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Homogeneous Integrable Legendrian Contact Structures in Dimension Five

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Abstract
We consider Legendrian contact structures on odd-dimensional complex analytic manifolds. We are particularly interested in integrable structures, which can be encoded by compatible complete systems of second order PDEs on a scalar function of many independent variables and considered up to point transformations. Using the techniques of parabolic differential geometry, we compute the associated regular, normal Cartan connection and give explicit formulas for the harmonic part of the curvature. The PDE system is trivializable by means of point transformations if and only if the harmonic curvature vanishes identically. In dimension five, the harmonic curvature takes the form of a binary quartic field, so there is a Petrov classification based on its root type. We give a complete local classification of all five-dimensional integrable Legendrian contact structures whose symmetry algebra is transitive on the manifold and has at least one-dimensional isotropy algebra at any point.

Keywords Legendrian structures · Symmetry algebra · Curvature module · Multiply transitive · Complete systems of PDEs

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1 Introduction

One of the main goals of this paper is to approach this complete classification of Levi non-degenerate real hypersurfaces in $\mathbb{C}^3$ that form the boundary (or rather its smooth part) of a possibly unbounded homogeneous domain in $\mathbb{C}^3$. It is well-known that all such bounded domains are symmetric. This result is attributed to Cartan [9], who however never published any details on this. A more recent classification of homogeneous bounded domains up to dimension 8 is present in [14]. We note that the first non-symmetric examples of bounded homogeneous domains are due to Pjateckii-Shapiro [22] and appear only in dimension 4. The unbounded case, however, is much less studied. One of the examples of such domains is given by the real hypersurface

$$\mathrm{Im}(z + w\bar{x}) + |x|^4 = 0,$$

known as the Winkelmann hypersurface [27]. This hypersurface separates $\mathbb{C}^3$ into two open domains, both of which are preserved by an 8-dimensional real Lie group of biholomorphic transformations, which is the maximal symmetry dimension of a non-spherical Levi non-degenerate hypersurface in $\mathbb{C}^3$.

Loboda [17–19] obtained the complete classification of all strictly pseudo-convex real hypersurfaces in $\mathbb{C}^3$ with large (of dimension 6 and higher) symmetry groups. He also classified such surfaces with 7-dimensional symmetry with indefinite Levi form. The classification of hypersurfaces with 6-dimensional symmetry and indefinite Levi form has been still open.

We approach this problem by considering its complex counterpart. Namely, the geometry of real analytic Levi non-degenerate hypersurfaces in $\mathbb{C}^3$ is closely related to complete systems of 2nd order complex PDEs on one function $u(x, y)$ of two variables. We refer to works of Sukhov [23,24], Gaussier, Merker [13,20] for more detail. Complete systems of 2nd order PDEs on one function of an arbitrary number of independent variables can be represented as differential geometric objects known as Legendrian contact geometries.

The present paper focuses on the detailed study of these geometries. In particular, we obtain the complete list of integrable Legendrian contact structures in dimension 5 with transitive symmetry algebras of dimension 6 and higher. The next step, a classification of homogeneous CR structures in dimension 5 and the corresponding homogeneous hypersurfaces in $\mathbb{C}^3$, will be treated in a separate paper [11].

A *Legendrian contact structure* $(M; E, F)$ is defined to be a splitting of a contact distribution C (on an odd-dimensional manifold M) into the direct sum of two subdistributions E, F that are maximally isotropic with respect to the naturally defined conformal symplectic structure on C. Such structures can be treated in both the real smooth and complex analytic categories. In the current paper, we assume that all our manifolds and related objects are complex analytic, although many results are also valid in the smooth category.

We shall exclusively deal with *integrable* Legendrian contact structures (or just ILC structures), which means that both isotropic subdistributions are completely integrable. The main sources of ILC structures are compatible complete systems of 2nd order PDEs on one unknown function of several variables (considered up to *point* transformations), i.e.

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = f_{ij}(x, u, \partial u), \qquad 1 \le i, j \le n,$$

and the complexifications of (Levi non-degenerate) CR structures of codimension 1.

The smallest dimension of a manifold with a Legendrian contact structure is 3. In this dimension both isotropic subdistributions are 1-dimensional and are automatically completely integrable. The corresponding ILC structures can be encoded by a single 2nd order ODE and have been well-studied starting from the pioneering work of Tresse [26] (see also [2,6,21]). Their real counterpart, CR structures on 3-dimensional real hypersurfaces in $\mathbb{C}^2$, have also been well-studied starting from the classical works of Cartan [7,8].

Legendrian contact structures belong to the class of so-called parabolic geometries. In particular, they enjoy a number of important properties derived from the general theory of parabolic geometries [4]: the existence of a natural Cartan connection, description of the principal invariants in terms of the representation theory of simple Lie algebras, finite-dimensional symmetry algebras, and the classification of submaximal symmetry dimensions [16]. Legendrian contact structures are modeled by the flag variety $\mathrm{Flag}_{1,n+1}(\mathbb{C}^{n+2})$ of pairs of incident lines and hyperplanes in $\mathbb{C}^{n+1}$ equipped with a natural action of $\mathrm{PGL}(n+2, \mathbb{C})$.

We note that in [25], Takeuchi studied the special class of Legendrian contact structures that are induced on the projective cotangent bundle $M = \mathcal{P}(T^*N)$ from a projective structure $(N, [\nabla])$. With the sole exception of the flat model, this induced structure on M is never an ILC structure. Thus, his study is transverse to our study here.

In the current paper we are mainly interested in the classification of *multiply transitive* ILC structures in dimension 5. The term "multiply transitive" means that the symmetry algebra of the ILC structure should be transitive on the manifold and should have a non-trivial isotropy subalgebra (i.e. at least one-dimensional) at each point. As our study here is local in nature, we may as well require these conditions in an open subset of the manifold.

In dimension 3, all multiply transitive ILC structures are flat. This reflects a well-known fact that any 2nd order ODE is either equivalent to the trivial equation $u''(x) = 0$ and has 8-dimensional symmetry algebra, or its symmetry algebra is at most 3-dimensional. In dimension 5 this is no longer the case, as, for example, the submaximally symmetric ILC structures have symmetry algebras of dimension 8 and are multiply transitive [16]. In fact, all ILC structures with 8 symmetries are locally equivalent. This leaves us with the classification of ILC structures with 6- and 7-dimensional symmetry.

As in the case of the geometry of scalar 2nd order ODEs, complete systems of 2nd order PDEs also admit a notion of duality that swaps the set of dependent and

independent variables with the space of constants of integration parametrizing the generic solution. This corresponds to swapping the two isotropic distributions defining the ILC structure. We classify ILC structures up to this duality and indicate which structures are self-dual, i.e., locally contact equivalent to their dual.

In his famous 1910 paper [5], Élie Cartan studied the geometry of rank two distributions on 5-manifolds having generic growth vector $(2, 3, 5)$. For such structures, Cartan solved the local equivalence problem and obtained a classification of all multiply transitive models.[1] While the equivalence problem was solved by means of *Cartan's equivalence method* [12], we bypass this step in our study of ILC structures by using the full power of parabolic geometry. Indeed, representation theory is used to quickly construct the full curvature module and set up the structure equations for the (regular, normal) Cartan geometry. Our classification of multiply transitive ILC structures implements Cartan's technique, which we refer to as *Cartan's reduction method*.

There is another striking similarity between ILC structures in dimension 5 and $(2, 3, 5)$ distributions. In both cases the fundamental invariant is represented by a single binary quartic. Similar to the Petrov classification for the Weyl curvature tensor in Lorentzian (conformal) geometry, we classify ILC structures in dimension 5 by the number and multiplicity of roots of this quartic. We also prove that non-flat multiply transitive structures may only have type N (a single root of multiplicity 4), type D (two roots of multiplicity 2), or type III (one simple root and one root of multiplicity 3). This is quite similar to Cartan's result [5] that all multiply transitive $(2, 3, 5)$-distributions have either type N or type D. We identify the maximal symmetry dimension for each Petrov type in Theorem 3.1.

The main result of our paper can be summarized as follows:

Theorem 1.1 *Any multiply transitive ILC structure in dimension 5 is locally equivalent to the ILC structure defined by one of PDE models in Table 1 or its dual.*

Remark 1.2 We denote by u_{11}, u_{12}, u_{22} the second order partial derivatives of the unknown function u, and use the notation $p = u_1, q = u_2$ for the first order derivatives.

The functions $\mathcal{F}_\mu$ and $\mathcal{G}_\kappa$ are defined as follows:

$$\mathcal{F}_\mu(z) = \begin{cases} z^\mu, & \mu \in \mathbb{C}\backslash\{0, 1\} \\ \ln(z), & \mu = 0 \\ z\ln(z), & \mu = 1 \\ \exp(z), & \mu = \infty \end{cases}$$

$$\mathcal{G}_\kappa(z) = \begin{cases} z^\kappa, & \kappa \in \mathbb{C} \\ \exp(z), & \kappa = \infty \end{cases}$$

In particular, the parameters μ, κ are both allowed to take the value ∞ if the contrary is not stated.

Remark 1.3 A checkmark or cross under the SD column indicates that *every* element in the indicated family is self-dual or not self-dual, respectively. The situation for N.6-2

[1] One inadvertent omission from Cartan's list was recently discovered in [10].

Table 1 Classification of all multiply transitive ILC structures in dimension 5

Model	SD	u_{11}	u_{12}	u_{22}	Remarks
O.15	✓	0	0	0	Flat model
N.8	✓	q^2	0	0	Unique submaximal
N.7-1	×	$q^2\mathcal{G}_\kappa(x)$	0	0	$\kappa \in \mathbb{C}_\infty\backslash\{0,-3\};\ \kappa \sim -3-\kappa$
N.7-2	✓	$\frac{1}{q}$	1	0	
N.6-1	✓	$\mathcal{F}_\mu(q)$	1	0	$\mu \in \mathbb{C}\backslash\{-1,2\}$
N.6-2	*	$\mathcal{F}_\mu(q)\mathcal{G}_\kappa(x)$	0	0	$\mu \in \mathbb{C}_\infty\backslash\{-1,2\}, \kappa \in \mathbb{C}_\infty\backslash\{0,-3\};$ $\mu \sim 1-\mu,\ \kappa \sim -3-\kappa$
D.7	✓	p^2	0	λq^2	$\lambda \in \mathbb{C}\backslash\{-1\};\ \lambda \sim \frac{1}{\lambda}$ for $\lambda \neq 0$
D.6-1	✓	$p^2-\frac{q^4}{4}$	$q(p-\frac{q^2}{2})$	$p-\frac{q^2}{2}$	
D.6-2	✓	$\mathcal{G}_\mu(p)$	0	0	$\mu \in \mathbb{C}_\infty\backslash\{0,1,2\}$
D.6-3	✓	$\lambda p^2 R$	$1+\lambda(pq-2u)R$	$\lambda q^2 R$	$R=\frac{\sqrt{u-pq}}{u^{3/2}},\ \lambda \in \mathbb{C}\backslash\{0,\pm\frac{1}{2}\};\ \lambda \sim -\lambda$
D.6-3$_\infty$	✓	$p^2\sqrt{1-2pq}$	$(pq-1)\sqrt{1-2pq}$	$q^2\sqrt{1-2pq}$	
D.6-4	×	0	$\frac{1+pq}{u}$	0	
III.6-1	×	$\frac{p}{x-q}$	0	0	
III.6-2	×	$2q(2p-qu)$	q^2	0	

Table 2 Symmetry algebras of multiply transitive ILC structures

Model	Derived series (DS)	Nilradical	Comments
N.8	[8,6,4,0]	6-dim, DS = [6,4,0], LCS = [6,4,3,1,0]	
N.7-1	[7,5,2,0]	5-dim, DS = LCS = [5,2,0]	
N.7-2	[7,6,6]	4-dim abelian	$(\mathfrak{so}_3 \ltimes \mathbb{C}^3) \times \mathbb{C}$
N.6-1	$\begin{cases} [6,5,2,0], & \mu \neq 0 \\ [6,4,1,0], & \mu = 0 \end{cases}$	5-dim, DS = [5,2,0], LCS = [5,2,1,0]	
N.6-2	[6,4,0]	$\begin{cases} \text{5-dim, DS} = [5,2,0], \\ \quad \text{LCS} = [5,2,1,0] \\ \text{4-dim abelian,} \end{cases}$, $\mu = \kappa = \infty$ otw	
D.7	$\begin{cases} [7,6,6], & \lambda \neq 0 \\ [7,6,4,3,3], & \lambda = 0 \end{cases}$	$\begin{cases} \text{1-dim abelian,} & \lambda \neq 0 \\ \text{3-dim Heisenberg,} & \lambda = 0 \end{cases}$	$\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathbb{C}$
D.6-1	[6,6]	1-dim	$\mathfrak{sl}_2 \ltimes \mathfrak{s}_3$, $\mathfrak{s}_3$ is Heisenberg
D.6-2	[6,4,1,0]	4-dim, DS = LCS = [4,1,0]	
D.6-3	[6,6]	0-dim	$\mathfrak{sl}_2 \times \mathfrak{sl}_2$
D.6-3_∞	[6,6]	3-dim abelian	$\mathfrak{so}_3 \ltimes \mathbb{C}^3$
D.6-4	[6,6]	0-dim	$\mathfrak{sl}_2 \times \mathfrak{sl}_2$
III.6-1	[6,4,2,0]	4-dim, DS = [4,2,0], LCS = [4,2,1,0]	
III.6-2	[6,5,5]	2-dim abelian	$\mathfrak{gl}_2 \ltimes \mathbb{C}^2$

is more complicated. The corresponding ILC structure is self-dual if and only if the parameters μ and κ satisfy $\mu - \kappa - 2 = 0$ or $\mu + \kappa + 1 = 0$ (see Table 14).

Remark 1.4 Equations from different items in this list correspond to inequivalent ILC structures. However, there are some additional equivalence relations on parameter spaces for multi-parameter equations within the same item. They are indicated in the last column of Table 1.

Our labeling abides by the following rules. The first letter (N, D, or III) denotes the type of the invariant binary quartic. The next digit (6, 7, or 8) refers to the dimension of the symmetry algebra. The final digit is a labeling of the equation within the given subclass. Finally, the case D.6-3_∞ is a limit of D.6-3 as the parameter λ tends to infinity.

Table 2 describes basic algebraic properties of symmetry algebras for obtained models.

The paper is organized as follows. In Sect. 2 we provide generalities concerning Legendrian contact structures, establish the relationship between ILC structures and compatible complete systems of 2nd order PDEs, discuss the notion of duality, define the (regular, normal) Cartan connection associated with a given ILC structure, and provide explicit formulas for the fundamental (harmonic) part of its curvature.

In Sect. 3 we specialize to 5-dimensional ILC structures, define the fundamental binary quartic and prove that ILC structures of types I and II cannot be multiply transitive. We also reconstruct the full curvature tensor of the Cartan geometry.

In Sect. 4 we proceed with the detailed Cartan analysis of the general regular, normal Cartan connection, which involves normalizing parts of the curvature and its derivatives, reducing the Cartan bundle and iterating the procedure. As we are interested only in multiply transitive ILC structures, we terminate this process as soon as the fibers become 0-dimensional. This leads us to the list of all possible structure equations for the reduced bundles. We integrate each of these structure equations and come up with the corresponding ILC model defined in terms of the system of 2nd order PDEs. Finally, in the Appendix we give the detailed Lie algebra isomorphisms establishing the correspondence between the Cartan equations of the reduced bundle and the model systems of 2nd order PDEs, the equivalence relations on the parameters and the duality.

2 Legendrian Contact Structures

2.1 Generalities

On any contact manifold (M, C), the contact distribution $C \subset TM$ is locally defined by the vanishing of a 1-form σ (unique up to multiplication by a non-vanishing function), and $d\sigma|_C$ is a (conformal) symplectic form. Given a splitting $C = E \oplus F \subset TM$ into transverse Legendrian subdistributions E and F, i.e. $d\sigma|_E = 0$ and $d\sigma|_F = 0$, we say $(M; E, F)$ is a *Legendrian contact (LC) structure*. Let $\dim(M) = 2n + 1$, so $n = \operatorname{rank}(E) = \operatorname{rank}(F)$. Two LC structures $(M; E, F)$ and $(\tilde{M}; \tilde{E}, \tilde{F})$ are *(locally) equivalent* if there exists a (local) diffeomorphism $\phi : M \to \tilde{M}$ such that $d\phi(E) = \tilde{E}$ and $d\phi(F) = \tilde{F}$. There is also a natural notion of duality of LC structures: the *dual* of $(M; E, F)$ is $(M; F, E)$.

