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STRONG DUAL PROBLEMS IN LINEAR COPOSITIVE OPTIMIZATION AND THEIR PROPERTIES

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The presentation is devoted to formulation of new strong dual problems for linear copositive optimization and analysis of their properties. Based on the recently introduced concept of the set of immobile indices, here we deduce an extended dual problems that satisfy the strong duality relations and do not require any additional regularity assumptions such as constraint qualifications.

Keywords: Duality Theory; Semidefinite programming; Copositive programming; strong duality.

Introduction

Duality theory is a rich and powerful area of Convex Optimization which is central to understanding sensitivity analysis and infeasibility issues as well as to development of numerical methods.

One says that a pair of dual problems (P) and (D) satisfies *strong duality relation/property* if under the assunption that prime problem (P) is feasible and $val(P) > -\infty$, dual problem (D) has an optimal solution and the duality gap is zero: $val(P) - val(D) = 0.$ Here and in what follows, for an optimization problem (P), $val(P)$ denotes the optimal value of its cost function.

Strong duality property is important both from theoretical and practical point of view since the majority of numerical methods (e.g. the Interior Point Method) are based on the assumption that the strong duality relation is satisfied. Violation of this property leads to great numerical difficulties.Hence, it is important to get strong dual formulations that do not need any regularity conditions.

1. Conic optimization problems

A linear conic optimization problems has a (general) form

$$\text{CP:} \qquad \min_{x} c^\top x \quad \text{s.t. } A(x) \in K,$$

where the decision variable is the $n-$ vector $x = (x_1, ..., x_n)^\top$; $\mathcal{A}(x) := \sum_{j=1}^{n} A_j x_j + A_0$ with given matrices $A_j \in S^p$, $j = 0,1,\ldots,n$; S^p is the space of $p \times p$ symmetric matrices, $K \subset S^p$ is a cone. For the conic problem (CP), the classical Lagrangian dual problem has the form

$$\text{LD:} \qquad \max_{U} -U \bullet A_0,\ s.t.\ U \bullet A_j = c_j,\ \ j = 1,2,\ldots,n,\ \ U \in K^*,$$

where $K^* := \{A \in S^p : A \bullet D \geq 0 \ \ \forall D \in K\}$ is the dual cone of cone K , $A \bullet D = \mathrm{trac}(AD)$.

The most important classes of Conic Optimization are Semidefinite Programming and Copositive Programming problems.

If in the conic problem (CP), we set $K := S_+^p$, we get a problem of Semidefinite Programming (SDP)

$$\text{SDP:} \qquad \min_{x} c^\top x \quad \text{s.t. } A(x) \in S_+^p.$$

Here $p > 1$, $\mathcal{S}_+^p$ is the cone of symmetric positive semidefinite $p \times p$ matrices.

SDP is a natural generalization of Linear Programming. SDP models have many theoretical and practical applications (see [1, 2]). SDP problems are rather well studied and there are several solvers.

Let $\mathcal{COP}^p \subset \mathcal{S}^p$ be the cone of copositive matrices defined as

$$\mathcal{COP}^p := \{D \in S^p : t^\top Dt \geq 0 \ \forall t \in \mathbb{R}_+^p\}.$$

If in the conic problem (CP) set $K := \mathcal{COP}^p$, then we get a linear Copositive Programming (CoP) problem

$$\text{COP:} \qquad \min_{x} c^\top x \quad \text{s.t. } A(x) \in \mathcal{COP}^p.$$

CoP is a relatively new field of conic optimization which is actively developed in recent years. CoP problems have important applications (see [3]), including $\mathcal{NP}$ -hard problems.

CoP problems can be considered as a generalization of SDP ones. Unfortunately, CoP problems are less studied than SDP problems. There are many open theoretical problems (see [4]). There exists no methods, nor efficient software for solving CoP problems as well. The reason for this is that the cone $\mathcal{COP}^p$ and its dual, the cone of *completely positive matrices:*

$$\mathcal{CP}^p := \text{conv}\{tt^\top : t \in \mathbb{R}^p_+\}, \tag{1}$$

are more complicated than the cone S^p_+. In fact, the cone S^p_+ is self-dual ($(S^p_+)^* = S^p_+$) and homogeneous (hence symmetric), facially exposed, and nice; but the copositive cone $\mathcal{COP}^p$ *does not* possess *all* mentioned here properties of S^p_+.

