the above conditions.

5. Sonine's Contribution to Fractional Calculus

5.1. Sonine's Fractional Derivative and Integral

In his polemical article [9], N. Ya. Sonine not only criticized Grünwald-Letnikov approach but also proposed another form of "general" fractional derivative. For his

formula, Sonine used generalization of the Cauchy integral (or, better to say, the Cauchy-type integral, since the below integral is defined for any continuous function f):

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{\Gamma(p+1)}{2\pi i} \int_{\gamma} \frac{f(\tau) d\tau}{(\tau-x)^{\alpha+1}}, \quad (29)$$

where γ is a closed simple smooth curve on the complex plane surrounding the point x (without loss of generality, one can assume that γ is the circle of radius r centered at the point x). This formula is really a good candidate for generalization of usual derivatives since, for $\alpha = p$, positive integer (29) gives the value of p -th derivative at the point x , assuming that the function f is p -time differentiable at x .

This formula was analyzed by Letnikov in his answer [7] on the remarks by Sonine. Letnikov proved that, under assumption that the function f is $(m+1) = ([\alpha] + 1)$ -times continuously differentiable inside the circle γ , Formula (29) coincides with Letnikov's Formula (28). Note that Letnikov discussed Sonine's Formula (29) under the stronger assumptions on the function f .

$$\begin{aligned} \frac{d^\alpha f(x)}{dx^\alpha} &= \frac{(-r)^{-\alpha} f(x+r)}{\Gamma(-\alpha+1)} + \frac{(-r)^{-\alpha+1} f'(x+r)}{\Gamma(-\alpha+2)} + \dots + \frac{(-r)^{-\alpha+m} f^{(m)}(x+r)}{\Gamma(-\alpha+m+1)} \\ &+ \frac{\Gamma(\alpha-m)}{2\pi r^{\alpha-m-1}} \int_0^{2\pi} f^{(m+1)}(x+re^{i\theta}) e^{-(\alpha-m-1)i\theta} d\theta. \end{aligned} \quad (30)$$

Since the last integral can be transformed to the form

$$\begin{aligned} &\frac{\Gamma(p-m)}{2\pi r^{\alpha-m-1}} \int_0^{2\pi} f^{(m+1)}(x+re^{i\theta}) e^{-(\alpha-m-1)i\theta} d\theta = \\ &\frac{(-1)^{-\alpha+m}}{\Gamma(-\alpha+m+1)} \int_{x+r}^x f^{(m+1)}(\tau) (\tau-x)^{-\alpha+m} d\tau, \end{aligned}$$

then Formula (29) coincides with the definition of the fractional derivative given by Letnikov (28). In Reference [9], Sonine concluded that his formula cannot coincide with Grünwald-Letnikov formula for $\alpha > 0$ without adding auxiliary function.

Sonine's definition of a fractional derivative of negative order (i.e., fractional integral) has been criticized by Letnikov in Reference [7]. Sonine used Leibniz's rule (composition formula) for fractional derivatives (which is generally not valid; see Reference [1]). A fractional integral by Sonine is defined as inverse operation for fractional derivative:

$$\frac{d^\alpha}{dx^\alpha} \frac{d^{-\alpha} f(x)}{dx^{-\alpha}} = f(x). \quad (31)$$

From this formula, Sonine concluded that it should exist as a so-called auxiliary function $\psi(x)$, satisfying the following relation:

$$\frac{d^\alpha \psi(x)}{dx^\alpha} = 0. \quad (32)$$

Sonine took the function $\psi(x)$ in the form

$$\psi(x) = A_1(x-a)^{\alpha-1} + A_2(x-a)^{\alpha-2} + \dots + A_k(x-a)^{\alpha-k},$$

where $A_j, j = 1, 2, \dots, k$, are arbitrary constants, $k = [p] + 1$, but a is not defined by Sonine. If Cauchy formula is taken as the definition of the derivative of an arbitrary order, then the auxiliary function has to satisfy the relation

$$\int_{\gamma} \frac{\psi(\tau) d\tau}{(\tau-x)^{\alpha+1}} = 0.$$

Since it was shown that by integration by parts Formula (29) is reduced to the definition of the fractional derivative (28) without any auxiliary function, then Letnikov concluded that the following alternative holds: either (1) there exists no auxiliary function of the above form, or (2) formula (29) cannot be taken as the general definition of fractional derivative of an arbitrary order. Instead, he said that his definition is free of necessity to add an auxiliary function (in spite of the fact that this definition is reduced to different a form than (27) and (28) whenever α is negative or positive, respectively).