Since E and F are Legendrian, then $[E, E] \subset C$ and $[F, F] \subset C$. The projections from C onto E and F induce maps $\tau_E : \Gamma(E) \times \Gamma(E) \to \Gamma(F)$ and $\tau_F : \Gamma(F) \times \Gamma(F) \to \Gamma(E)$ that obstruct the integrability of E and F. The structure is *semi-integrable* or *integrable* according to whether one or both of τ_E, τ_F are identically zero. In the latter case, we call it an *ILC structure*.

Proposition 2.1 *Given any contact manifold (M, C) of dimension $2n + 1$ and a rank n integrable subdistribution $V \subset C$, we may choose local coordinates (x^i, u, p_i) on M such that contact form is $\sigma = du - p_i dx^i$ and $V = \operatorname{span}\{\partial_{p_i}\}$.*

Proof Since V is integrable and rank n, then by the Frobenius theorem there exist local coordinates $\{x^i\}_{i=1}^{2n+1}$ such that $V = \ker\{dx^1 = \ldots = dx^{n+1}\}$. Hence, $C = \ker\{\sigma\}$, where $\sigma = \lambda_1 dx^1 + \ldots + \lambda_{n+1} dx^{n+1}$. The contact condition $(d\sigma)^n \wedge \sigma \neq 0$ implies that not all λ_i can simultaneously vanish, so WLOG $\lambda_{n+1} \neq 0$ locally, and after rescaling we may assume $\lambda_{n+1} = 1$. Now define $u = x^{n+1}$ and $p_i = \lambda_i$. The contact condition guarantees that (x^i, u, p_i) is indeed a coordinate system. □

Suppose that $V := F$ is integrable, i.e. the LC structure is semi-integrable. By Proposition 2.1, there exist functions $f_{ij} = f_{ij}(x^k, u, p_\ell)$ with $f_{ij} = f_{ji}$ (since E is Legendrian) such that

$$E = \operatorname{span}\{\mathcal{D}_i := \partial_{x^i} + p_i \partial_u + f_{ij} \partial_{p_j}\}, \qquad V = \operatorname{span}\{\partial_{p_i}\}. \tag{2.1}$$

Equivalently, we are studying the geometry of the system of scalar 2nd order PDE

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = f_{ij}(x, u, \partial u), \qquad 1 \leq i, j \leq n, \tag{2.2}$$

considered up to *point transformations*. These are contact transformations that preserve the (vertical) bundle V. All such transformations are precisely the prolongations of arbitrary diffeomorphisms in the (x^i, u) variables. The system (2.2) is overdetermined if $n > 1$. If $n = 1$, then (2.2) is a single 2nd order ODE, whose point geometry has been well-studied [26].

Remark 2.2 Consider the jet spaces $J^k = J^k(\mathbb{C}^n, \mathbb{C})$ and projections $\pi^k_\ell : J^k \to J^\ell$. On J^2, the contact system is $\{du - p_i dx^i, dp_i - p_{ij} dx^j\}$, expressed in standard jet coordinates. Pulling back to a submanifold $\mathcal{E}$ defined by $p_{ij} = f_{ij}(x^k, u, p_\ell)$ yields the subbundle E in (2.1). The restriction $\pi^2_1|_{\mathcal{E}} : \mathcal{E} \to J^1$ is a local diffeomorphism. The subbundle V in (2.1) is tangent to the fibers of $\pi^1_0 \circ \pi^2_1$.

Lemma 2.3 *The PDE system (2.2) is compatible if and only if E in (2.1) is integrable.*

Proof It is easy to see that $[\mathcal{D}_i, \mathcal{D}_j] \in E$ if and only if $[\mathcal{D}_i, \mathcal{D}_j] = 0$, which happens if and only if $\mathcal{D}_j f_{ik} = \mathcal{D}_i f_{jk}$ for $1 \leq i, j, k \leq n$. This is exactly the compatibility condition of (2.2). □

2.2 Duality

If the dual LC structures $(M; E, F)$ and $(M; F, E)$ are equivalent, then we say that the structure is *self-dual*. For ILC structures, the notion of duality generalizes the classical duality for 2nd order ODE [6]. Namely, for the ILC structure $(M; E, V)$ given by (2.1), we can (by Proposition 2.1) find coordinates (y^i, v, q_i) for the dual ILC structure $(M; V, E)$, i.e.

$$V = \operatorname{span}\left\{\partial_{y^i} + q_i \partial_v + \tilde{f}_{ij} \partial_{q_j}\right\}, \quad E = \operatorname{span}\{\partial_{q_i}\}.$$

Then $\frac{\partial^2 v}{\partial x^i \partial x^j} = \tilde{f}_{ij}$ is the *dual system* to (2.2) (and is well-defined only up to point transformations).

Example 2.4 The simplest example of an ILC structure is the flat model $u_{ij} = 0$. The Legendre transformation $(y^i, v, q_k) = (p_i, u - p_j x^j, -x^k)$, is a contact (but non-point) transformation which swaps the E and V subbundles, so this structure is self-dual.

Example 2.5 For ILC structures when $n = 2$, we have the self-dual D.7 systems:

$$\mathcal{S}_\lambda : \quad u_{11} = p^2, \quad u_{12} = 0, \quad u_{22} = \lambda q^2, \qquad \lambda \in \mathbb{C}\backslash\{-1\},$$

where $p = u_1$ and $q = u_2$. For fixed λ, a self-duality, i.e. a swap $(E, V) \mapsto (V, E)$, is exhibited by

$$\Phi(x, y, u, p, q) = \begin{cases} \left(-\lambda(x + \frac{1}{p}), -\left(y + \frac{1}{\lambda q}\right), -u + \ln(-p) + \frac{1}{\lambda}\ln(-q), \frac{p}{\lambda}, q\right), & \lambda \neq 0; \\ \left(-\left(x + \frac{1}{p}\right), -q, -u + qy + \ln(-p), p, -y\right), & \lambda = 0 \end{cases}$$

Moreover, $\mathcal{S}_\lambda \cong \mathcal{S}_{1/\lambda}$ when $\lambda \neq 0$ via the transformation $\Phi(x, y, u, p, q) = (y, x, \lambda u, \lambda q, \lambda p)$.

As in the case of dual 2nd order ODEs, the dual ILC structures can be constructed in terms of the corresponding PDE models via swapping the space of independent and dependent variables with the space of integration constants parametrizing solutions of a given compatible PDE. In more detail, the general solution of any compatible system (2.2) is parametrized by $n + 1$ constants of integration and can be written as:

$$F(x^i, u; a^j, b) = 0, \qquad 1 \leq i, j \leq n. \tag{2.3}$$

We can consider this as an $(n+1)$-parameter family of hypersurfaces in (x^i, u)-space with parameter space (a^j, b). On the other hand, we can (locally) regard b as a function of a^j, so that (2.3) can be interpreted as an $(n+1)$-parameter family of hypersurfaces in (a^j, b)-space with parameter space (x^i, u). This is the solution space of a well-defined compatible system of 2nd order PDE's on $b(a^j)$.

Algorithmically, we construct the dual PDE system by differentiating (2.3) with respect to a^j (regarding x^i, u as constants and b as a function of a^j), solving the obtained system of $n + 1$ equations with respect to x^i, u and substituting the solution into the second order derivatives of (2.3) with respect to a^j.

Example 2.6 In the simplest example of the flat equation $u_{ij} = 0$ the general solution is given by:

$$u = a^1 x^1 + \ldots a^n x^n + b.$$

Treating b as a function of a^j, differentiating this solution twice and excluding x^i, u we get the same flat equation $b_{ij} = 0$. This again demonstrates the self-duality of the flat model.

Example 2.7 The III.6-1 system $u_{11} = \frac{p}{x-q}$, $u_{12} = u_{22} = 0$ has general solution

$$u = -ay + c - b(x + a)^2, \qquad a, b, c \in \mathbb{C}.$$

Regarding c as a function of a, b and treating x, y, u as parameters, we have $c_a = y + 2b(x + a)$, $c_b = (x + a)^2$, and

$$c_{aa} = 2b, \quad c_{ab} = 2(x + a) = \pm 2\sqrt{c_b}, \quad c_{bb} = 0.$$

WLOG, the $\pm$ ambiguity can be eliminated: the corresponding PDE systems are equivalent via the point transformation $(a, b, c) \mapsto (-a, b, c)$. Thus, the dual system to III.6-1 is

$$u_{11} = 2y, \quad u_{12} = 2\sqrt{q}, \quad u_{22} = 0.$$

Our classification indicates that III.6-1 is not self-dual (but a priori this is not at all obvious).

2.3 LC Structures as Parabolic Geometries

There is an equivalence of categories between (holomorphic) LC structures $(M; E, F)$ and (regular, normal) parabolic geometries $(\mathcal{G} \to M, \omega)$ of a fixed type (G, P) [4]. Here, $G = \mathrm{PGL}(n+2, \mathbb{C})$ acts on the flag variety of pairs of incident lines and hyperplanes:

$$G/P \cong \mathrm{Flag}_{1,n+1}\left(\mathbb{C}^{n+2}\right) = \{(\ell, \pi) : \pi(\ell) = 0\} \subset \mathbb{CP}^{n+1} \times \left(\mathbb{CP}^{n+1}\right)^*,$$

and $P \subset G$ is the parabolic subgroup that is the stabilizer of a chosen origin. Since $A \in \mathrm{GL}(n+2, \mathbb{C})$ and λA (for $\lambda \in \mathbb{C}^\times$) have the same action on G/P, we will instead use $G = \mathrm{SL}(n+2, \mathbb{C})$. The kernel of this action is isomorphic to the cyclic group $\mathbb{Z}_{n+2}$, generated by multiples of the identity matrix by $(n+2)$-th roots of unity. In terms of Lie algebras, P corresponds to the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}_{n+2}$ defined by the contact grading:

$$\mathfrak{sl}_{n+2} = \left\{ \begin{bmatrix} a & U & \gamma \\ X & A & W \\ \beta & Y & b \end{bmatrix} : \begin{array}{l} b = -a - \mathrm{tr}(A), \\ a \in \mathbb{C},\ A \in \mathfrak{gl}_n, \\ \text{etc.} \end{array} \right\} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \overbrace{\mathfrak{g}_0 \oplus \underbrace{\mathfrak{g}_1 \oplus \mathfrak{g}_2}_{\mathfrak{p}_+}}^{\mathfrak{p}}. \tag{2.4}$$

The reductive part $G_0 \subset P$ has corresponding subalgebra $\mathfrak{g}_0 \cong \mathbb{C}^2 \times \mathfrak{sl}_n$ (corresponding to the diagonal blocks (a, A, b)) and there is a unique element $Z \in \mathcal{Z}(\mathfrak{g}_0)$ that induces the grading. We refer to the eigenvalues of Z on a particular $\mathfrak{g}_0$-module as its *homogeneities*.

At the origin $o \in G/P$, we have $T_o(G/P) \cong \mathfrak{g}/\mathfrak{p}$. Define the subspaces $E_o, F_o \subset T_o(G/P)$ (or subspaces in $\mathfrak{g}_{-1}/\mathfrak{p}$) corresponding to X, Y in (2.4) respectively. The induced G-invariant structure $(G/P; E, F)$ is the *flat* LC structure, and $(G \to G/P, \omega_G)$ is the *flat* model, where ω_G is the Maurer–Cartan form on G. The dimension of the Lie algebra of (infinitesimal) symmetries of the flat model is $\dim(G) = n^2 + 4n + 3$.

A Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) is a curved analogue of the flat model. It consists of a principal P-bundle $\mathcal{G} \to M$ equipped with a Cartan connection $\omega \in \Omega^1(\mathcal{G}; \mathfrak{g})$. This means:

(CC.1) $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for any $u \in \mathcal{G}$;
(CC.2) $R_p^*\omega = \mathrm{Ad}_{p^{-1}} \circ \omega$ for any $p \in P$;
(CC.3) $\omega(\zeta_A) = A$ for any $A \in \mathfrak{p}$, where $\zeta_A(u) = \frac{d}{dt}\big|_{t=0} R_{\exp(tA)}(u)$, i.e., ζ_A is the fundamental vertical vector field corresponding to A.

The curvature of $(\mathcal{G} \to M, \omega)$ is the 2-form $K = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}; \mathfrak{g})$. Using the framing of $T\mathcal{G}$ provided by ω yields a P-equivariant function $\kappa : \mathcal{G} \to \bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$ that descends to $\kappa : \mathcal{G} \to \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ since K is horizontal. For parabolic geometries, the Killing form on $\mathfrak{g}$ yields a P-module isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$, so we obtain a function $\kappa : \mathcal{G} \to \bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{g}$. The geometry is

- *regular* if κ is valued in the subspace of $\bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ consisting of positive homogeneities;
- *normal* if $\partial^*\kappa = 0$, where ∂^* is the Lie algebra homology differential.

2.4 Harmonic Curvature

For regular, normal parabolic geometries, since $(\partial^*)^2 = 0$, we may quotient κ by $\mathrm{im}(\partial^*)$ to obtain $\kappa_H : \mathcal{G} \to \frac{\ker(\partial^*)}{\mathrm{im}(\partial^*)}$. This fundamental curvature quantity is called *harmonic curvature* and is a complete obstruction to flatness of the geometry. The P-module $\frac{\ker(\partial^*)}{\mathrm{im}(\partial^*)}$ is completely reducible, so $\mathfrak{p}_+$ acts trivially. By a result of Kostant [4,15], the 2-cochains $C^2(\mathfrak{g}_-, \mathfrak{g})$ admit the (orthogonal) $\mathfrak{g}_0$-module decomposition

$$C^2(\mathfrak{g}_-, \mathfrak{g}) = \overbrace{\mathrm{im}(\partial^*) \oplus \underbrace{\ker(\Box)}_{\ker(\partial)} \oplus \mathrm{im}(\partial)}^{\ker(\partial^*)}, \tag{2.5}$$

where ∂ is the Lie algebra differential, and $\Box = \partial\partial^* + \partial^*\partial$ is the Kostant Laplacian. Thus,

$$\frac{\ker(\partial^*)}{\mathrm{im}(\partial^*)} \cong \ker(\Box) \cong \frac{\ker(\partial)}{\mathrm{im}(\partial)} =: H^2(\mathfrak{g}_-, \mathfrak{g}).$$

The $\mathfrak{g}_0$-module structure of the Lie algebra cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g})$ is completely described by Kostant's Bott–Borel–Weil theorem [1,4,15]. For LC structures with $n \geq 2$, $H^2(\mathfrak{g}_-, \mathfrak{g})$ decomposes into three $\mathfrak{g}_0$-irreps

$$H^2(\mathfrak{g}_-, \mathfrak{g}) = \mathbb{W} \oplus \mathbb{T}_1 \oplus \mathbb{T}_2$$

having homogeneities $+2, +1, +1$ respectively. The $\mathbb{T}_1$ and $\mathbb{T}_2$ components of κ_H are precisely the torsions τ_E and τ_F (see Sect. 2.1), and these vanish in the ILC case. Results from twistor theory (see [3]) indicate that the LC structures with trivial $\mathbb{W}$ and $\mathbb{T}_2$ components for κ_H correspond to projective structures. This is the case that was studied by Takeuchi [25].

2.5 Parametric Computations of Harmonic Curvature

Consider a semi-integrable LC structure $(M; E, V)$ given by (2.1). We will give an explicit formula for the $\mathbb{W}$-component of κ_H.

We use the following coframe for computations on the manifold M:

$$\theta^i = \mathrm{d}x^i, \qquad \pi_i = \mathrm{d}p_i - f_{ij}\,\mathrm{d}x^j, \qquad \sigma = \mathrm{d}u - p_i\,\mathrm{d}x^i, \qquad 1 \le i, j \le n,$$

so that

$$E = \ker\{\sigma, \pi_i\}, \qquad V = \ker\{\sigma, \theta^i\}.$$

The differential of an arbitrary function F is defined by the formula:

$$\mathrm{d}F = \frac{\mathrm{d}F}{\mathrm{d}x^i}\theta^i + \frac{\partial F}{\partial p_i}\pi_i + \frac{\partial F}{\partial u}\sigma,$$

where $\frac{\mathrm{d}}{\mathrm{d}x^i} := \mathcal{D}_i$ (see (2.1)) is the total derivative with respect to x^i.

Let $(\mathcal{G}, \omega)$ be any *regular* Cartan geometry of type (G, P) with underlying structure $(M; E, V)$ and curvature K. Let $E_a{}^b \in \mathfrak{gl}_{n+2}$ denote the element with 1 in the a-th row and b-column and 0 otherwise. Here, we let $0 \le a, b \le n+1$. If $s : M \to \mathcal{G}$ is any (local) section, write

$$s^*\omega = \omega^a{}_b\, E_a{}^b, \quad s^*K = K^a{}_b\, E_a{}^b,$$

where $K^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$.