One of important open problems for CoP is to formulate dual problems that satisfy strong duality property. The aim of this presentation is to consider and analyze such formulations.

2. Strong dual formulations for SDP and cop problems

In paper [5] for the semidefinite programming problem (SDP), it was obtained the Extended Dual Problem

$$\max -(U + W_{m_0}) \bullet A_0,$$

$$\text{s.t.} \quad (U_m + W_{m-1}) \bullet A_j = 0,\ j = 0,1,...,n,\ m = 1,...,m_0, \tag{2}$$

$$\text{ED-R:} \qquad (U + W_{m_0}) \bullet A_j = c_j,\ j = 1,2,...,n; \quad W_0 = \mathbb{O}_p, \tag{3}$$

$$U \in S^p_+,\ \begin{pmatrix} U_m & W_m \\ W_m^\top & I \end{pmatrix} \in S^{2p}_+,\ m = 1,...,m_0,$$

where $m_0 \geq 0$ is a finite integer, $\mathbb{O}_p$ is the $p \times p$ null matrix

This dual is stated completely in terms of the data of the original program (SDP), and the pair of dual problems (SDP) and (ED-R) satisfies strong duality relation without any additional assumptions.

The aim of our study is to formulate for CoP dual problems which have a form that is similar to the problem (ED-R) and satisfy the strong duality property.

For linear CoP problem (COP), the corresponding Lagrangian dual problem has the form

LDP: $$\max_U -U \bullet A_0, \text{ s.t. } U \bullet A_j = c_j,\ j=1,2,\ldots,n, U \in \mathcal{CP}^p,$$

where $\mathcal{CP}^p$ is the dual cone to the cone $\mathcal{COP}^p$ defined in (1).

It is known that under the Slater condition the duality gap for dual pair (COP) and (LDP) is zero. But without the Slater condition, the duality gap can be positive.

Based on an approach proposed in [6, 7], we formulated in [8] a new *extended* dual problem for (COP) in the form

EDP: $$\max -(U + W_{m_0}) \bullet A_0, \text{ s.t. (2), (3) and}$$

$$U \in \mathcal{CP}^p, \begin{pmatrix} U_m & W_m \\ W_m^{\top} & D_m \end{pmatrix} \in \mathcal{CP}^{2p}, m = 1,\ldots,m_0, \tag{4}$$

with the dual variables $U_m \in S^p, W_m \in \mathbb{R}^{p\times p}, D_m \in S^p, m = 1,\ldots,m_0; U \in \mathcal{CP}^p$.

Theorem 1 in [8] justifies that pair of problems (COP) and (EDP) satisfies strong duality relation: if $val(\text{COP}) > -\infty$, then there exists a finite $m_0 \geq 0$ such that the dual problem (EDP) has an optimal solution and $val(\text{COP}) = val(\text{EDP})$.

If we compare the strong duals (ED-R) for problem SDP and (EDP) for CoP problem, we can see that they have a similar structure. The only natural expected difference is that for SDP formulations we use the cone S^p_+ (which is self-dual), and for the CoP formulations use the cone $\mathcal{COP}^p$ (for the primal problem) and the cone $\mathcal{CP}^p$ (which is dual to $\mathcal{COP}^p$) for its dual.

In paper [9], we obtined an estimate for integer m_0 : $0 \leq m_0 \leq p^* = \min\{2n, p(p+1)/2\}$.

For the problem (COP), let us formulate other dual problems and compare their properties. We start with a dual problem that was proposed and justified in our recently submitted paper.

Given a finite integer $m_0 \geq 0$, let us consider the following problem:

DP: $$\max -(U + W_{m_0}) \bullet A_0, \quad \text{s.t. (2), (3) and}$$

$$U_m \in \mathcal{CP}^p, W_m \in (\mathcal{F}(U_m))^* \ \forall m = 1,\ldots,m_0, \tag{5}$$

where $\mathcal{F}(U) := \{D \in \mathcal{COP}^p : D \bullet U = 0\}$ is the exposed face of $\mathcal{COP}^p$ generated by $U \in \mathcal{CP}^p$.