5.2. Sonine Kernel and Sonine Integral Equations

In one of his pioneering articles [18], Abel presented solution to the integral equation

$$\int_0^x \frac{\varphi(\tau) d\tau}{(x-\tau)^{1-\alpha}} = F(x), \quad 0 < \alpha < 1. \quad (33)$$

The main component of Abel's method was the following identity:

$$\int_0^x f(t) dt = \frac{\sin \pi \alpha}{\pi} \int_0^x \frac{dt}{(x-t)^{1-\alpha}} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^\alpha}, \quad (34)$$

where $f(x) = F'(x)$.

Sonine tried to generalize Abel's identity (34) in order to solve more general equation than integral Equation (33). He looked for a pair of functions $\sigma(x), \psi(x)$ satisfying the identity

$$\int_0^x f(t) dt = \int_0^x \psi(x-t) dt \int_0^t f(\tau) \sigma(t-\tau) d\tau, \quad (35)$$

or, i.e., a pair of functions generating integral representation of unity

$$1 = \int_0^x \psi(x-t) dt \int_0^t \sigma(t-\tau) d\tau. \quad (36)$$

Sonine described in Reference [19] a possible form of the functions $\sigma(x), \psi(x)$,

$$\sigma(t) = \frac{t^{-p}}{\Gamma(1-p)} \sum_{k=0}^{\infty} a_k t^k, \quad \psi(t) = \frac{t^{-q}}{\Gamma(1-q)} \sum_{k=0}^{\infty} b_k t^k,$$

where $p+q=1$, and coefficients a_k, b_k are defined by the following relations

$$a_0 b_0 = 1, \quad \sum_{k=0}^n \Gamma(k+p) \Gamma(q+n-k) a_{n-k} b_k = 0, \quad n = 1, 2, \dots$$

He also applied relation (36) for representation of the solution to the first kind of integral equations with one of these functions as the kernel:

$$\int_0^x \sigma(x-\tau) \varphi(\tau) d\tau = f(x). \quad (37)$$

Both functions $\sigma(x), \psi(x)$ are known as Sonine kernels, and integral Equation (37), generalizing Abel integral Equation (33), is called a Sonine integral equation. In modern language (see, e.g., Reference [20]), a locally integrable function $\sigma(x)$ is called the Sonine kernel if there exists another locally integrable function $\psi(x)$ such that the following identity holds:

$$\int_0^x \sigma(x-\tau) \psi(\tau) d\tau = 1, \quad x > 0. \quad (38)$$

In fact, the function $\psi(x)$ is also called the Sonine kernel (sometimes, these functions are called the associated Sonine kernels).

Several special examples of Sonine kernel are presented in Reference [21]. We can mention also Reference [20], in which the properties of the Sonine kernel are discussed in modern setting. Several difficulties which one has to overcome by formal application of Sonine's approach to the solution of the corresponding integral equations are discovered. Possible ways to overcome these difficulties are shown.

In Reference [22], the general fractional integrals and derivatives of arbitrary order are introduced, along with study some of their basic properties and particular cases. First, a suitable generalization of the Sonine condition is presented, and some important classes of the kernels that satisfy this condition are introduced. In the introduction of the general fractional integrals and derivatives, the author follows a recent approach by Kochubei [23]. The general fractional integrals and derivatives with Sonine kernel are defined in the Riemann-Liouville form (see Reference [21,22] and references therein)

$$(\mathbb{I}_\sigma f)(x) = \int_0^x \sigma(x-t)f(t)dt, \quad x > 0, \quad (39)$$

$$(\mathbb{D}_\psi f)(x) = \frac{d}{dx} \int_0^x \psi(x-t)f(t)dt, \quad x > 0, \quad (40)$$

where the functions $\sigma(x), \psi(x)$ are associated Sonine kernels. Operators (39) and (40) are discussed in Reference [21–23] under different conditions on Sonine kernels, and the constructions are not only similar to Riemann-Liouville-type fractional integrals and derivatives but also to Dzhrbashian-Caputo-type and to Marshaud-type.