Lemma 2.8 *There exists a section $s\colon M \to \mathcal{G}$ such that $s^*\omega$ satisfies*

$$\omega^{n+1}{}_0 = \sigma, \quad \omega^i{}_0 = \theta^i, \quad \omega^{n+1}{}_i = \pi_i, \quad \omega^0{}_0 \equiv 0 \mod \{\theta^i, \pi_i\}$$

Proof Consider a section $s\colon M \to \mathcal{G}$. Since ω is regular, the negative part of $s^*\omega$ is an adapted coframe, i.e.

$$\omega^{n+1}{}_0 = e\sigma, \quad \omega^{n+1}{}_i = g^j{}_i\pi_j + g_i\sigma, \quad \omega^i{}_0 = h^i{}_j\theta^j + h^i\sigma.$$

An arbitrary section $\tilde{s}$ is given in terms of a function $h\colon M \to P$ such that $\tilde{s} = s \cdot h$. This satisfies:

$$\tilde{s}^*\omega = h^{-1}\left(s^*\omega\right)h + h^{-1}dh.$$

Since $h^{-1}dh$ term is $\mathfrak{p}$-valued, the negative part of $s^*\omega$ transforms via the adjoint action.

Using the G_0-action, we can normalize $e = 1$ and $g^j{}_i = \delta^j{}_i$. Since

$$\begin{aligned} K^{n+1}{}_0 &= \mathrm{d}\omega^{n+1}{}_0 + \omega^{n+1}{}_a \wedge \omega^a{}_0 \equiv \mathrm{d}\sigma + \omega^{n+1}{}_i \wedge \omega^i{}_0 \\ &\equiv (-\pi_i + h^j{}_i \pi_j) \wedge \theta^i \mod \sigma, \end{aligned}$$

and regularity implies $K^{n+1}{}_0 \equiv 0 \mod \sigma$, then $h^i{}_j = \delta^i{}_j$. Using the action of subgroup of P corresponding to $\mathfrak{g}_1$, we can normalize $g_i = 0, h^i = 0$. Similarly, using the subgroup of P corresponding to $\mathfrak{g}_2$, we can normalize $\omega^0{}_0 \equiv 0 \mod \{\theta^i, \pi_j\}$. □

With respect to such a section, write

$$\omega^a{}_b = r^a{}_{bi}\,\theta^i + s^a{}_b{}^i\,\pi_i + t^a{}_b\,\sigma.$$

To obtain the harmonic part of the normal curvature, it is sufficient to compute normalization conditions only in homogeneities 1 and 2. For any regular, normal parabolic geometry, the lowest homogeneity curvature component is harmonic [4]. Thus, all curvature components in homogeneity 1 must vanish except the coefficients of $\theta^j \wedge \theta^k$ in $K^{n+1}{}_i$, and this corresponds to the torsion of our semi-integrable structure. Recalling that $\omega^{n+1}{}_{n+1} = -\omega^0{}_0 - \omega^i{}_i$ since ω is $\mathfrak{sl}_{n+2}$-valued, we have:

$$\begin{aligned} K^{n+1}{}_0 &= \mathrm{d}\omega^{n+1}{}_0 + \omega^{n+1}{}_a \wedge \omega^a{}_0 = \omega^{n+1}{}_0 \wedge \omega^0{}_0 + \omega^{n+1}{}_{n+1} \wedge \omega^{n+1}{}_0 \\ &= \sigma \wedge (2\omega^0{}_0 + \omega^i{}_i) \\ &= (2r^0{}_{0j} + r^i{}_{ij})\sigma \wedge \theta^j + (2s^0{}_0{}^j + s^i{}_i{}^j)\sigma \wedge \pi_j \\ K^i{}_0 &= \mathrm{d}\omega^i{}_0 + \omega^i{}_a \wedge \omega^a{}_0 \equiv \omega^i{}_0 \wedge \omega^0{}_0 + \omega^i{}_j \wedge \omega^j{}_0 \mod \sigma \\ &\equiv (r^i{}_{[jk]} + r^0{}_{0[j}\delta^i{}_{k]})\theta^k \wedge \theta^j + (s^i{}_j{}^k - s^0{}_0{}^k\delta^i{}_j)\pi_k \wedge \theta^j \mod \sigma \\ K^{n+1}{}_i &= \mathrm{d}\omega^{n+1}{}_i + \omega^{n+1}{}_a \wedge \omega^a{}_i \equiv \mathrm{d}\pi_i + \pi_j \wedge \omega^j{}_i + \omega^{n+1}{}_{n+1} \wedge \pi_i \mod \sigma \\ &\equiv \frac{\mathrm{d}f_{ij}}{dx^k}\theta^j \wedge \theta^k + \left(r^j{}_{ik} + (r^0{}_{0k} + r^l{}_{lk})\delta^j{}_i - \frac{\partial f_{ik}}{\partial p_j}\right)\pi_j \wedge \theta^k \\ &\quad + \left(s^j{}_i{}^k + (s^0{}_0{}^k + s^l{}_l{}^k)\delta^j{}_i\right)\pi_j \wedge \pi_k \mod \sigma \end{aligned}$$

We confirm that the coefficient of $\theta^j \wedge \theta^k$ in $K^{n+1}{}_i$ is indeed the obstruction $\mathcal{D}_k f_{ij} - \mathcal{D}_j f_{ik}$ to integrability of E. All remaining terms above are zero, so we get:

$$\begin{aligned} &s^j{}_i{}^k = 0, \quad s^0{}_0{}^i = 0, \quad r^0{}_{0i} = -\frac{1}{n+2}\frac{\partial f_{ij}}{\partial p_j}, \\ &r^i{}_{jk} = \frac{\partial f_{jk}}{\partial p_i} - \delta^i{}_j \frac{1}{n+2}\frac{\partial f_{lk}}{\partial p_l}. \end{aligned} \tag{2.6}$$

Proceed now to homogeneity 2. Using (2.6), we compute:

$$K^i{}_0 = s^i{}_{n+1}{}^j \pi_j \wedge \sigma + \left(r^i{}_{n+1,j} - t^i{}_j\right)\theta^j \wedge \sigma \tag{2.7}$$

$$K^{n+1}{}_i = \frac{\mathrm{d} f_{ij}}{\mathrm{d}x^k}\theta^j \wedge \theta^k + \left(\frac{\partial f_{ij}}{\partial u} - r^0{}_{ij}\right)\theta^j \wedge \sigma + \left(t^j{}_i - s^0{}_i{}^j + \delta^j_i t^k{}_k\right)\pi_j \wedge \sigma \tag{2.8}$$

$$K^0{}_0 \equiv \left(\frac{\mathrm{d}r^0{}_{0i}}{\mathrm{d}x^j} + r^0{}_{ij}\right)\theta^j \wedge \theta^i + \left(\frac{\partial r^0{}_{0i}}{\partial p_j} + s^0{}_i{}^j\right)\pi_j \wedge \theta^i \qquad \mathrm{mod}\ \sigma \tag{2.9}$$

$$K^i{}_j \equiv \left(\frac{\mathrm{d}r^i{}_{jl}}{\mathrm{d}x^k} + \delta^i{}_k r^0{}_{jl} + r^i{}_{pk} r^p{}_{jl}\right)\theta^k \wedge \theta^l + \left(\frac{\partial r^i{}_{jl}}{\partial p_k} - \delta^k{}_l t^i{}_j - \delta^k{}_j r^i{}_{n+1,l} - \delta^i{}_l s^0{}_j{}^k\right)\pi_k \wedge \theta^l \tag{2.10}$$

$$+ s^i{}_{n+1}{}^k \pi_k \wedge \pi_j \qquad \mathrm{mod}\ \sigma \tag{2.11}$$

To obtain the pullback $s^*\kappa : M \to \bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ of the curvature function $\kappa : \mathcal{G} \to \bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{g}$, we note that the framing provided by ω together with P-equivariancy of κ allows us to identify $\sigma = \omega^{n+1}{}_0$, $\theta^i = \omega^i{}_0$ and $\pi_i = \omega^{n+1}{}_i$ with $(E_j{}^0)^*$, $(E_{n+1}{}^j)^*$ and $(E_0{}^{n+1})^*$ respectively. A form B on $\mathfrak{gl}_{n+2}$ which is defined by $B(X, Y) = \mathrm{tr}(XY)$ and is proportional to the Killing form on $\mathfrak{gl}_{n+2}$ induces a P-module isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$. This allows us to make the replacements

$$\theta^j \leftrightarrow E_0{}^j, \quad \pi_j \leftrightarrow E_j{}^{n+1}, \quad \sigma \leftrightarrow E_0{}^{n+1}$$

in the curvature 2-form K. The homology differential $\partial^* : \bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{g} \to \mathfrak{p}_+ \otimes \mathfrak{g}$ is defined on decomposable elements as

$$\partial^*(X \wedge Y \otimes v) = -Y \otimes [X, v] + X \otimes [Y, v] - [X, Y] \otimes v.$$

We introduce a bi-grading on $\bigwedge^\bullet \mathfrak{p}_+ \otimes \mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{gl}_{n+2}$ be Cartan subalgebra for the standard upper-triangular Borel subalgebra. Let also $Z_i \in \mathfrak{h}$, $1 \le i \le n+1$ be a dual basis to the simple roots basis $\alpha_i \in \mathfrak{h}^*$, $1 \le i \le n+1$. Then the pair (Z_1, Z_{n+1}) induces bi-grading $X \to (a_1, a_{n+1})$ where $[Z_i, X] = a_i X$ for $i = 1, n+1$. Homogeneity of an element X is equal to $a_1 + a_{n+1}$ since $Z = Z_1 + Z_{n+1}$ where Z is a grading element. Moreover, since ∂^* is P-equivariant map it respects bi-grading.

In order to compute harmonic curvature it is sufficient to use only $\partial^*\kappa_{(1,1)} = 0$ and $\partial^*\kappa_{(0,2)} = 0$ normality conditions. Using (2.7)–(2.10) and $K^{n+1}{}_{n+1} = -K^0{}_0 - K^i{}_i$ we compute:

$$0 = \partial^*\kappa_{(1,1)} = \left(r^i{}_{n+1,j} - t^i{}_j\right)\left(E_0{}^{n+1} \otimes \left(E_i{}^j - \delta_i{}^j E_0{}^0\right) - E_0{}^j \otimes E_i{}^{n+1}\right) + \left(t^j{}_i - s^0{}_i{}^j + \delta^j{}_i t^k{}_k\right)$$

$$
\begin{aligned}
&\left(-{E_0}^{n+1}\otimes\left({E_j}^i-{\delta_j}^i{E_{n+1}}^{n+1}\right)+{E_j}^{n+1}\otimes{E_0}^i\right)\\
&+\left(\frac{\partial {r^0}_{0i}}{\partial p_j}+{s^0}_i{}^j\right)\left(-{E_j}^{n+1}\otimes{E_0}^i+{\delta^i}_j{E_0}^{n+1}\otimes{E_0}^0\right)\\
&+\left(\frac{\partial {r^i}_{jl}}{\partial p_k}-{\delta^k}_l{t^i}_j-{\delta^k}_j{r^i}_{n+1,l}-{\delta^i}_l{s^0}_j{}^k\right)\\
&\left({E_0}^l\otimes{\delta^j}_k{E_i}^{n+1}+{E_k}^{n+1}\otimes{\delta^l}_i{E_0}^j+{\delta^l}_k{E_0}^{n+1}\otimes{E_i}^j\right)\\
&+\left(\frac{\partial {r^0}_{0l}}{\partial p_k}+\frac{\partial {r^i}_{il}}{\partial p_k}-{\delta^k}_l{t^i}_i-r^k_{n+1,l}\right)\\
&\left({E_0}^l\otimes{E_k}^{n+1}-{\delta_k}^l{E_0}^{n+1}\otimes{E_{n+1}}^{n+1}\right)\\
=&\left(\frac{\partial {r^j}_{ki}}{\partial p_k}+\frac{\partial {r^0}_{0i}}{\partial p_j}+\frac{\partial {r^k}_{ki}}{\partial p_j}-(n+2){r^j}_{n+1,i}-{\delta^j}_i\left({s^0}_k{}^k+{t^k}_k\right)\right)\\
&{E_0}^i\otimes{E_j}^{n+1}\\
&+\left(\frac{\partial {r^i}_{jk}}{\partial p_k}-(n+2){t^i}_j-{\delta^i}_j{t^k}_k\right){E_0}^{n+1}\otimes\left({E_i}^j-{\delta_i}^j{E_{n+1}}^{n+1}\right)\\
&+\left(\frac{\partial {r^0}_{0i}}{\partial p_i}+{t^i}_i+{s^0}_i{}^i-{r^i}_{n+1,i}\right){E_0}^{n+1}\otimes\left({E_0}^0-{E_{n+1}}^{n+1}\right)\\
&+\left(\frac{\partial {r^k}_{ik}}{\partial p_j}-\frac{\partial {r^0}_{0i}}{\partial p_j}-(n+2){s^0}_i{}^j+{\delta^j}_i\left({t^k}_k-{r^k}_{n+1,k}\right)\right){E_j}^{n+1}\otimes{E_0}^i,\\
0=\partial^*\kappa_{(0,2)}=&-{s^i}_{n+1}{}^j{E_j}^{n+1}\otimes{E_i}^{n+1}\\
&+{s^i}_{n+1}{}^k(1-n){E_k}^{n+1}\otimes{E_i}^{n+1}-{s^i}_{n+1}{}^j{E_j}^{n+1}\otimes{E_i}^{n+1}\\
&+{s^i}_{n+1}{}^j{E_i}^{n+1}\otimes{E_j}^{n+1}=\left({s^j}_{n+1}{}^i-(n+1){s^i}_{n+1}{}^j\right){E_j}^{n+1}\otimes{E_i}^{n+1}.
\end{aligned}
$$

Substituting (2.6) we obtain linear system of equations on coefficients of normal regular Cartan connection:

$$
\begin{aligned}
0&=\frac{\partial^2 f_{ik}}{\partial p_j\partial p_k}-(n+2){r^j}_{n+1,i}-{\delta^j}_i\left({s^0}_k{}^k+{t^k}_k\right),\\
0&=\frac{\partial^2 f_{jk}}{\partial p_i\partial p_k}-{\delta^i}_j\frac{1}{n+2}\frac{\partial^2 f_{lk}}{\partial p_l\partial p_k}-(n+2){t^i}_j-{\delta^i}_j,{t^k}_k\\
0&=-\frac{1}{n+2}\frac{\partial^2 f_{ij}}{\partial p_i\partial p_j}+{t^i}_i+{s^0}_i{}^i-{r^i}_{n+1,i},\\
0&=\frac{\partial^2 f_{ik}}{\partial p_j\partial p_k}-(n+2){s^0}_i{}^j+{\delta^j}_i\left({t^k}_k-{r^k}_{n+1,k}\right).
\end{aligned}
$$

Solving the linear system of equations above we get the homogeneity 2 coefficients of normal Cartan connection needed for the computation of κ_H:

$$t^i{}_j = \frac{1}{n+2}\frac{\partial^2 f_{jk}}{\partial p_i \partial p_k} - \delta^i_j \frac{1}{(n+2)(n+1)}\frac{\partial^2 f_{lk}}{\partial p_l \partial p_k}, \tag{2.12}$$

$$r^i{}_{n+1,j} = t^i{}_j, \tag{2.13}$$

$$s^0{}_i{}^j = \frac{1}{n+2}\frac{\partial^2 f_{ik}}{\partial p_j \partial p_k}, \tag{2.14}$$

$$s^i{}_{n+1}{}^j = 0. \tag{2.15}$$

From Kostant's theorem we know that $\mathbb{W}$ has the lowest weight vector $\phi_0 = E_0{}^1 \wedge E_n{}^{n+1} \otimes E_n{}^1$. The element ϕ_0 belongs to the module $\mathbb{V}$ generated by:

$$w_l{}^k{}_i{}^j = E_0{}^k \wedge E_l{}^{n+1} \otimes \left(E_i{}^j - \delta_i{}^j E_{n+1}{}^{n+1}\right).$$

The module $\mathbb{W}$ is the submodule of $\mathbb{V}$ consisting of tensors that are trace-free in (i, j), symmetric in (l, i), and symmetric in (k, j). We denote coefficients of κ which corresponds to $w_l{}^k{}_i{}^j$ as $W^l{}_k{}^i{}_j$ and assume that

$$T^k{}_i{}^l{}_j = -\frac{\partial^2 f_{ij}}{\partial p_k \partial p_l}.$$

and $T^l{}_j = T^i{}_i{}^l{}_j$, $T = T^i{}_i{}^j{}_j$. According to (2.10)

$$W^l{}_k{}^i{}_j = -\left(\frac{\partial r^i{}_{jk}}{\partial p_l} - \delta^l{}_k t^i{}_j - \delta^l{}_j r^i{}_{n+1,k} - \delta^i{}_k r^l{}_{j,-1}\right).$$

Using (2.6) and (2.12)–(2.14) we obtain that $W^l{}_k{}^i{}_j$ is equal to trace-free part of tensor $T^l{}_k{}^i{}_j$:

$$\begin{aligned} W^l{}_k{}^i{}_j = T^l{}_k{}^i{}_j &- \frac{1}{n+2}\left(\delta^i{}_j T^l{}_k + \delta^l{}_k T^i{}_j + \delta^l{}_j T^i{}_k + \delta^i{}_k T^l{}_j\right) \\ &+ \frac{1}{(n+2)(n+1)}\left(\delta^l{}_k \delta^i{}_j + \delta^l{}_j \delta^i{}_k\right) T. \end{aligned}$$

Coefficients $W^l{}_k{}^i{}_j$ are symmetric in (i, l), (j, k), trace-free in (i, j) and form the $\mathbb{W}$-component of κ_H.