Theorem 1. *Let the problem* (COP) *be consistent and* $val(\text{COP}) > -\infty$.

Then there exists m_0, $0 \le m_0 \le p^*$ *such that for the pair of problems* (COP) *and* (DP)*, the strong duality relations hold true.*

Moreover, one can show that in general the set of feasible solutions of the dual problem (DP) is bigger than the set of feasible solutions of the problem (EDP).

Some other strong dual for Conic Optimization problems was considered in [10]. Theorem 2 from [10], applied to the problem (COP), is as follows.

Theorem 2. *For all large enough integer* m_0*, problem*

$$\text{FDP:} \quad \begin{aligned} &\max(-Y_{m_0+1} \bullet A_0), \quad \text{s.t. } Y_m \bullet A_j = 0,\ j = 0,1,...,n,\ m = 1,...,m_0; \\ &\qquad\qquad Y_{m_0+1} \bullet A_j = c_j,\ j = 1,2,...,n; \\ &\qquad\qquad (Y_1, Y_2, ..., Y_{m_0+1}) \in \mathrm{F}R_{m_0+1}(\mathcal{COP}^p) \end{aligned} \tag{6}$$

*is a strong dual for problem (*COP*). Here for integer* $k \ge 1$, $\mathrm{F}R_k(\mathcal{K})$ *denotes a facial reduction cone of order* k of a cone $\mathcal{K}$:

$$\mathrm{F}R_k(\mathcal{K}) := \{(Y_1, Y_2, ..., Y_k) : Y_1 \in \mathcal{K}^*, Y_m \in (\mathcal{K} \cap Y_1^{\perp} \cap ... \cap Y_{m-1}^{\perp})^*, m = 2,...,k\}.$$

Thus, the variables of the problem (FDP) (the dual variables) belong to the facial reduction cone of order $m_0 + 1$ of the cone $\mathcal{COP}^p$. It was shown in [10] that for any $k \ge 1$, the cone $\mathrm{F}R_k(\mathcal{COP}^p)$ is convex and, for any $k > 1$, it is not closed.

The following lemma shows that the set of feasible solutions of the problem (FDP) is wider than the set of feasible solutions of the problem (DP).

Lemma 1. *Let* $(W_0, U_m, W_m, m = 1,...,m_0, U)$ *be a feasible solution of the problem (*DP*). Then* $(Y_1 = U_1, Y_m = U_m + W_{m-1}, m = 2,...,m_0, Y_{m_0+1} = U + W_{m_0})$ *is a feasible solution of the problem* (FDP).

Thus we have considered several dual problems for the copositive problem (COP) that satisfy strong duality relation without any additional assumptions. Having compared these dual problems, we can state the following.

1) The problems (EDP), (DP), and (FDP) differ from each other in constraints (4), (5), and (6).

2) The problem (EDP) can be considered as a completely positive problem. The problems (DP) and (FDP) are conic problems whose variables belong to the cones $\mathrm{F}R_2(\mathcal{COP}^p)$ and $\mathrm{F}R_{m_0+1}(\mathcal{COP}^p)$, respectively.

3) The dual problems (EDP) and (DP), contain m_0 separate explicite conditions (4) and (5), respectively, for each $m = 1,...,m_0$. In the problem (FDP),

instead of these m_0 constraints, there is a unique, but more complex constraint (6) in a recursive form (this constraint can be considered as a kind of "aggregation" of the mentioned above "simple" constraints in the problem (EDP)).

4) The facial reduction cone $\mathrm{F}R_{m_0+1}(\mathcal{COP}^p)$ used in the problem (FDP) is not explicitly described. The dimension of this cone is large, which greatly complicates the solution of this problem.

5) Each feasible solution of the problem (EDP) generates a feasible solution of the problem (DP), and each feasible solution of the latter problem generates a feasible solution of the problem (FDP).

Conclusions

The main contribution of the presentation consists in considering some new dual problems for the copositive problem and comparing their properties.The results provide templates for creating other strong dual formulations for linear/convex copositive problems. These formulations can be used for a variety of purposes, both theoretical and practical.

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