5.3. Higher Order Hypergeometric Functions

The main research interest by Sonine was to study the properties of several classes of special functions. His results served as an impetus for the development of the theory of cylindrical functions (or Bessel-type functions) in the second half of the XIX century. These results are based on the achievements by C. Neumann, O. Schlömilch, E. Lommel, H. Hankel, N. Nielsen, L. Schlafli, L. Gegenbauer, and others (see, e.g., Reference [24,25]). Sonine defined in Reference [26] the cylindrical functions $S_\nu(z)$ as a partial solutions to the following system of functional-differential equations:

$$\begin{cases} S_{\nu+1}(z) + 2S'_\nu(z) - S_{\nu-1}(z) = 0, \\ 2\nu S_\nu(z) = z[S_{\nu-1}(z) + S_{\nu+1}(z)], \\ S_1(z) = -S'_0(z), \end{cases} \quad (41)$$

where z is the complex variable, and ν is an arbitrary complex parameter. Sonine proved that these partial solutions admit an integral representation:

$$S_\nu(z) = \frac{1}{2\pi i} \int_a^b \exp\left\{\frac{z}{2}\left(t - \frac{1}{t}\right)\right\} \frac{dt}{t^{\nu+1}}. \quad (42)$$

He found four possible cases for the limits of integration, namely: (1) $\infty \cdot \alpha, \infty \cdot \beta$; (2) $-\frac{0}{\alpha}, -\frac{0}{\beta}$; (3) $-\frac{0}{\alpha}, \infty \cdot \beta$; (4) $\text{Im}(za) = \pm\infty, \text{Im}(zb) = \pm\infty$, where, in cases (1)–(3), $\text{Re}(za) < 0$, $\text{Re}(zb) < 0$, but, in case (4) $\text{Re}(\nu) > 0$. Sonine denoted the functions obtained in these four cases by $S_\nu^{(k)}(z)$ and showed that

$$S_\nu^{(1)}(z) = J_\nu(z), \quad S_\nu^{(2)}(z) = e^{-\nu\pi i} J_{-\nu}(z), \quad S_\nu^{(3)}(z) = \frac{1}{2} H_\nu^{(1)}(z), \quad S_\nu^{(4)}(z) = J_\nu(z).$$

The above integral representation (42) is called Sonine integral representation. It is a source for obtaining new representations for cylindrical functions (see Reference [25]), as well as for calculation of certain definite integrals. Among these integrals are those known as the first and the second Sonine integrals, respectively (or classical Sonine formulas); see, e.g., Reference [27]:

$$J_{\nu+\mu+1}(aq) = \frac{q^\nu}{2^\nu \Gamma(\nu+1) a^{\nu+\mu+1}} \int_0^a J_\mu(qx) (a^2 - x^2)^\nu x^{\mu+1} dx, \quad (43)$$

$$\int_0^a J_\mu(qx) J_\nu[z\sqrt{a^2 - x^2}] (a^2 - x^2)^{\frac{\nu}{2}} x^{\mu+1} dx = a^{\nu+\mu+1} q^\mu z^\nu \frac{J_{\nu+\mu+1}(a\sqrt{q^2 + z^2})}{(\sqrt{q^2 + z^2})^{\nu+\mu+1}}, \quad (44)$$

where $\operatorname{Re} \nu, \operatorname{Re} \mu > -1$. Sonine formulas find interest in different questions of analysis (e.g., in Dunkl theory, as in Reference [28], or in the study of Levy processes [29]).

There exist several multivariate extensions of the classical Sonine integral representation for Bessel functions of some index $\mu + \nu$ in terms of such functions of lower index μ (see, e.g., Reference [30]). For Bessel functions on matrix cones, Sonine formulas involve beta densities $\beta_{\mu,nu}$ on the cone and go already back to Herz.

Several important results dealing with properties of Γ -function were obtained by Sonine during his career. They are based on the study of the solution to the difference equation

$$F(x+1) - F(x) = f(x). \quad (45)$$

In these works, Sonine followed the idea by Binet (1838), who examined the relation

$$\log \Gamma(x+1) - \log \Gamma(x) = \log x.$$

Sonine found [31], in particular, the form of the remainder factor in the product representation of Γ -function

$$\Gamma(x+1) = \frac{n!(n+1)^x}{(x+1)(x+2)\cdots(x+n)} \frac{\left(1 + \frac{x}{n+1}\right)^{x+n+\theta}}{\left(1 + \frac{1}{n+1}\right)^{x(1+n+\theta)}}, \quad x \in \mathbb{R}, \quad 0 < \theta < 1. \quad (46)$$

In his article on Bernoulli polynomials, Sonine obtained one more representation, related to Γ -functions (this formula was rediscovered by Ch. Hermite in 1895)

$$\log \frac{\Gamma(x+y)}{\Gamma(y)} = x \log y + \sum_{k=2}^n \frac{(-1)^k \varphi_k(x)}{(k-1)ky^{k-1}} + R_n(x, y), \quad (47)$$

where $\varphi_k(x)$ are Bernoulli polynomials defined by Sonine using difference equation

$$\varphi_k(x+1) - \varphi_k(x) = kx^{k-1}, \quad \varphi_k(0) = 0, \quad k = 1, 2, \dots$$

Reference [32] contains a number of the most important articles by N. Ya. Sonine, as well as a survey on his other research.