We summarize computations of this subsection in the following theorem.

Theorem 2.9 *With respect to the section s defined by Lemma 2.8, the $\mathbb{W}$ component of the harmonic curvature of the regular, normal connection for semi-integrable LCS given by 2.1 is*

$$W^l{}_k{}^i{}_j E_0{}^k \wedge E_l{}^{n+1} \otimes \left(E_i{}^j - \delta_i{}^j E_{n+1}{}^{n+1}\right), \tag{2.16}$$

where $W^l{}_k{}^i{}_j$ *is the trace-free part of the tensor*

$$T^l{}_k{}^i{}_j = -\frac{\partial^2 f_{kj}}{\partial p_l \partial p_i}.$$

3 ILC Structures in Dimension Five

Henceforth, we specialize to the $n = 2$ ILC case, which corresponds to *compatible* PDE systems

$$u_{11} = F, \quad u_{12} = G, \quad u_{22} = H,$$

where F, G, H are functions of (x, y, u, p, q) with $p = u_1$ and $q = u_2$. Equivalently, E and V as in (2.1) (with $f_{11} = F$, $f_{12} = G$, $f_{22} = H$) are both integrable.

Let us fix notation for $\mathfrak{p}$. Take the standard (upper triangular) Borel subalgebra, diagonal Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_4$, and simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1} \in \mathfrak{h}^*$ for $i = 1, 2, 3$. The dual basis $Z_1, Z_2, Z_3 \in \mathfrak{h}$ to the simple roots is given by

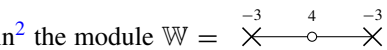
$$Z_1 = \operatorname{diag}\left(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right), \quad Z_2 = \operatorname{diag}\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right),$$
$$Z_3 = \operatorname{diag}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\right).$$

The grading element adapted to P is $Z := Z_1 + Z_3$. Use linear coordinates on $\mathfrak{p}$:

$$\begin{bmatrix} \frac{3z_1+z_2}{4} & t_1 & t_2 & t_5 \\ 0 & v_1 + \frac{z_2-z_1}{4} & v_2 & t_3 \\ 0 & v_3 & -v_1 + \frac{z_2-z_1}{4} & t_4 \\ 0 & 0 & 0 & -\frac{z_1+3z_2}{4} \end{bmatrix} \in \mathfrak{p}. \tag{3.1}$$

We have $\mathfrak{g}_0 = \mathcal{Z}(\mathfrak{g}_0) \times (\mathfrak{g}_0)_{ss} \cong \mathbb{C}^2 \times \mathfrak{sl}_2$, where $\mathbb{C}^2 = \operatorname{span}\{Z_1, Z_3\}$. In terms of the standard basis $\{E_a{}^b\}_{0 \le a,b \le 3}$ of $\mathfrak{gl}_4$, a standard $\mathfrak{sl}_2$-triple spanning the semisimple part $(\mathfrak{g}_0)_{ss} \subset \mathfrak{p}$ is given by:

$$\mathsf{H} := E_1{}^1 - E_2{}^2, \qquad \mathsf{X} := E_1{}^2, \qquad \mathsf{Y} := E_2{}^1. \tag{3.2}$$

For ILC structures, κ_H takes values in[2] the module $\mathbb{W} =$  (in the notation of [1]). With respect to (Z_1, Z_3), $\mathbb{W}$ has bi-grading $(+1, +1)$ so that its homogeneity is $+2$. As $\mathfrak{sl}_2$-modules, $\mathbb{W} \cong \bigodot^4(\mathbb{C}^2)$, i.e., the space of binary quartics in r, s, say. Hence, κ_H (up to sign) is:

[2] In terms of $\mathfrak{sl}_4$ fundamental weights $\{\lambda_i\}$, $\mathbb{W}$ has *lowest* weight $3\lambda_1 - 4\lambda_2 + 3\lambda_3 = \alpha_1 - \alpha_2 + \alpha_3$ by the "minus lowest weight" convention [1].

$$\kappa_H = F_{qq}\mathsf{r}^4 + 2(F_{pq} - G_{qq})\mathsf{r}^3\mathsf{s} + (F_{pp} - 4G_{pq} + H_{qq})\mathsf{r}^2\mathsf{s}^2 + 2(H_{pq} - G_{pp})\mathsf{r}\mathsf{s}^3 + H_{pp}\mathsf{s}^4. \tag{3.3}$$

Strictly speaking, this is the pullback of κ_H by a (local) section $s\colon M \to \mathcal{G}$. Since P_+ acts vertically trivially on $\ker(\partial^*)/\operatorname{im}(\partial^*)$, (3.3) is canonically defined only up to a G_0-transformation.

3.1 Petrov Classification

As in the Petrov classification of the Weyl tensor in 4-dimensional Lorentzian (conformal) geometry and the classification of $(2, 3, 5)$-distributions [5], ILC structures can be classified based on the (pointwise) root type of the binary quartic field (3.3). We use the same notation for types as in the Petrov classification, e.g., type N and D indicate a single quadruple root and a pair of double roots, respectively.

Any ILC structure admits at most a 15-dimensional symmetry algebra and 15 is realized only on (an open subset of) the flat model (up to local isomorphism). Among (regular, normal) parabolic geometries $(\mathcal{G} \to M, \omega)$ of a given type (G, P), Kruglikov and The [16] gave a general method for finding the submaximal symmetry dimension, i.e. the maximal symmetry dimension for any non-flat structure, and for ILC structures this dimension is eight. These techniques can also be used to determine the maximal symmetry dimension for ILC structures with constant root type. We briefly outline their method. A non-trivial root type corresponds to a G_0-orbit $\{0\} \neq \mathcal{O} \subset \mathbb{W}$ (or in type I, a collection of G_0-orbits). Defining $\mathfrak{a}^\phi = \mathfrak{g}_- \oplus \mathfrak{ann}(\phi)$ for non-flat ILC structures, we have:

$$\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \leq \max\{\dim(\mathfrak{a}^\phi) : \phi \in \mathcal{O}\} = 5 + \max\{\dim(\mathfrak{ann}(\phi)) : \phi \in \mathcal{O}\}. \tag{3.4}$$

Since $\dim(\mathfrak{ann}(\phi))$ is constant along G_0-orbits, it suffices to evaluate it on a cross-section.

Theorem 3.1 *Among ILC structures with constant root type, we have:*

Root type	O	N	D	III	II	I
Max. sym. dim.	15	8	7	6	5	5
Sharp?	✓	✓	✓	✓	✓	✓

Proof See Table 1 for type N, D, III models with the stated symmetry dimensions.

A Type I model with 5-dimensional symmetry is given by:

$$u_{11} = 6S^{5/3} - 6uS^{4/3} + 2(u^2 - q)S - 2pq, \quad u_{12} = 3S^{4/3} - 2uS - q^2, \quad u_{22} = 2S,$$

where $S = p + uq$. Its harmonic curvature is given by the quartic:

$$\kappa_H = -\frac{4}{3}\mathsf{r}(u\mathsf{r}+\mathsf{s})(\mathsf{r}-(u\mathsf{r}+\mathsf{s})S^{-1/3})(3\mathsf{r}-2(u\mathsf{r}+\mathsf{s})S^{-1/3}),$$

which has four distinct roots on the open set $\{S \neq 0\}$. The equation is invariant with respect to the action of $\mathfrak{sl}_2 \ltimes \mathbb{C}^2$ generated by:

$$\begin{gathered}\partial_x, \quad \partial_y, \quad x\partial_y + \partial_u - q\partial_p, \quad 2x\partial_x + y\partial_y - u\partial_u - 3p\partial_p - 2q\partial_q,\\ x^2\partial_x + xy\partial_y + (y - xu)\partial_u - (u + 3xp + yq)\partial_p + (1 - 2xq)\partial_q.\end{gathered}$$

Next, consider

$$u_{11} = p^{\lambda+2}q^{\mu}, \quad u_{12} = p^{\lambda+1}q^{\mu+1}, \quad u_{22} = p^{\lambda}q^{\mu+2},$$

which is type II when $\lambda, \mu \neq 0$, $\lambda + \mu \neq 0, 1$ according to

$$\kappa_H = p^{\lambda-2}q^{\mu-2}(p\mathsf{r} - q\mathsf{s})^2(\mu(\mu-1)p^2\,\mathsf{r}^2 + 2\lambda\mu pq\,\mathsf{rs} + \lambda(\lambda-1)q^2\,\mathsf{s}^2).$$

The symmetry algebra is generated by the 5 vector fields:

$$\begin{gathered}\partial_x, \quad \partial_y, \quad \partial_u, \quad -\mu(x\partial_x - p\partial_p) + \lambda(y\partial_y - q\partial_q),\\ (1+\lambda+\mu)(x\partial_x + y\partial_y) + (\lambda+\mu)u\partial_u - p\partial_p - q\partial_q.\end{gathered}$$

Now we establish upper bounds. Up to scale, representative elements in the G_0-orbits are

$$\text{N: } \mathsf{s}^4; \quad \text{D: } \mathsf{r}^2\mathsf{s}^2; \quad \text{III: } \mathsf{r}\mathsf{s}^3; \quad \text{II: } \mathsf{r}^2\mathsf{s}(\mathsf{r}-\mathsf{s}); \quad \text{I: } \mathsf{rs}(\mathsf{r}-\mathsf{s})(\mathsf{r}-c\mathsf{s}), \quad c \in \mathbb{C}\backslash\{0,1\}.$$

The annihilators of the above elements, cf. (3.2), are spanned by:

$$\begin{aligned}&\text{N: } Z_1 - Z_3,\ Y,\ H + 4Z_1; \quad &&\text{D: } Z_1 - Z_3,\ H;\\ &\text{III: } Z_1 - Z_3,\ H + 2Z_1; \quad &&\text{II, I: } Z_1 - Z_3.\end{aligned}$$

By (3.4), the result is proved for N and D, while for III, II, I the upper bound is one more than in the stated result. For the latter, we show that the upper bound is never realizable.

Consider the type III orbit and assume there is a model with 7-dimensional symmetry algebra $\mathfrak{s}$. According to [16, Corollary 3.4.8] (in particular, ILC structures are "prolongation-rigid"), $\mathfrak{s}$ admits a natural filtration $\mathfrak{s} = \mathfrak{s}^{(-2)} \supset \mathfrak{s}^{(-1)} \supset \mathfrak{s}^{(0)}$ with associated-graded Lie algebra isomorphic to $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{a}_0$, where $\mathfrak{a}_0$ is the above annihilator of the type III orbit and $\mathfrak{g}_{-1}$, $\mathfrak{g}_{-2}$ are graded subspaces of $\mathfrak{g} = \mathfrak{sl}_4$. In other words, $\mathfrak{s}$ is a *filtered deformation* of the above 7-dimensional graded Lie algebra.

Any such deformation is necessarily invariant with respect to $\mathfrak{s}^{(0)} = \mathfrak{a}_0$. Fix a basis $T_1 = Z_1 - Z_3, T_2 = H + 2Z_1$ in $\mathfrak{a}_0$. Its action on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2}$ diagonalizes

with pairs of eigenvalues $(-1,-1),(-1,-3),(1,-1),(1,1)$ and $(0,-2)$ respectively. Denote by $E_1 = {E_1}^0, E_2 = {E_2}^0, F_1 = {E_3}^1, F_2 = {E_3}^2, U = -{E_3}^0$ the corresponding eigenvectors of this action. Then all possible deformations of $\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{a}_0$ preserving the filtration and the action of $\mathfrak{a}_0$ have the form:

$$
\begin{aligned}
&[T_1,E_1]=-E_1,\quad [T_1,E_2]=-E_2,\quad [T_1,F_1]=F_1,\quad [T_1,F_2]=F_2,\\
&[T_2,E_1]=-E_1,\quad [T_2,E_2]=-3E_2,\quad [T_2,F_1]=-F_1,\quad [T_2,F_2]=F_2,\\
&[T_2,U]=-2U,\\
&[E_1,F_1]=U,\quad [E_1,F_2]=aT_1+bT_2,\quad [E_2,F_2]=U,\quad [F_2,U]=cF_1.
\end{aligned}
$$

However, due to Jacobi identity we get $a=b=c=0$. Thus, there are no non-trivial deformations in Type III case, thereby contradicting [[15], Proposition 4.2.2], and dimension 7 of symmetry algebra is not realized.

Similarly, for types I and II we have the one-dimensional annihilator $\mathfrak{a}_0$ spanned by $T = Z_1 - Z_3$. Using the same argument, we get a 4-parameter family of non-trivial deformations $\mathfrak{s}$ given by:

$$
\begin{aligned}
&[T,E_1]=-E_1,\quad [T,E_2]=-E_2,\quad [T,F_1]=F_1,\quad [T,F_2]=F_2,\\
&[E_1,F_1]=U+a_{11}T,\quad [E_1,F_2]=a_{12}T,\quad [E_2,F_1]=a_{21}T,\\
&[E_2,F_2]=U+a_{22}T,\\
&[E_1,U]=-a_{22}E_1+a_{12}E_2,\quad [E_2,U]=a_{21}E_1-a_{11}E_2,\\
&[F_1,U]=a_{22}F_1-a_{21}F_2,\quad [F_2,U]=-a_{12}F_1+a_{11}F_2.
\end{aligned}
$$

Replacing U by $U+\lambda T$, we may assume that $a_{22}=-a_{11}$. Each of these deformations $\mathfrak{s}$ defines an S-invariant ILC structure on the homogeneous space S/S_0, where S is the corresponding Lie group and S_0 is the subgroup corresponding to the 1-dimensional subalgebra spanned by T. The linear map $\alpha:\mathfrak{s}\to\mathfrak{sl}_4$ given by

$$
\begin{aligned}
&E_1\mapsto {E_1}^0-\frac{1}{2}a_{11}{E_1}^3-\frac{1}{2}a_{12}{E_2}^3,\quad E_2\mapsto {E_2}^0-\frac{1}{2}a_{21}{E_1}^3+\frac{1}{2}a_{11}{E_2}^3,\\
&F_1\mapsto {E_0}^1-\frac{1}{2}a_{11}{E_0}^1-\frac{1}{2}a_{21}{E_0}^2,\quad F_2\mapsto {E_0}^2-\frac{1}{2}a_{12}{E_0}^1+\frac{1}{2}a_{11}{E_0}^2,\\
&U\mapsto -{E_3}^0-\frac{1}{2}(a_{11}{E_1}^1+a_{21}{E_1}^2+a_{12}{E_2}^1-a_{11}{E_2}^2)-\frac{1}{4}(a_{11}^2+a_{12}a_{21}){E_0}^3
\end{aligned}
$$

is in fact a Lie algebra homomorphism. Hence, all these deformations are in fact trivial and yield the flat ILC structure [4, Sect. 1.5.15-16]. This contradicts the type I or II assumption. □

We exclude types II and I from further consideration, since no multiply transitive models exist.

3.2 Curvature Module

Since ILC structures are torsion-free geometries, a result of Čap [3, Sect. 3.2 corollary] implies that the curvature function κ takes values in the P-module $\mathbb{K} \subset \bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{p}$ generated by $\mathbb{W}$. We refer to $\mathbb{K}$ as the *curvature module*.

From Kostant's theorem, $\mathbb{W}$ has lowest weight vector $\phi_0 = {E_0}^1 \wedge {E_2}^3 \otimes {E_2}^1 \leftrightarrow \mathsf{s}^4$, and we generate all of $\mathbb{W}$ by applying the raising operator ${E_1}^2 \leftrightarrow \mathsf{r}\partial_{\mathsf{s}}$. The result of applying the raising operators ${E_0}^1$, ${E_0}^2$, ${E_1}^3$, ${E_2}^3$, ${E_0}^3 \in \mathfrak{p}_+$ to $\mathbb{W}$ is given in Table 3. Introduce coordinates on $\mathbb{K}$ (26-dimensional):

Quartic	$A = A_1\mathsf{r}^4 + 4A_2\mathsf{r}^3\mathsf{s} + 6A_3\mathsf{r}^2\mathsf{s}^2 + 4A_4\mathsf{r}\mathsf{s}^3 + A_5\mathsf{s}^4$
Cubic	$B = 4(B_1\mathsf{r}^3 + 3B_2\mathsf{r}^2\mathsf{s} + 3B_3\mathsf{r}\mathsf{s}^2 + B_4\mathsf{s}^3)\mathsf{w}_1$
	$B' = 4(B_5\mathsf{r}^3 + 3B_6\mathsf{r}^2\mathsf{s} + 3B_7\mathsf{r}\mathsf{s}^2 + B_8\mathsf{s}^3)\mathsf{w}_2$
Quadratic	$C = 6(C_1\mathsf{r}^2 + 2C_2\mathsf{r}\mathsf{s} + C_3\mathsf{s}^2)\mathsf{w}_1^2$
	$\widehat{C} = 12(C_4\mathsf{r}^2 + 2C_5\mathsf{r}\mathsf{s} + C_6\mathsf{s}^2)\mathsf{w}_1\mathsf{w}_2$
	$C' = 6(C_7\mathsf{r}^2 + 2C_8\mathsf{r}\mathsf{s} + C_9\mathsf{s}^2)\mathsf{w}_2^2$
Linear	$D = 12(D_1\mathsf{r} + D_2\mathsf{s})\mathsf{w}_1^2\mathsf{w}_2$
	$D' = 12(D_3\mathsf{r} + D_4\mathsf{s})\mathsf{w}_1\mathsf{w}_2^2$

3.3 Structure Equations

Write the Cartan connection $\omega \in \Omega^1(\mathcal{G}; \mathfrak{g})$ as

$$\omega = [{\omega^a}_b] = \begin{bmatrix} \frac{3\zeta_1+\zeta_2}{4} & \tau_1 & \tau_2 & \tau_5 \\ \varpi_1 & \nu_1 + \frac{\zeta_2-\zeta_1}{4} & \nu_2 & \tau_3 \\ \varpi_2 & \nu_3 & -\nu_1 + \frac{\zeta_2-\zeta_1}{4} & \tau_4 \\ \varpi_5 & \varpi_3 & \varpi_4 & -\frac{\zeta_1+3\zeta_2}{4} \end{bmatrix}.$$

Decompose $K = {K^a}_b {E_a}^b$, where ${K^a}_b \in \Omega^2(\mathcal{G})$. By torsion-freeness, ${K^1}_0 = {K^2}_0 = {K^3}_0 = {K^3}_1 = {K^3}_2 = 0$, and for ILC structures ${K^0}_0 = {K^3}_3 = 0$. The structure equations are $d{\omega^a}_b = -{\omega^a}_c \wedge {\omega^c}_b + {K^a}_b$, i.e.

$$\begin{aligned}
d\tau_1 &= (\nu_1 - \zeta_1) \wedge \tau_1 + \nu_3 \wedge \tau_2 - \tau_5 \wedge \varpi_3 + {K^0}_1 & d\varpi_1 &= (\zeta_1 - \nu_1) \wedge \varpi_1 - \nu_2 \wedge \varpi_2 - \tau_3 \wedge \varpi_5 \\
d\tau_2 &= \nu_2 \wedge \tau_1 - (\zeta_1 + \nu_1) \wedge \tau_2 - \tau_5 \wedge \varpi_4 + {K^0}_2 & d\varpi_2 &= -\nu_3 \wedge \varpi_1 + (\nu_1 + \zeta_1) \wedge \varpi_2 - \tau_4 \wedge \varpi_5 \\
d\tau_3 &= \tau_5 \wedge \varpi_1 - (\nu_1 + \zeta_2) \wedge \tau_3 - \nu_2 \wedge \tau_4 + {K^1}_3 & d\varpi_3 &= (\nu_1 + \zeta_2) \wedge \varpi_3 + \nu_3 \wedge \varpi_4 + \tau_1 \wedge \varpi_5 \\
d\tau_4 &= \tau_5 \wedge \varpi_2 - \nu_3 \wedge \tau_3 + (\nu_1 - \zeta_2) \wedge \tau_4 + {K^2}_3 & d\varpi_4 &= \nu_2 \wedge \varpi_3 + (\zeta_2 - \nu_1) \wedge \varpi_4 + \tau_2 \wedge \varpi_5 \\
d\tau_5 &= -\tau_1 \wedge \tau_3 - \tau_2 \wedge \tau_4 - (\zeta_1 + \zeta_2) \wedge \tau_5 + {K^0}_3 & d\varpi_5 &= \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4 + (\zeta_1 + \zeta_2) \wedge \varpi_5
\end{aligned}$$

$$\begin{aligned}
d\zeta_1 &= -\tfrac{3}{2}\tau_1 \wedge \varpi_1 - \tfrac{3}{2}\tau_2 \wedge \varpi_2 + \tfrac{1}{2}\tau_3 \wedge \varpi_3 + \tfrac{1}{2}\tau_4 \wedge \varpi_4 - \tau_5 \wedge \varpi_5 \\
d\zeta_2 &= \tfrac{1}{2}\tau_1 \wedge \varpi_1 + \tfrac{1}{2}\tau_2 \wedge \varpi_2 - \tfrac{3}{2}\tau_3 \wedge \varpi_3 - \tfrac{3}{2}\tau_4 \wedge \varpi_4 - \tau_5 \wedge \varpi_5 \\
d\nu_1 &= \tfrac{1}{2}\tau_1 \wedge \varpi_1 - \tfrac{1}{2}\tau_2 \wedge \varpi_2 - \tfrac{1}{2}\tau_3 \wedge \varpi_3 + \tfrac{1}{2}\tau_4 \wedge \varpi_4 - \nu_2 \wedge \nu_3 + \tfrac{1}{2}{K^1}_1 - \tfrac{1}{2}{K^2}_2 \\
d\nu_2 &= \tau_2 \wedge \varpi_1 - 2\nu_1 \wedge \nu_2 - \tau_3 \wedge \varpi_4 + {K^1}_2 \\
d\nu_3 &= \tau_1 \wedge \varpi_2 + 2\nu_1 \wedge \nu_3 - \tau_4 \wedge \varpi_3 + {K^2}_1
\end{aligned}$$

Table 3 The curvature module for ILC structures

(Z_1, Z_3)-grade	Label	2-chain
(+1,+1)	r^4	$E_0{}^2 \wedge E_1{}^3 \otimes E_1{}^2$
	$4\mathsf{r}^3\mathsf{s}$	$-\Omega \otimes E_1{}^2 - E_0{}^2 \wedge E_1{}^3 \otimes \mathsf{H}$
	$6\mathsf{r}^2\mathsf{s}^2$	$-E_0{}^1 \wedge E_2{}^3 \otimes E_1{}^2 - E_0{}^2 \wedge E_1{}^3 \otimes E_2{}^1 + \Omega \otimes \mathsf{H}$
	$4\mathsf{r}\mathsf{s}^3$	$\Omega \otimes E_2{}^1 + E_0{}^1 \wedge E_2{}^3 \otimes \mathsf{H}$
	s^4	$E_0{}^1 \wedge E_2{}^3 \otimes E_2{}^1$
(+2,+1)	$4\mathsf{r}^3\mathsf{w}_1$	$E_0{}^2 \wedge E_0{}^3 \otimes E_1{}^2 + E_0{}^2 \wedge E_1{}^3 \otimes E_0{}^2$
	$12\mathsf{r}^2\mathsf{s}\mathsf{w}_1$	$-E_0{}^1 \wedge E_0{}^3 \otimes E_1{}^2 - E_0{}^2 \wedge E_1{}^3 \otimes E_0{}^1 - \Omega \otimes E_0{}^2 - E_0{}^2 \wedge E_0{}^3 \otimes \mathsf{H}$
	$12\mathsf{r}\mathsf{s}^2\mathsf{w}_1$	$-E_0{}^2 \wedge E_0{}^3 \otimes E_2{}^1 - E_0{}^1 \wedge E_2{}^3 \otimes E_0{}^2 + \Omega \otimes E_0{}^1 + E_0{}^1 \wedge E_0{}^3 \otimes \mathsf{H}$
	$4\mathsf{s}^3\mathsf{w}_1$	$E_0{}^1 \wedge E_0{}^3 \otimes E_2{}^1 + E_0{}^1 \wedge E_2{}^3 \otimes E_0{}^1$
(+1,+2)	$4\mathsf{r}^3\mathsf{w}_2$	$E_1{}^3 \wedge E_0{}^3 \otimes E_1{}^2 + E_1{}^3 \wedge E_0{}^2 \otimes E_1{}^3$
	$12\mathsf{r}^2\mathsf{s}\mathsf{w}_2$	$E_2{}^3 \wedge E_0{}^3 \otimes E_1{}^2 + E_1{}^3 \wedge E_0{}^2 \otimes E_2{}^3 + \Omega \otimes E_1{}^3 - E_1{}^3 \wedge E_0{}^3 \otimes \mathsf{H}$
	$12\mathsf{r}\mathsf{s}^2\mathsf{w}_2$	$-E_1{}^3 \wedge E_0{}^3 \otimes E_2{}^1 - E_2{}^3 \wedge E_0{}^1 \otimes E_1{}^3 + \Omega \otimes E_2{}^3 - E_2{}^3 \wedge E_0{}^3 \otimes \mathsf{H}$
	$4\mathsf{s}^3\mathsf{w}_2$	$-E_2{}^3 \wedge E_0{}^3 \otimes E_2{}^1 - E_2{}^3 \wedge E_0{}^1 \otimes E_2{}^3$
(+3,+1)	$6\mathsf{r}^2\mathsf{w}_1^2$	$E_0{}^2 \wedge E_0{}^3 \otimes E_0{}^2$
	$12\mathsf{r}\mathsf{s}\mathsf{w}_1^2$	$E_0{}^3 \wedge E_0{}^2 \otimes E_0{}^1 + E_0{}^3 \wedge E_0{}^1 \otimes E_0{}^2$
	$6\mathsf{s}^2\mathsf{w}_1^2$	$E_0{}^1 \wedge E_0{}^3 \otimes E_0{}^1$
(+2,+2)	$12\mathsf{r}^2\mathsf{w}_1\mathsf{w}_2$	$E_1{}^3 \wedge E_0{}^3 \otimes E_0{}^2 + E_0{}^3 \wedge E_0{}^2 \otimes E_1{}^3 + E_1{}^3 \wedge E_0{}^2 \otimes E_0{}^3$
	$24\mathsf{r}\mathsf{s}\mathsf{w}_1\mathsf{w}_2$	$E_0{}^3 \wedge E_0{}^2 \otimes E_2{}^3 - E_0{}^3 \wedge E_2{}^3 \otimes E_0{}^2 + \Omega \otimes E_0{}^3 + E_0{}^3 \wedge E_1{}^3 \otimes E_0{}^1 - E_0{}^3 \wedge E_0{}^1 \otimes E_1{}^3$
	$12\mathsf{s}^2\mathsf{w}_1\mathsf{w}_2$	$E_0{}^1 \wedge E_0{}^3 \otimes E_2{}^3 + E_0{}^3 \wedge E_2{}^3 \otimes E_0{}^1 + E_0{}^1 \wedge E_2{}^3 \otimes E_0{}^3$
(+1,+3)	$6\mathsf{r}^2\mathsf{w}_2^2$	$E_0{}^3 \wedge E_1{}^3 \otimes E_1{}^3$
	$12\mathsf{r}\mathsf{s}\mathsf{w}_2^2$	$E_0{}^3 \wedge E_1{}^3 \otimes E_2{}^3 + E_0{}^3 \wedge E_2{}^3 \otimes E_1{}^3$
	$6\mathsf{s}^2\mathsf{w}_2^2$	$E_0{}^3 \wedge E_2{}^3 \otimes E_2{}^3$
(+3,+2)	$12\mathsf{r}\mathsf{w}_1^2\mathsf{w}_2$	$E_0{}^3 \wedge E_0{}^2 \otimes E_0{}^3$
	$12\mathsf{s}\mathsf{w}_1^2\mathsf{w}_2$	$E_0{}^1 \wedge E_0{}^3 \otimes E_0{}^3$
(+2,+3)	$12\mathsf{r}\mathsf{w}_1\mathsf{w}_2^2$	$E_0{}^3 \wedge E_1{}^3 \otimes E_0{}^3$
	$12\mathsf{s}\mathsf{w}_1\mathsf{w}_2^2$	$E_0{}^3 \wedge E_2{}^3 \otimes E_0{}^3$

Notation: $\Omega := E_0{}^1 \wedge E_1{}^3 - E_0{}^2 \wedge E_2{}^3$
$\mathfrak{p}$-module description: Degree 4 polynomials in $\mathsf{r}, \mathsf{s}, \mathsf{w}_1, \mathsf{w}_2$ modulo $\mathsf{w}_1^3, \mathsf{w}_2^3, \mathsf{w}_1^3\mathsf{w}_2, \mathsf{w}_1^2\mathsf{w}_2^2, \mathsf{w}_1\mathsf{w}_2^3$
$\mathfrak{sl}_2$-action: $\mathsf{H} = E_1{}^1 - E_2{}^2 \leftrightarrow \mathsf{r}\partial_\mathsf{r} - \mathsf{s}\partial_\mathsf{s}$, $\mathsf{X} = E_1{}^2 \leftrightarrow \mathsf{r}\partial_\mathsf{s}$, $\mathsf{Y} = E_2{}^1 \leftrightarrow \mathsf{s}\partial_\mathsf{r}$
$\mathfrak{p}_+$-action: $E_0{}^1 \leftrightarrow \mathsf{w}_1\partial_\mathsf{r}$, $E_0{}^2 \leftrightarrow \mathsf{w}_1\partial_\mathsf{s}$, $E_1{}^3 \leftrightarrow -\mathsf{w}_2\partial_\mathsf{s}$, $E_2{}^3 \leftrightarrow \mathsf{w}_2\partial_\mathsf{r}$, $E_0{}^3 \leftrightarrow 0$

To convert from $\kappa : \mathcal{G} \to \bigwedge^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ to $K \in \Omega^2(\mathcal{G}; \mathfrak{g})$, the Killing form on $\mathfrak{sl}_4$ induces $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$:

$$E_0{}^1 \leftrightarrow \omega^1{}_0 = \varpi_1, \quad E_0{}^2 \leftrightarrow \omega^2{}_0 = \varpi_2, \quad E_1{}^3 \leftrightarrow \omega^3{}_1 = \varpi_3,$$
$$E_2{}^3 \leftrightarrow \omega^3{}_2 = \varpi_4, \quad E_0{}^3 \leftrightarrow \omega^3{}_0 = \varpi_5.$$

Writing $\varpi_{kl} := \varpi_k \wedge \varpi_l$ for $1 \le k, l \le 5$, we have

$$
\begin{aligned}
K^1{}_1 &= -K^2{}_2 = -A_2\varpi_{23} + A_3(\varpi_{13} - \varpi_{24}) + A_4\varpi_{14} - B_2\varpi_{25} + B_3\varpi_{15} - B_6\varpi_{35} - B_7\varpi_{45}\\
K^1{}_2 &= +A_1\varpi_{23} - A_2(\varpi_{13} - \varpi_{24}) - A_3\varpi_{14} + B_1\varpi_{25} - B_2\varpi_{15} + B_5\varpi_{35} + B_6\varpi_{45}\\
K^2{}_1 &= -A_3\varpi_{23} + A_4(\varpi_{13} - \varpi_{24}) + A_5\varpi_{14} - B_3\varpi_{25} + B_4\varpi_{15} - B_7\varpi_{35} - B_8\varpi_{45}\\
K^0{}_1 &= -B_2\varpi_{23} + B_3(\varpi_{13} - \varpi_{24}) + B_4\varpi_{14} - C_2\varpi_{25} + C_3\varpi_{15} - C_5\varpi_{35} - C_6\varpi_{45}\\
K^0{}_2 &= +B_1\varpi_{23} - B_2(\varpi_{13} - \varpi_{24}) - B_3\varpi_{14} + C_1\varpi_{25} - C_2\varpi_{15} + C_4\varpi_{35} + C_5\varpi_{45}\\
K^1{}_3 &= -B_5\varpi_{23} + B_6(\varpi_{13} - \varpi_{24}) + B_7\varpi_{14} - C_4\varpi_{25} + C_5\varpi_{15} - C_7\varpi_{35} - C_8\varpi_{45}\\
K^2{}_3 &= -B_6\varpi_{23} + B_7(\varpi_{13} - \varpi_{24}) + B_8\varpi_{14} - C_5\varpi_{25} + C_6\varpi_{15} - C_8\varpi_{35} - C_9\varpi_{45}\\
K^0{}_3 &= -C_4\varpi_{23} + C_5(\varpi_{13} - \varpi_{24}) + C_6\varpi_{14} - D_1\varpi_{25} + D_2\varpi_{15} - D_3\varpi_{35} - D_4\varpi_{45}
\end{aligned}
$$

Recall from Sect. 2.3 that κ is P-equivariant. Let λ be the P-representation $\mathbb{K}$. Then

$$
R_p^*\kappa = \lambda(p^{-1}) \cdot \kappa \quad \Rightarrow \quad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} R^*_{\exp(\epsilon X)}\kappa = -\lambda'(X) \cdot \kappa.
$$

We let δ refer to the infinitesimal $\mathfrak{p}$-action. Given $X \in \mathfrak{p}$ as in (3.1), we obtain Table 4.