6. Nekrasov's Contribution to Fractional Calculus

In Reference [8], Nekrasov proposed a new definition of the general differentiation. In fact, this definition includes Letnikov's definition as a special case. The main idea by Nekrasov was to define the derivative by using integration along closed contour L crossing the point x and encircling a group of singular points of the differentiable function $f(x)$. This definition gives, in fact, the differentiation with respect to a doubly connected domain, which is free of the singular points of $f(x)$, and contains the above said contour L . Therefore, Nekrasov used the ideas by Sonine (to take into account the properties of the analytic continuation of a given function and to apply the properties of functions in complex domains). The main aim of Nekrasov's construction is to extend the class of functions to which the general differentiation can be applied.

It should be noted that the construction proposed by Nekrasov is fairly complicated and needs to use properties of the functions on Riemann surfaces. It follows from the properties of functions to which Nekrasov tried to apply his definition. The starting point of his construction is the notion of classes (q, μ) of function. Let L be a closed contour encircled a group of singular points of the function $f(z)$. Let the function $f(z)$ have the

following property: if the point z makes a complete detour along L in counter clockwise direction, then the function $f(z)$, continuously changing, gains the multiplier $e^{2\pi qi}$. Then, this function is of class $(q, 0)$. Thus, any function of the class $(q, 0)$ can be represented in the form $f(z) = (z - a)^q \phi(z)$, where a lies inside L , and $\phi(z)$ is of the class $(0, 0)$. The function of the form $f(z) = (z - a)^q \log^\mu z \phi(z)$, with $\phi(z)$ being of the class $(0, 0)$, is said to belong to the class (q, μ) (with q being the power index, and μ being the logarithmic index which is supposed to be nonnegative integer number). It is clear that, if the function $f(z)$ belongs to the class (q, μ) , then it belongs to any class $(q \pm m, \mu)$, $m \in \mathbb{N}$. Clearly, this definition depends on the choice of the contour L .

The function $f(z)$, which can be represented in form of a sum of finite number n of functions belonging to different classes with respect to the contour L , is said to be reducible to n classes (or simply reducible).

Let the function $f(z)$ be reducible to n classes with zero logarithmic indices, i.e.,

$$f(z) = f_0(z) + f_1(z) + \dots + f_{n-1}(z), \quad (48)$$

where $f_j(z)$ is of class $(q_j, 0)$. Then, we have the following representation:

$$f(z) + \int_{(L^k)}^{(z)} \frac{df(t)}{dt} dt = \alpha_0^k f_0(z) + \alpha_1^k f_1(z) + \dots + \alpha_{n-1}^k f_{n-1}(z), \quad (49)$$

where integration is performed along the contour L , traversable k -times in counter clockwise direction starting from the point z , $\alpha_j = e^{2\pi q_j i}$. By assigning the values $0, 1, \dots, n-1$ to the parameter k , we obtain the system of equations sufficient for determination of functions $f_0(z), f_1(z), \dots, f_{n-1}(z)$.

Let the function $f(z)$ be reducible to n classes with non-zero logarithmic indices, i.e.,

$$f(z) = \sum_{s=0}^{n_0-1} (z-a)^{q_{s,0}} \phi_{s,0}(z) + \sum_{s=0}^{n_1-1} (z-a)^{q_{s,1}} \log(z-a) \phi_{s,1}(z) + \dots + \sum_{s=0}^{n_\mu-1} (z-a)^{q_{s,\mu}} \log^\mu(z-a) \phi_{s,\mu}(z), \quad (50)$$

where $n_0 + n_1 + \dots + n_\mu = n$, and all functions $\phi_{s,j}(z)$ are of class $(0, 0)$. Then, we have the following representation:

$$f(z) + \int_{(L^k)}^{(z)} \frac{df(t)}{dt} dt = \sum_{s=0}^{n_0-1} \alpha_{s,0}^k (z-a)^{q_{s,0}} \phi_{s,0}(z) + \sum_{s=0}^{n_1-1} \alpha_{s,1}^k (z-a)^{q_{s,1}} \{2k\pi i + \log(z-a)\} \phi_{s,1}(z) + \dots + \sum_{s=0}^{n_\mu-1} \alpha_{s,\mu}^k (z-a)^{q_{s,\mu}} \{2k\pi i + \log(z-a)\}^\mu \phi_{s,\mu}(z), \quad (51)$$

where integration is performed along the contour L , traversable k -times in counter clockwise direction starting from the point z , $\alpha_{s,j} = e^{2\pi q_{s,j} i}$. By assigning the values $0, 1, \dots, n-1$ to the parameter k , we obtain the system of equations sufficient for determination of functions $\phi_{s,0}(z), \phi_{s,1}(z), \dots, \phi_{s,n-1}(z)$.

Therefore, in both cases, we have a finite sum representation of the function $f(z)$ of the considered form. Now, the question is to define the integral/derivative of arbitrary order of each components of the representation (48) or (50). Moreover, any function $f(z)$ of the class (q, μ) can be determined as the following limit:

$$f(z) = \lim_{h \rightarrow 0} \left[(z-a)^q \left(\frac{(z-a)^h - 1}{h} \right)^\mu \right]. \quad (52)$$

Thus, we have the limit of the finite sum of functions belonging to the classes $(q, 0), (q+h, 0), \dots, (q+\mu h, 0)$. Therefore, the definition of the integral/derivative of ar-

bitrary order of any reducible function can be completely described if one can define the definition of a function of class $(q, 0)$. Nekrasov noted that his construction of his derivative generally speaking cannot be rigorously defined in the case when $f(z)$ is reducible to infinite number of classes.

For this definition, Nekrasov used Letnikov's formulas. The only difference is that the contour of integration is now specially deformed curve on the Riemann surfaces (which depend on the order of the derivative, i.e., can be either finite-sheeted or infinite-sheeted).

In Reference [33], Nekrasov applied his construction to determinate the solution of the following differential equation:

$$\sum_{s=0}^n (a_s + b_s x) x^s \frac{d^s y}{dx^s} = 0, \quad (53)$$

which is highly related to the generalized hypergeometric function ${}_pF_q(z)$.

7. Conclusions

In this article, we analyzed some important results by Russian mathematicians of the end of the XIX century, namely A. V. Letnikov, N. Ya. Sonine, and P. A. Nekrasov. The main attention is paid to their contribution related to Fractional Calculus and Special Functions Theory. Some of these results are presented for the first time in English.

Our article serves to clarify the beginning steps of the development of Fractional Calculus. We believe that it would be useful and interesting for members of fractional society.

Short Biographies

Letnikov Alexey Vasil'evich (1837–1888), Russian mathematician. A short biography.

A. V. Letnikov was born on 1 January 1837, in Moscow, Russia. When Alexey was 8 years old, his father died. His mother tried to give education to Alexey and his sister. The mother sent Alexey to a grammar school in 1847. In spite of his evident abilities, he was not too successful in education. Therefore, he was moved to Konstantin's land-surveyors institute (full-time provisional military-type institute). That was a second rank educational establishment. Its director discovered high interest of Alexey to mathematics and supported his growth in the subject. The director decided to prepare him to the career of a teacher in mathematics in Konstantin's land-surveyors institute. To get the corresponding position, Letnikov was sent to Moscow University and studied mathematics there for two years (1856–1858) as an extern student.

After graduation, he was sent to Paris in order to extend his knowledge in the most known mathematical center for two years and to study the structure and the content of the technical education in France. In Paris, Letnikov attended lectures of many well-known mathematicians (Liouville among them) in the Ecole Polytechnique, College de France and Sorbonne.

Returning from Paris in December 1860, he was appointed as a teacher in the engineering class of the Konstantin's land-surveyors institute and started to teach Probability Theory. Letnikov actively participated in mathematical life in Moscow. In particular, he was among the founders of Moscow Mathematical Society in 1864. In 1863, it was approved a new Statute of Higher Education. Among other regulations, it was supposed to enlarge a number of chairs at universities and to recruit new university teachers. To get a position at university, one needed either to pass graduation gymnasium's exams or to receive the degree at a foreign university. Letnikov decided to use the second option. In 1867, he defended his PhD, "Über die Bedingungen der Integrabilität einiger Differential-Gleichungen", at Leipzig University. In 1868, he got a position at recently reopened Imperial Technical College (now Bauman's Technical University). Letnikov was working in this College up to 1883, when he moved to Alexandrov's Commercial College, sharing this job with a part-time teacher in Konstantin's land-surveyors institute and in the Imperial Technical College.