On $\mathcal{G}$, the curvature coefficients A, B, C, D will satisfy structure equations that also account for variation in the horizontal direction. These are immediately deduced from Table 4. For example,

$$
\mathrm{d}(A_1) = \delta(A_1) + \alpha_1 = (\zeta_1 + \zeta_2 + 4\nu_1)A_1 + 4\nu_2 A_2 + \alpha_1. \tag{3.5}
$$

Here, α_1 is a semi-basic form, i.e., it is a linear combination (with coefficients that are functions on $\mathcal{G}$) of the ϖ_i. We have abused notation in (3.5) by taking a slightly different meaning for $\delta(A_1)$: we have taken the corresponding formula in Table 4 and replaced Lie algebra parameters by their corresponding forms in the Cartan connection. This abuse is justified by axiom (CC.3) in the Cartan connection definition. Similarly, we will write

$$
\begin{aligned}
&\mathrm{d}(A_i) = \delta(A_i) + \alpha_i, \quad \mathrm{d}(B_i) = \delta(B_i) + \beta_i, \quad \mathrm{d}(C_i) = \delta(C_i) + \gamma_i,\\
&\mathrm{d}(D_i) = \delta(D_i) + \delta_i.
\end{aligned}
$$

(The repetition of δ in the last formula is slightly unfortunate, but should not cause much confusion.)

3.4 Duality

The pullback of the subbundles $E, V \subset TM$ via the projection $\pi : \mathcal{G} \to M$ are

$$
\pi^{-1}(E) = \{\varpi_3 = \varpi_4 = \varpi_5 = 0\}, \quad \pi^{-1}(V) = \{\varpi_1 = \varpi_2 = \varpi_5 = 0\}.
$$

Table 4 Vertical derivatives of curvature coefficients

$$
\begin{aligned}
-\delta A_1 &= (z_1+z_2+4v_1)A_1+4v_2A_2\\
-\delta A_2 &= v_3A_1+(z_1+z_2+2v_1)A_2+3v_2A_3\\
-\delta A_3 &= 2v_3A_2+(z_1+z_2)A_3+2v_2A_4\\
-\delta A_4 &= 3v_3A_3+(z_1+z_2-2v_1)A_4+v_2A_5\\
-\delta A_5 &= 4v_3A_4+(z_1+z_2-4v_1)A_5\\
-\delta B_1 &= (2z_1+z_2+3v_1)B_1+3v_2B_2+t_1A_1+t_2A_2\\
-\delta B_2 &= v_3B_1+(2z_1+z_2+v_1)B_2+2v_2B_3+t_1A_2+t_2A_3\\
-\delta B_3 &= 2v_3B_2+(2z_1+z_2-v_1)B_3+v_2B_4+t_1A_3+t_2A_4\\
-\delta B_4 &= 3v_3B_3+(2z_1+z_2-3v_1)B_4+t_1A_4+t_2A_5\\
-\delta B_5 &= (z_1+2z_2+3v_1)B_5+3v_2B_6+t_4A_1-t_3A_2\\
-\delta B_6 &= v_3B_5+(z_1+2z_2+v_1)B_6+2v_2B_7+t_4A_2-t_3A_3\\
-\delta B_7 &= 2v_3B_6+(z_1+2z_2-v_1)B_7+v_2B_8+t_4A_3-t_3A_4\\
-\delta B_8 &= 3v_3B_7+(z_1+2z_2-3v_1)B_8+t_4A_4-t_3A_5\\
-\delta C_1 &= (3z_1+z_2+2v_1)C_1+2v_2C_2+2t_1B_1+2t_2B_2\\
-\delta C_2 &= v_3C_1+(3z_1+z_2)C_2+v_2C_3+2t_1B_2+2t_2B_3\\
-\delta C_3 &= 2v_3C_2+(3z_1+z_2-2v_1)C_3+2t_1B_3+2t_2B_4\\
-\delta C_4 &= (2z_1+2z_2+2v_1)C_4+2v_2C_5+t_4B_1-t_3B_2+t_1B_5+t_2B_6\\
-\delta C_5 &= v_3C_4+(2z_1+2z_2)C_5+v_2C_6+t_4B_2-t_3B_3+t_1B_6+t_2B_7\\
-\delta C_6 &= 2v_3C_5+(2z_1+2z_2-2v_1)C_6+t_4B_3-t_3B_4+t_1B_7+t_2B_8\\
-\delta C_7 &= (z_1+3z_2+2v_1)C_7+2v_2C_8+2t_4B_5-2t_3B_6\\
-\delta C_8 &= v_3C_7+(z_1+3z_2)C_8+v_2C_9+2t_4B_6-2t_3B_7\\
-\delta C_9 &= 2v_3C_8+(z_1+3z_2-2v_1)C_9+2t_4B_7-2t_3B_8\\
-\delta D_1 &= (3z_1+2z_2+v_1)D_1+v_2D_2+t_4C_1-t_3C_2+2t_1C_4+2t_2C_5\\
-\delta D_2 &= v_3D_1+(3z_1+2z_2-v_1)D_2+t_4C_2-t_3C_3+2t_1C_5+2t_2C_6\\
-\delta D_3 &= (2z_1+3z_2+v_1)D_3+v_2D_4+2t_4C_4-2t_3C_5+t_1C_7+t_2C_8\\
-\delta D_4 &= v_3D_3+(2z_1+3z_2-v_1)D_4+2t_4C_5-2t_3C_6+t_1C_8+t_2C_9
\end{aligned}
$$

These are interchanged by the duality transformation, a representative of which is

$$
\begin{aligned}
\iota :&(\varpi_1,\varpi_2,\varpi_3,\varpi_4,\varpi_5,\zeta_1,\zeta_2,\nu_1,\nu_2,\nu_3,\tau_1,\tau_2,\tau_3,\tau_4,\tau_5)\\
&\mapsto (\varpi_3,\varpi_4,\varpi_1,\varpi_2,-\varpi_5,\zeta_2,\zeta_1,-\nu_1,-\nu_3,-\nu_2,\tau_3,\tau_4,\tau_1,\tau_2,-\tau_5),
\end{aligned}
$$

which induces

$$
\begin{aligned}
(A_1,A_2,A_3,A_4,A_5)&\mapsto (A_5,-A_4,A_3,-A_2,A_1)\\
(B_1,B_2,B_3,B_4,B_5,B_6,B_7,B_8)&\mapsto (-B_8,B_7,-B_6,B_5,B_4,-B_3,B_2,-B_1)\\
(C_1,C_2,C_3,C_4,C_5,C_6,C_7,C_8,C_9)&\mapsto (C_9,-C_8,C_7,-C_6,C_5,-C_4,C_3,-C_2,C_1)\\
(D_1,D_2,D_3,D_4)&\mapsto (D_4,-D_3,-D_2,D_1)
\end{aligned}
$$

In particular, the induced action on the quartic is realizable by a G_0-transformation, namely that induced by $\rho : (\mathsf{r},\mathsf{s}) \mapsto (\mathsf{s},-\mathsf{r})$. Since any G_0-transformation preserves root type, this proves:

Proposition 3.2 *The duality transformation preserves root type.*

However, ρ induces the following transformation on B, C, D coefficients:

$$\begin{gathered}(B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8) \mapsto (-B_4, B_3, -B_2, B_1, -B_8, B_7, -B_6, B_5)\\ (C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9) \mapsto (C_3, -C_2, C_1, C_6, -C_5, C_4, C_9, -C_8, C_7)\\ (D_1, D_2, D_3, D_4) \mapsto (-D_2, D_1, -D_4, D_3)\end{gathered}$$

Note that the composition $\rho \circ \iota$ preserves A and induces

$$\begin{gathered}(B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8) \mapsto (-B_5, -B_6, -B_7, -B_8, B_1, B_2, B_3, B_4)\\ (C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9) \mapsto (C_7, C_8, C_9, -C_4, -C_5, -C_6, C_1, C_2, C_3)\\ (D_1, D_2, D_3, D_4) \mapsto (D_3, D_4, -D_1, -D_2)\end{gathered}$$

4 Cartan Analysis

Starting with the (regular, normal) Cartan geometry $(\mathcal{G} \to M, \omega)$ which is an equivalent description of any ILC structure, the goal is to classify all homogeneous subbundles of total dimension at least six that are obtained via natural reductions of the structure group P. We give an outline of how this is achieved in the type N case. The analysis for types D and III are similar, so we only provide a few details on how the analysis is begun in these cases. Types II and I do not contain any multiply transitive structures. The reader interested in the full details of the Cartan analysis is encouraged to examine the Maple files that accompany the arXiv submission of this paper.

4.1 Type N Reduction

Using the P-action (G_0-action), we can always normalize $A = \mathsf{s}^4$, i.e., $A_5 = 1,\ A_4 = A_3 = A_2 = A_1 = 0$.[3] Now $0 = d(A_i)$ are equivalent to:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0, \quad \nu_2 = \alpha_4, \quad \nu_1 = \frac{1}{4}(\zeta_1 + \zeta_2 - \alpha_5). \tag{4.1}$$

Differentiating the ν_1-relation in (4.1) yields the vertical action on coefficients in $\alpha_5 = a_{5j}\varpi_j$. (More precisely, we calculate $0 = \mathrm{d}(\nu_1 - \frac{1}{4}(\zeta_1+\zeta_2-\alpha_5)) \wedge \varpi_i \wedge \varpi_j \wedge \varpi_k \wedge \varpi_\ell$, where $1 \le i, j, k, l \le 5$.)

[3] This normalization is always possible working over $\mathbb{C}$, but over $\mathbb{R}$ we would have two possibilities: $A = \pm\mathsf{s}^4$.

$$\delta a_{51} = \left(\frac{z_2 - 3z_1}{4}\right) a_{51} + (a_{52} - 4a_{41})v_3 - 3t_1,$$
$$\delta a_{52} = -\left(\frac{5z_1 + z_2}{4}\right) a_{52} - 4a_{42}v_3 + t_2,$$
$$\delta a_{54} = \left(\frac{z_1 - 3z_2}{4}\right) a_{54} - (a_{53} + 4a_{44})v_3 - 3t_4,$$
$$\delta a_{53} = -\left(\frac{z_1 + 5z_2}{4}\right) a_{53} - 4a_{43}v_3 + t_3,$$
$$\delta a_{55} = -a_{55}(z_1 + z_2) - 4a_{45}v_3 - a_{53}t_1 - a_{54}t_2 + a_{51}t_3 + a_{52}t_4 - 2t_5.$$

The t_j induce translations on a_{5j}, so we can always normalize $\alpha_5 = 0$. This forces $t_i = \lambda_i v_3$, where $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-\frac{4}{3}a_{41},\ 4a_{42},\ 4a_{43},\ -\frac{4}{3}a_{44},\ -2a_{45})$. Hence, there exists functions T_{ij} such that

$$\tau_i = \lambda_i v_3 + \sum_j T_{ij}\varpi_j. \tag{4.2}$$

We have reduced to a 3-D structure algebra (with parameters v_3, z_1, z_2). We will show that:

Theorem 4.1 *Any multiply transitive type N structure with the normalizations* $A = \mathsf{s}^4$ *and* $\alpha_5 = 0$ *satisfies* $B = C = D = 0$.

The integrability conditions $0 = d^2 v_i \wedge \varpi_5$ $(i = 1, 2, 3)$ force

$$B_1 = B_2 = B_5 = B_6 = 0, \quad a_{41} = -2B_4, \quad a_{42} = -2B_3,$$
$$a_{43} = +2B_7, \quad a_{44} = -2B_8,$$

and $0 = \mathrm{d}(B_1) = \mathrm{d}(B_2) = \mathrm{d}(B_5) = \mathrm{d}(B_6)$ are equivalent to:

$$\beta_1 = \beta_5 = 0, \quad \beta_2 = 2B_3\alpha_4, \quad \beta_6 = 2B_7\alpha_4.$$

Moreover, B_3 and B_7 are relative invariants:

$$\delta B_3 = -\left(\frac{7z_1 + 3z_2}{4}\right) B_3, \qquad \delta B_4 = 5B_3 v_3 - \left(\frac{5z_1 + z_2}{4}\right) B_4,$$
$$\delta B_7 = -\left(\frac{3z_1 + 7z_2}{4}\right) B_7, \qquad \delta B_8 = 5B_7 v_3 - \left(\frac{z_1 + 5z_2}{4}\right) B_8.$$

If $B_3 B_7$ is nowhere vanishing, we can normalize $(B_3, B_4, B_7) = (1, 0, 1)$. This trivializes the structure algebra and so such structures admit at most five symmetries (henceforth excluded since these are not multiply transitive). We have the following

trichotomy:[4]

Condition	Bound on symmetry dimension
$B_3 = B_7 = 0$	8
$(B_3, B_7) \neq (0,0),\ B_3 B_7 = 0$	6
$B_3 B_7 \neq 0$	5

Lemma 4.2 *No structures with 6 symmetries exist when* $(B_3, B_7) \neq (0,0),\ B_3 B_7 = 0$.

Proof By duality, take $B_3 \neq 0$ and $B_7 = 0$. Normalizing $(B_3, B_4) = (1, 0)$ forces $z_2 = -\frac{7}{3} z_1$, $v_3 = 0$. The structure algebra is reduced to $\mathrm{diag}(\frac{z_1}{6}, -\frac{7z_1}{6}, -\frac{z_1}{2}, \frac{3z_1}{2})$, and this acts trivially vertically under the 6 symmetry assumption. Hence, $B_8 = 0$. From $0 = \mathrm{d}(B_3) = \mathrm{d}(B_4) = \mathrm{d}(B_7) = \mathrm{d}(B_8)$,

$$\zeta_2 = -\frac{7}{3}\zeta_1 + \frac{4}{3}\beta_3, \qquad \nu_3 = \frac{1}{5}(\tau_2 - \beta_4), \qquad \beta_7 = 0, \qquad \beta_8 = -\tau_3.$$

All coefficients with nonzero (vertical) scaling weight with respect to z_1 must vanish. Differentiating the relations on $\zeta_2, \nu_1, \nu_2, \nu_3, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5$, we conclude from these weights that

$$\beta_3 = \beta_4 = \beta_8 = \tau_2 = \tau_3 = \tau_4 = \nu_3 = 0, \quad a_{45} = 0, \quad T_{ij} = 0,$$
$$(i, j) \notin \{(1,5), (5,1)\}$$

But differentiating $\nu_3 = 0$ then yields the contradiction $0 = \varpi_1 \wedge \varpi_4 - (T_{15} + 1)\varpi_2 \wedge \varpi_5$. □

Thus, $B_3 = B_7 = 0$ for multiply transitive structures. Now $0 = d(B_3) = d(B_7)$ implies:

$$\beta_3 = B_4 \alpha_4, \quad \beta_7 = B_8 \alpha_4.$$

Moreover, B_4, B_8 are relative invariants.

Case	Condition	Bound on symmetry dimension
(a)	$B_4 = B_8 = 0$	8
(b)	$(B_4, B_8) \neq (0,0),\ B_4 B_8 = 0$	7
(c)	$B_4 B_8 \neq 0$	6

Lemma 4.3 *Any multiply transitive type N structure with normalization* $A = \mathsf{s}^4$ *satisfies* $B = 0$.

[4] Implicitly, this trichotomy depends on B_3 and B_7 have locally constant type, i.e., the stated invariant conditions are true locally. For (multiply) transitive structures, this is always true.

Proof Suppose $B_4B_8 \neq 0$. Normalizing $B_4 = B_8 = 1$ forces $z_1 = z_2 = 0$. Hence, $\zeta_i = Z_{ij}\varpi_j$. For multiply transitive structures, Z_{ij} are constant, $\varpi_1, \ldots, \varpi_5, \upsilon_3$ linearly independent, and υ_3 must act vertically trivially. This forces $C_i = 0\,(i \neq 3, 6, 9)$, $D_1 = D_3 = 0$,

$$T_{23} = T_{32} = -\frac{16}{9}, \quad T_{22} = T_{33} = -\frac{16}{3}, \quad Z_{12} = -Z_{23} = 4,$$
$$Z_{13} = -Z_{22} = \frac{4}{3}, \quad a_{45} = -\frac{32}{9},$$

and several more linear relations between T_{ij} and Z_{ij}. Since all B, C, D coefficients must be constant, apply d to get further relations. Imposing Bianchi identities yields $C_3 = C_9 = \frac{8}{3}$ and $C_6 = \frac{4}{9}$. A contradiction is then obtained from $0 = d^2\tau_1 \wedge \varpi_3 \wedge \varpi_4 \neq 0$.

The case $B_4 \neq 0$, $B_8 = 0$ is more involved, but similarly yields a contradiction. □

Given $B = 0$, the conditions $0 = d^2\upsilon_i$ imply $C_j = 0$ for $j \neq 3, 6, 9$, and

$$a_{45} = C_6, \quad b_{42} = -C_3, \quad b_{43} = 2C_6, \quad b_{82} = -2C_6, \quad b_{83} = C_9, \quad b_{81} = -b_{44}.$$

Now imposing $0 = d(B_4) = d(B_8)$ and $0 = d(\upsilon_2 - \alpha_4)$, we obtain $C_3 = C_6 = C_9 = 0$, so $C = 0$, and

$$T_{2i} = 0\ (i \neq 1), \quad T_{3j} = 0\ (j \neq 4), \quad T_{21} = b_{41}, \quad T_{34} = -b_{84},$$
$$b_{44} = b_{45} = b_{85} = 0.$$

For $i = 1, \ldots, 9$, $0 = d(C_i)$ implies $\gamma_i = 0$. Then $0 = d^2\tau_1 = d^2\tau_2$ implies $D = 0$. For $i = 1, \ldots, 4$, $0 = d(D_i)$ implies $\delta_i = 0$. Now $0 = d(\upsilon_1 - \frac{1}{4}(\zeta_1 + \zeta_2))$ implies relations among the T_{ij}. We obtain:

$$\tau_2 = T_{21}\varpi_1, \quad \tau_3 = -3T_{43}\varpi_4, \quad \tau_5 = T_{51}\varpi_1 + T_{54}\varpi_4 + T_{55}\varpi_5,$$
$$\tau_1 = T_{11}\varpi_1 - \frac{T_{21}}{3}\varpi_2 + T_{41}\varpi_4 + \frac{2}{3}T_{51}\varpi_5,$$
$$\tau_4 = T_{41}\varpi_1 + T_{43}\varpi_3 + T_{44}\varpi_4 + \frac{2}{3}T_{54}\varpi_5,$$

and all other T_{ij} not appearing above are zero. Differentiating the τ_i-relations ($i = 1, \ldots, 4$), we obtain the vertical action:

$$\delta T_{21} = -2T_{21}z_1, \quad \delta T_{11} = \left(\frac{-3z_1 + z_2}{2}\right)T_{11} + \frac{2}{3}T_{21}\upsilon_3$$
$$\delta T_{43} = -2T_{43}z_2, \quad \delta T_{44} = \left(\frac{z_1 - 3z_2}{2}\right)T_{44} + 2T_{43}\upsilon_3$$

Lemma 4.4 *With normalizations as above, we must have* $T_{21} = T_{43} = T_{51} = T_{54} = T_{55} = 0$.

Proof If $T_{21}T_{43} \neq 0$, then there are at most 5 symmetries. If $T_{21} \neq 0$ and $T_{43} = 0$, normalize $T_{21} = 1$ and $T_{11} = 0$, and write $\zeta_1 = Z_{1j}\varpi_j$, $\nu_3 = V_{3j}\varpi_j$. We have at most 6 symmetries, and for 6 the residual structure algebra (generated by z_2) must act vertically trivially. This forces $\zeta_1 = \nu_3 = 0$ and $T_{41} = T_{44} = T_{51} = T_{54} = T_{55} = 0$. But $0 = d\zeta_1 = -2\varpi_1 \wedge \varpi_2 \neq 0$ yields a contradiction. The $T_{43} \neq 0$, $T_{21} = 0$ case similarly yields a contradiction. Thus, we conclude that $T_{21} = T_{43} = 0$ and hence $\tau_2 = \tau_3 = 0$. From $0 = d\tau_2 = d\tau_3$, we obtain $T_{51} = T_{54} = T_{55} = 0$. □

Summary: For multiply transitive type N structures, we have reduced to an 8-dimensional subbundle of the original Cartan bundle (given the normalizations $A = \mathsf{s}^4$ and $\alpha_5 = 0$).

- *Curvature coefficients* All A, B, C, D are zero, except $A_5 = 1$.
- *Coframe* $\varpi_1, \ldots, \varpi_5, \zeta_1, \zeta_2, \nu_3$. Relations on other forms:

$$\nu_1 = \frac{1}{4}(\zeta_1 + \zeta_2), \quad \nu_2 = \tau_2 = \tau_3 = \tau_5 = 0, \quad \tau_1 = T_{11}\varpi_1 + T_{41}\varpi_4,$$
$$\tau_4 = T_{41}\varpi_1 + T_{44}\varpi_4.$$

- Among $\alpha_i, \beta_j, \gamma_k, \delta_\ell$, the only possibly non-trivial forms are $\beta_4 = \tau_2$ and $\beta_8 = -\tau_3$.
- All Bianchi identities are satisfied, e.g., $0 = d^2\nu_1$, etc.
- *Structure group* $\begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & s & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix}$, where $r_1r_2r_3r_4 = 1$, $r_1{r_3}^2 = {r_2}^2r_4$, i.e., $r_1 = \frac{1}{{r_2}^4{r_4}^3}$, $r_3 = {r_2}^3{r_4}^2$. This induces $\begin{cases} \tilde{T}_{11} = \frac{{r_2}^2}{{r_1}^2}T_{11} = {r_2}^{10}{r_4}^6T_{11} \\ \tilde{T}_{41} = \frac{r_2r_4}{r_1r_3}T_{41} = {r_2}^2{r_4}^2T_{41} \\ \tilde{T}_{44} = \frac{{r_4}^2}{{r_3}^2}T_{44} = \frac{1}{{r_2}^6{r_4}^2}T_{44} \end{cases} \Rightarrow$
$\begin{cases} \delta(T_{11}) = \left(\frac{z_2-3z_1}{2}\right)T_{11} \\ \delta(T_{41}) = -\left(\frac{z_1+z_2}{2}\right)T_{41} \\ \delta(T_{44}) = \left(\frac{z_1-3z_2}{2}\right)T_{44} \end{cases}$.
- Let Φ be the duality transformation:

$$\varpi_1 \leftrightarrow \varpi_4, \quad \varpi_2 \leftrightarrow \varpi_3, \quad \varpi_5 \mapsto -\varpi_5, \quad \tau_1 \leftrightarrow \tau_4, \quad \tau_2 \leftrightarrow \tau_3, \quad \tau_5 \mapsto -\tau_5,$$
$$\zeta_1 \leftrightarrow \zeta_2, \quad \nu_1 \text{ fixed}, \quad \nu_2 \mapsto -\nu_2, \quad \nu_3 \mapsto -\nu_3.$$

  This preserves A and induces $(\tau_1, \tau_4) \mapsto (\tau_4, \tau_1)$ and so $(T_{11}, T_{41}, T_{44}) \mapsto (T_{44}, T_{41}, T_{11})$.

The case analysis based on the relative invariants T_{11}, T_{41}, T_{44} is straightforward. Table 5 summarizes this classification and Table 6 contains the structure equations obtained.

Some care is required to deduce any redundancy of parameters appearing in the structure equations. Consider the case $T_{11} \neq 0$. Normalize $T_{11} = 1$, so ${r_4}^6 = \frac{1}{{r_2}^{10}}$,

Table 5 Classification of multiply transitive type N structures

	$T_{11} = T_{44} = 0$	$T_{11} \neq 0, T_{44} = 0$	$T_{11}T_{44} \neq 0$
$T_{41} = 0$	N.8	N.7-1	N.6-2
$T_{41} \neq 0$	N.7-2	None	N.6-1

and $\zeta_2 = 3\zeta_1 + Z_{21}\varpi_1 + Z_{24}\varpi_4$ is forced. Quotienting the structure group by $\mathrm{diag}(r_2, r_2, r_2, r_2)$ (since these act trivially), we may WLOG take the diagonal to be $\mathrm{diag}\left(\frac{1}{{r_2}^5{r_4}^3}, 1, {r_2}^2{r_4}^2, \frac{r_4}{r_2}\right)$. Let Q be the residual group below.

(1) N.6-1: $T_{41} = 1$ and $\zeta_1 = Z_{11}\varpi_1 + Z_{14}\varpi_4$. Then $Q \cong \mathbb{Z}_2$, generated by $E = \mathrm{diag}(-1, 1, 1, -1)$.

$$\tilde{T}_{44} = T_{44}, \quad \tilde{Z}_{j1} = -Z_{j1}, \quad \tilde{Z}_{j4} = -Z_{j4} \qquad (j = 1, 2).$$

Let $Z_{24} = 4a$, so $\pm a$ yield the same structure. We must $a(a^2 + 1) \neq 0$ and

$$Z_{11} = \frac{1 - 2a^2}{a}, \quad Z_{14} = \frac{-a(2a^2 + 1)}{a^2 + 1}, \quad T_{44} = \frac{a^4}{(a^2 + 1)^2}, \quad Z_{21} = \frac{4(a^2 - 1)}{a}.$$

Thus, $a^2 \in \mathbb{C}\backslash\{0, -1\}$ is the essential parameter.

(2) N.6-2: $T_{44} = 1$ and $\zeta_1 = -\frac{3}{8}Z_{21}\varpi_1 + Z_{14}\varpi_4$. Write $Z_{21} = -4a$ and $Z_{14} = \frac{b}{2}$. Here, $Q \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $M_1 = \mathrm{diag}(-1, 1, 1, -1)$ and $M_2 = \mathrm{diag}(1, 1, -1, 1)$. Then:

$$M_1: \quad (\tilde{a}, \tilde{b}) = (-a, -b); \qquad M_2: \quad (\tilde{a}, \tilde{b}) = (a, -b).$$

Thus, $(a^2, b^2) \in \mathbb{C} \times \mathbb{C}$ is the essential parameter.

(3) N.7-1: $T_{41} = T_{44} = 0$, $\zeta_2 = 4a\varpi_1 + 3\zeta_1$. Here, Q has diagonal $\mathrm{diag}\left(\frac{1}{{r_2}^5{r_4}^3}, 1, {r_2}^2{r_4}^2, \frac{r_4}{r_2}\right)$, with $\epsilon := {r_2}^5{r_4}^3 = \pm 1$. Induced action: $\tilde{a} = \epsilon a$. Thus, $a^2 \in \mathbb{C}$ is the essential parameter.

All type N structure equations are given in Table 6.

4.2 Type *D* Reduction

Normalize $A = 6\mathsf{r}^2\mathsf{s}^2$, i.e., $A_3 = 1, A_5 = A_4 = A_2 = A_1 = 0$. Now $0 = d(A_i)$ implies:

$$\nu_2 = \frac{1}{3}\alpha_2, \quad \nu_3 = \frac{1}{3}\alpha_4, \quad \zeta_2 = -\zeta_1 + \alpha_3, \quad \alpha_1 = \alpha_5 = 0. \tag{4.3}$$

Differentiating the ζ_2-relation above yields the vertical action on coefficients in $\alpha_3 = a_{3j}\varpi_j$:

Table 6 Multiply transitive type N structure equations

Model	SD	Structure equations	Embedding into Cartan bundle
N.8	✓	$\mathrm{d}\varpi_1 = \frac{3}{4}\zeta_1 \wedge \varpi_1 - \frac{1}{4}\zeta_2 \wedge \varpi_1$ $\mathrm{d}\varpi_2 = \frac{5}{4}\zeta_1 \wedge \varpi_2 + \frac{1}{4}\zeta_2 \wedge \varpi_2 - \nu_3 \wedge \varpi_1$ $\mathrm{d}\varpi_3 = \frac{1}{4}\zeta_1 \wedge \varpi_3 + \frac{5}{4}\zeta_2 \wedge \varpi_3 + \nu_3 \wedge \varpi_4$ $\mathrm{d}\varpi_4 = -\frac{1}{4}\zeta_1 \wedge \varpi_4 + \frac{3}{4}\zeta_2 \wedge \varpi_4$ $\mathrm{d}\varpi_5 = \zeta_1 \wedge \varpi_5 + \zeta_2 \wedge \varpi_5 + \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4$ $\mathrm{d}\nu_3 = \frac{1}{2}\zeta_1 \wedge \nu_3 + \frac{1}{2}\zeta_2 \wedge \nu_3 + \varpi_1 \wedge \varpi_4$ $\mathrm{d}\zeta_1 = 0$ $\mathrm{d}\zeta_2 = 0$	$\nu_1 = \frac{1}{4}(\zeta_1 + \zeta_2)$ $\nu_2 = 0$ $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = 0$
N.7-1	×	$d\varpi_1 = 0$ $\mathrm{d}\varpi_2 = 2\zeta_1 \wedge \varpi_2 - \nu_3 \wedge \varpi_1 + a\,\varpi_1 \wedge \varpi_2$ $\mathrm{d}\varpi_3 = 4\zeta_1 \wedge \varpi_3 + \nu_3 \wedge \varpi_4 + \varpi_1 \wedge \varpi_5 + 5a\,\varpi_1 \wedge \varpi_3$ $\mathrm{d}\varpi_4 = 2\zeta_1 \wedge \varpi_4 + 3a\,\varpi_1 \wedge \varpi_4$ $\mathrm{d}\varpi_5 = 4\zeta_1 \wedge \varpi_5 + \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4 + 4a\,\varpi_1 \wedge \varpi_5$ $\mathrm{d}\nu_3 = 2\zeta_1 \wedge \nu_3 - 2a\,\nu_3 \wedge \varpi_1 + \varpi_1 \wedge \varpi_2 + \varpi_1 \wedge \varpi_4$ $\mathrm{d}\zeta_1 = 0$	$\zeta_2 = 3\zeta_1 + 4a\varpi_1$ $\nu_1 = \zeta_1 + a\varpi_1$ $\tau_1 = \varpi_1$ $\nu_2 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = 0$ (Parameter: $a^2 \in \mathbb{C}$)
N.7-2	✓	$d\varpi_1 = \zeta_1 \wedge \varpi_1$ $\mathrm{d}\varpi_2 = \zeta_1 \wedge \varpi_2 - \nu_3 \wedge \varpi_1 - \varpi_1 \wedge \varpi_5$ $\mathrm{d}\varpi_3 = -\zeta_1 \wedge \varpi_3 + \nu_3 \wedge \varpi_4 + \varpi_4 \wedge \varpi_5$ $\mathrm{d}\varpi_4 = -\zeta_1 \wedge \varpi_4$ $\mathrm{d}\varpi_5 = \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4$ $\mathrm{d}\nu_3 = -\varpi_2 \wedge \varpi_4 - \varpi_1 \wedge \varpi_3 + \varpi_1 \wedge \varpi_4$ $\mathrm{d}\zeta_1 = 2\varpi_1 \wedge \varpi_4$	$\zeta_2 = -\zeta_1$ $\tau_1 = \varpi_4$ $\tau_4 = \varpi_1$ $\nu_1 = \nu_2 = \tau_2 = \tau_3 = \tau_5 = 0$
N.6-1	✓	$\mathrm{d}\varpi_1 = a\varpi_1 \wedge \varpi_4$ $\mathrm{d}\varpi_2 = -\nu_3 \wedge \varpi_1 - \frac{(3a^2-1)}{a}\varpi_1 \wedge \varpi_2 - \varpi_1 \wedge \varpi_5 + \frac{a(3a^2+1)}{(a^2+1)}\varpi_2 \wedge \varpi_4 - \frac{a^4}{(a^2+1)^2}\varpi_4 \wedge \varpi_5$ $\mathrm{d}\varpi_3 = \nu_3 \wedge \varpi_4 - \frac{(3a^2+1)}{a}\varpi_1 \wedge \varpi_3 + \varpi_1 \wedge \varpi_5 + \frac{a(3a^2-1)}{(a^2+1)}\varpi_3 \wedge \varpi_4 + \varpi_4 \wedge \varpi_5$ $\mathrm{d}\varpi_4 = -\frac{(a^2+1)}{a}\varpi_1 \wedge \varpi_4$ $d\varpi_5 = \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4 - 4a\left(\varpi_1 + \frac{a^2}{(a^2+1)}\varpi_4\right) \wedge \varpi_5$ $\mathrm{d}\nu_3 = 2a\nu_3 \wedge \left(\varpi_1 + \frac{a^2}{(a^2+1)}\varpi_4\right) + \varpi_1 \wedge \varpi_2 - \varpi_1 \wedge \varpi_3 + \varpi_1 \wedge \varpi_4 - \varpi_2 \wedge \varpi_4 + \frac{a^4}{(a^2+1)^2}\varpi_3 \wedge \varpi_4$	$\zeta_1 = -\frac{(2a^2-1)}{a}\varpi_1 - \frac{a(2a^2+1)}{(a^2+1)}\varpi_4$ $\zeta_2 = -\frac{(2a^2+1)}{a}\varpi_1 - \frac{a(2a^2-1)}{(a^2+1)}\varpi_4$ $\nu_1 = -a\left(\varpi_1 + \frac{a^2}{(a^2+1)}\varpi_4\right)$ $\tau_1 = \varpi_1 + \varpi_4$ $\tau_4 = \varpi_1 + \frac{a^4}{(a^2+1)^2}\varpi_4$ $\nu_2 = \tau_2 = \tau_3 = \tau_5 = 0$ (Parameter: $a^2 \in \mathbb{C}\backslash\{0, -1\}$)