It was active time for him, and he was awarded the degree of a state councillor, got the order of Saint Stanislav, and was appointed in 1884 as a corresponding member of St.-Petersburg Academy of Sciences (by recommendation of V. Imshenetsky, V. Bunyakovsky, and O. Backlund). At the end of 1880s, Letnikov should have received a state pension and was supposed to leave teaching and to concentrate on the research. He was dreaming of getting a position at the Moscow University. It was not to happen since, at the opening ceremony of a new building of Alexandrov's Commercial College, he caught a cold. He had no serious illness before and continued to deliver lectures. But, this time, the illness was strong enough, and he died on 27 February 1888.

Sonine Nikolai Yakovlevich (1849–1915), Russian mathematician. A short biography.

N.Ya.Sonine was born on February 10th, 1849, in Tula, Russia.

He studied at Physical-mathematical Faculty of Moscow University (1865–1869). After graduation, he continued research in Moscow University for two years and, in 1871, defended his Master Thesis, “On expansion of functions into infinite series”. In June 1871, he became Associate Professor (dozent) of Warsaw University.

In 1873, he was sent to Paris to continue research study. In Paris, he attended lectures by Liouville, Hermite, Bertrand, Serre and Darboux. In September 1874, in Moscow University, N. Ya. Sonine defended his PhD Thesis, “On integration of partial differential equations of the second order”. In 1877, he became extra-ordinary Professor of Warsaw University and, in 1879, became an ordinary Professor of Warsaw University. In 1891, he resigned from his position at Warsaw University, but he still continued his research. In 1891, N. Ya. Sonine was elected a corresponding member of Academy of Sciences, and, in 1893, he became an academician of St.-Petersburg Academy of Sciences (by recommendation of P. L. Chebyshev). In 1890, he was awarded by V. Ya. Bunyakovsky Prize for the Best Results in Mathematics.

Starting in 1899, N. Ya. Sonine occupied different administrative positions, mostly in the education. He died on 18 February 1915, in St.-Petersburg.

Nekrasov Pavel Alekseevich (1853–1924), Russian mathematician and philosopher. A short biography.

P. A. Nekrasov was born on 1st (13th) February 1853, in Ryazan region, Russia.

After graduation at Ryazan Orthodox seminary in 1874, he entered Physical-mathematical faculty of the Moscow University. In 1878, he graduated from the Moscow University with degree of the candidate of sciences and was left at the department of pure mathematics for preparation to the professorship. From August 1879, P. A. Nekrasov shared his research with teaching mathematics at the private Voskresensky's real school. In 1883, he defended his master thesis, “Study of the equation $u''' - pu'' - q = 0$ ”. For this work, he was awarded by V. Ya. Bunyakovsky Prize for the Best Results in Mathematics.

In 1885, P. A. Nekrasov became a Privatdozent in the Moscow University (having defended his Russian PhD “On Lagrange series” in 1886), and, in 1886, he got the position of an associate professor (extraordinary professor) at Moscow University. In 1890, he received a full professorship. In 1893, he became the rector of Moscow University. After his term as the rector, he actually wanted to retire, but he was not allowed to. He also taught, from 1885–1891, the Probability Theory and the Higher Mathematics in the Land-surveyors institute.

Starting in 1898, he performed only administrative duties for the Ministry of Education (he was curator of the Moscow University and responsible for the schools in Moscow and the surrounding area) and moved in 1905 to Saint Petersburg as a member of the Council of the Ministry of Education. After the Russian Revolution, he tried to adapt himself to the new realities, dealt with mathematical economics (which he lectured in 1918–1919), and studied Marxism. He died of pneumonia on 20 December 1924, in Moscow.

Author Contributions: Conceptualization, S.R. and M.D. Writing—review and editing, S.R. and M.D. The authors made equal contributions to this article. Both authors have read and agreed to the published version of the manuscript.

Funding: The work has been supported by Belarusian Fund for Fundamental Scientific Research through the grant F20R-083.

Data Availability Statement: The study did not report any data.

Conflicts of Interest: The authors declare no conflict of interest.

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