Table 6 continued

Model	SD	Structure equations	Embedding into Cartan bundle
N.6-2	$\checkmark: a^2 = b^2$	$\mathrm{d}\varpi_1 = 0$	$\zeta_1 = \frac{3a}{2}\varpi_1 + \frac{b}{2}\varpi_4$
	$\times: a^2 \neq b^2$	$\mathrm{d}\varpi_2 = -\nu_3 \wedge \varpi_1 + 2a\varpi_1 \wedge \varpi_2 - b\varpi_2 \wedge \varpi_4 - \varpi_4 \wedge \varpi_5$	$\zeta_2 = \frac{a}{2}\varpi_1 + \frac{3b}{2}\varpi_4$
		$\mathrm{d}\varpi_3 = +\nu_3 \wedge \varpi_4 + a\varpi_1 \wedge \varpi_3 - 2b\varpi_3 \wedge \varpi_4 + \varpi_1 \wedge \varpi_5$	$\nu_1 = \frac{a}{2}\varpi_1 + \frac{b}{2}\varpi_4$
		$\mathrm{d}\varpi_4 = 0$	$\tau_1 = \varpi_1,\ \tau_4 = \varpi_4$
		$\mathrm{d}\varpi_5 = \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4 + 2a\varpi_1 \wedge \varpi_5 + 2b\varpi_4 \wedge \varpi_5$	$\nu_2 = \tau_2 = \tau_3 = \tau_5 = 0$
		$\mathrm{d}\nu_3 = -\nu_3 \wedge (a\varpi_1 + b\varpi_4) + \varpi_1 \wedge \varpi_2 + \varpi_3 \wedge \varpi_4 + \varpi_1 \wedge \varpi_4$	(Parameter: $(a^2, b^2) \in \mathbb{C} \times \mathbb{C}$)

$$\delta a_{31} = a_{31}(v_1 - z_1) - t_1, \quad \delta a_{32} = -a_{32}(v_1 + z_1) - t_2,$$
$$\delta a_{33} = a_{33}(-v_1 + z_1) - t_3, \delta a_{34} = a_{34}(v_1 + z_1) - t_4,$$
$$\delta a_{35} = a_{32}t_4 + a_{31}t_3 - a_{34}t_2 - a_{33}t_1 - 2t_5.$$

Normalize $\alpha_3 = 0$, so $\tau_i = T_{ij}\varpi_j$. We have reduced to the 2-dimensional structure algebra $\mathrm{diag}\left(\frac{z_1}{2}, v_1 - \frac{z_1}{2}, -v_1 - \frac{z_1}{2}, \frac{z_1}{2}\right)$, so all type D structures admit at most seven symmetries. Using duality and the G_0-map $(\mathsf{r}, \mathsf{s}) \mapsto (\mathsf{s}, -\mathsf{r})$, we can assume that B_1 or B_2 is nonzero, or $B = 0$.

For the 7-symmetry case, the 2-dimensional structure algebra must act trivially. This forces:

- only C_5 (necessarily constant) to survive among B, C, D coefficients;
- $\alpha_2 = \alpha_4 = 0$ (so $\nu_2 = \nu_3 = 0$);
- all T_{ij} to vanish except $T_{13}, T_{24}, T_{31}, T_{42}, T_{55}$ (necessarily constants).

From $0 = \mathrm{d}\nu_2 = \mathrm{d}(\zeta_1 + \zeta_2) = \mathrm{d}(\tau_1 - T_{13}\varpi_3) = \mathrm{d}(\tau_2 - T_{24}\varpi_4)$, we obtain

$$T_{31} = T_{13} = C_5 - \frac{1}{2}, \quad T_{42} = T_{24} = -C_5 - \frac{1}{2}, \quad T_{55} = (C_5)^2 + \frac{1}{4}.$$

Now, $0 = \mathrm{d}(B_i) = \mathrm{d}(C_j) = \mathrm{d}(D_k)$ forces $\beta_i = 0$ $(i = 1, 4, 5, 8)$, and $\gamma_j = 0$, and

$$(\beta_2, \beta_3, \beta_6, \beta_7) = (\tau_2, \tau_1, -\tau_3, \tau_4), \quad (\delta_1, \delta_2, \delta_3, \delta_4) = 2C_5(\beta_2, \beta_3, \beta_6, \beta_7).$$

This yields model D.7. The 2-dimensional structure group is generated by $\mathrm{diag}(r_1, \frac{r_2}{r_1}, \frac{1}{r_1 r_2}, r_1)$ $(r_1, r_2 \in \mathbb{C}^\times)$, along with $\mathrm{diag}(e^{i\pi/4}, e^{i\pi/4}, -e^{i\pi/4}, e^{i\pi/4})$ and $\mathrm{diag}\left(1, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 1\right)$.[5] Only this last transformation acts non-trivially on C_5, i.e., $C_5 \mapsto -C_5$. Thus, $(C_5)^2 \in \mathbb{C}$ is the essential parameter.

The 6-symmetry case proceeds similarly, but is very tedious, particularly for the $B_2 \neq 0$ case that leads to model D.6-2. All (multiply transitive) type D structure equations are given in Table 7. As mentioned earlier, full details of the Cartan analysis are given in the Maple files accompanying the arXiv submission of this article.

[5] The latter two correspond to $(\mathsf{r}, \mathsf{s}) \mapsto (\mathsf{r}, -\mathsf{s})$ and $(\mathsf{r}, \mathsf{s}) \mapsto (-\mathsf{s}, \mathsf{r})$.

Table 7 Multiply transitive type D structure equations

Model	SD	Structure equations	Embedding into Cartan bundle
D.7 ($a^2 \in \mathbb{C}$)	✓	$\mathrm{d}\varpi_1 = (\zeta_1 - \nu_1 + (a - \frac{1}{2})\varpi_5) \wedge \varpi_1$ $\mathrm{d}\varpi_2 = (\zeta_1 + \nu_1 - (a + \frac{1}{2})\varpi_5) \wedge \varpi_2$ $\mathrm{d}\varpi_3 = (-\zeta_1 + \nu_1 - (a - \frac{1}{2})\varpi_5) \wedge \varpi_3$ $\mathrm{d}\varpi_4 = (-\zeta_1 - \nu_1 + (a + \frac{1}{2})\varpi_5) \wedge \varpi_4$ $\mathrm{d}\varpi_5 = \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4$ $\mathrm{d}\zeta_1 = (2a - 1)\varpi_1 \wedge \varpi_3 - (2a + 1)\varpi_2 \wedge \varpi_4$ $\mathrm{d}\nu_1 = (-a + \frac{3}{2})\varpi_1 \wedge \varpi_3 - (a + \frac{3}{2})\varpi_2 \wedge \varpi_4$	$\zeta_2 = -\zeta_1,\ \nu_2 = \nu_3 = 0$ $\tau_1 = (a - \frac{1}{2})\varpi_3$ $\tau_2 = -(a + \frac{1}{2})\varpi_4$ $\tau_3 = (a - \frac{1}{2})\varpi_1$ $\tau_4 = -(a + \frac{1}{2})\varpi_2$ $\tau_5 = (a^2 + \frac{1}{4})\varpi_5$
D.6-1	✓	$d\varpi_1 = (3\varpi_5 - 4\nu_1) \wedge \varpi_1$ $\mathrm{d}\varpi_2 = (\frac{3}{2}\varpi_5 - 2\nu_1) \wedge \varpi_2 + \sqrt{2}\,\varpi_1 \wedge \varpi_4$ $\mathrm{d}\varpi_3 = (4\nu_1 - 3\varpi_5) \wedge \varpi_3$ $\mathrm{d}\varpi_4 = (2\nu_1 - \frac{3}{2}\varpi_5) \wedge \varpi_4 + \sqrt{2}\,\varpi_3 \wedge \varpi_2$ $\mathrm{d}\varpi_5 = \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4$ $\mathrm{d}\nu_1 = -\frac{1}{4}\varpi_1 \wedge \varpi_3 + \frac{3}{4}\varpi_2 \wedge \varpi_4$	$\nu_2 = -\sqrt{2}\varpi_2,\ \nu_3 = \sqrt{2}\varpi_4$ $\zeta_2 = -\zeta_1 = -\frac{7}{4}\varpi_5 + 3\nu_1$ $\tau_1 = \frac{5}{4}\varpi_3,\ \tau_2 = -\frac{1}{4}\varpi_4$ $\tau_3 = \frac{5}{4}\varpi_1,\ \tau_4 = -\frac{1}{4}\varpi_2$ $\tau_5 = \frac{13}{16}\varpi_5$
D.6-2 ($a \neq 1, \frac{2}{3}$)	✓	$d\varpi_1 = -2\nu_1 \wedge \varpi_1 + (-a + \frac{2}{3})\varpi_1 \wedge \varpi_2 - \frac{9a}{(3a-2)(a-1)}\varpi_1 \wedge \varpi_4 - \frac{(3a+5)}{2(a-1)}\varpi_1 \wedge \varpi_5$ $\mathrm{d}\varpi_2 = -\frac{6}{(3a-2)(a-1)}\varpi_1 \wedge \varpi_3 - \frac{9a}{(3a-2)(a-1)}\varpi_2 \wedge \varpi_4 - \frac{2}{a-1}\varpi_2 \wedge \varpi_5 + \frac{18}{(3a-2)(a-1)^2}\varpi_4 \wedge \varpi_5$	$\zeta_2 = -\zeta_1 = \nu_1 - a\varpi_2 - \frac{9a}{(a-1)(3a-2)}\varpi_4 - \frac{15a^2 + 23a - 14}{4(a-1)(3a-2)}\varpi_5$ $\nu_2 = -\frac{2}{3}\varpi_1$

Table 7 continued

Model	SD	Structure equations	Embedding into Cartan bundle
		$\mathrm{d}\varpi_3 = 2\nu_1 \wedge \varpi_3 - a\varpi_2 \wedge \varpi_3 + \frac{3}{a-1}\varpi_3 \wedge \varpi_4 + \frac{3a+5}{2(a-1)}\varpi_3 \wedge \varpi_5$	$\nu_3 = -\frac{6}{(a-1)(3a-2)}\varpi_3$
		$\mathrm{d}\varpi_4 = -\frac{2}{3}\varpi_1 \wedge \varpi_3 - a\varpi_2 \wedge \varpi_4 + (\frac{4}{9} - \frac{2}{3}a)\varpi_2 \wedge \varpi_5 + \frac{2}{a-1}\varpi_4 \wedge \varpi_5$	$\tau_1 = \frac{3a^2-5a-6}{4(a-1)(3a-2)}\varpi_3$
		$\mathrm{d}\varpi_5 = \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4$	$\tau_2 = (\frac{4}{9} - \frac{2}{3}a)\varpi_2 - \frac{15a^2-a+2}{4(a-1)(3a-2)}\varpi_4 - \frac{2(3a+1)}{3(a-1)}\varpi_5$
		$\mathrm{d}\nu_1 = \frac{9a^2-15a-2}{4(3a-2)(a-1)}\varpi_1 \wedge \varpi_3 - \frac{27a^2-21a+10}{4(3a-2)(a-1)}\varpi_2 \wedge \varpi_4 - \frac{2(3a-1)}{3(a-1)}\varpi_2 \wedge \varpi_5 + \frac{6(3a-1)}{(a-1)^2(3a-2)}\varpi_4 \wedge \varpi_5$	$\tau_3 = \frac{3a^2-5a-6}{4(a-1)(3a-2)}\varpi_1$
			$\tau_4 = -\frac{15a^2-a+2}{4(a-1)(3a-2)}\varpi_2 - \frac{18}{(3a-2)(a-1)^2}\varpi_4 - \frac{6(3a+1)}{(3a-2)(a-1)^2}\varpi_5$
			$\tau_5 = -\frac{3a+1}{3(a-1)}\varpi_2 - \frac{3(3a+1)}{(3a-2)(a-1)^2}\varpi_4 + \frac{117a^4-462a^3-407a^2+604a-44}{16(a-1)^2(3a-2)^2}\varpi_5$

Table 7 continued

Model	SD	Structure equations	Embedding into Cartan bundle
D.6-3 ($a \neq 0$)	✓	$d\varpi_1 = \frac{3}{2}\varpi_1 \wedge \varpi_5 - \nu_1 \wedge \varpi_1 - \frac{1}{2}a\varpi_4 \wedge \varpi_5$	$\zeta_2 = -\zeta_1 = \varpi_5,\ \nu_2 = \nu_3 = 0$
		$\mathrm{d}\varpi_2 = \frac{3}{2}\varpi_2 \wedge \varpi_5 + \nu_1 \wedge \varpi_2 - \frac{1}{2}a\varpi_3 \wedge \varpi_5$	$\tau_1 = \frac{1}{2}a\varpi_2 - \frac{1}{2}\varpi_3$
		$\mathrm{d}\varpi_3 = \frac{1}{2}a\varpi_2 \wedge \varpi_5 - \frac{3}{2}\varpi_3 \wedge \varpi_5 + \nu_1 \wedge \varpi_3$	$\tau_2 = \frac{1}{2}a\varpi_1 - \frac{1}{2}\varpi_4$
		$\mathrm{d}\varpi_4 = \frac{1}{2}a\varpi_1 \wedge \varpi_5 - \frac{3}{2}\varpi_4 \wedge \varpi_5 - \nu_1 \wedge \varpi_4$	$\tau_3 = -\frac{1}{2}\varpi_1 + \frac{1}{2}a\varpi_4$
		$\mathrm{d}\varpi_5 = \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4$	$\tau_4 = -\frac{1}{2}\varpi_2 + \frac{1}{2}a\varpi_3$
		$\mathrm{d}\nu_1 = -\frac{1}{2}a\varpi_1 \wedge \varpi_2 + \frac{3}{2}\varpi_1 \wedge \varpi_3 - \frac{3}{2}\varpi_2 \wedge \varpi_4 + \frac{1}{2}a\varpi_3 \wedge \varpi_4$	$\tau_5 = \frac{1}{4}(1-a^2)\varpi_5$
D.6-4	×	$d\varpi_1 = \frac{3}{2}\varpi_1 \wedge \varpi_5 - \nu_1 \wedge \varpi_1$	$\zeta_2 = -\zeta_1 = \varpi_5,\ \nu_2 = \nu_3 = 0$
		$d\varpi_2 = \frac{3}{2}\varpi_2 \wedge \varpi_5 + \nu_1 \wedge \varpi_2$	$\tau_1 = -\frac{1}{2}\varpi_2 - \frac{1}{2}\varpi_3$
		$\mathrm{d}\varpi_3 = -\frac{1}{2}\varpi_2 \wedge \varpi_5 - \frac{3}{2}\varpi_3 \wedge \varpi_5 + \nu_1 \wedge \varpi_3$	$\tau_2 = -\frac{1}{2}\varpi_1 - \frac{1}{2}\varpi_4$
		$\mathrm{d}\varpi_4 = -\frac{1}{2}\varpi_1 \wedge \varpi_5 - \frac{3}{2}\varpi_4 \wedge \varpi_5 - \nu_1 \wedge \varpi_4$	$\tau_3 = -\frac{1}{2}\varpi_1$
		$d\varpi_5 = \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4$	$\tau_4 = -\frac{1}{2}\varpi_2$
		$\mathrm{d}\nu_1 = \frac{1}{2}\varpi_1 \wedge \varpi_2 + \frac{3}{2}\varpi_1 \wedge \varpi_3 - \frac{3}{2}\varpi_2 \wedge \varpi_4$	$\tau_5 = \frac{1}{4}\varpi_5$

4.3 Type III Reduction

Normalize $A = \mathsf{rs}^3$, i.e., $A_4 = \frac{1}{4}$, $A_5 = A_3 = A_2 = A_1 = 0$. Then $0 = d(A_i)$ implies:

$$\alpha_1 = \alpha_2 = 0, \quad \nu_2 = 2\alpha_3, \quad \nu_1 = \frac{1}{2}(\zeta_1 + \zeta_2) - 2\alpha_4, \quad \nu_3 = \alpha_5.$$

Differentiating the ν_i-relations above yields

$$\begin{aligned} &2\delta a_{31} = -(3z_1 + z_2)a_{31} + t_2, \quad 2\delta a_{34} = -(z_1 + 3z_2)a_{34} - t_3, \\ &2\delta a_{41} = (z_2 - z_1)a_{41} - t_1, \quad 2\delta a_{44} = (z_1 - z_2)a_{44} - t_4. \end{aligned}$$

Normalize $a_{31} = a_{34} = a_{41} = a_{44} = 0$ (so $t_1 = t_2 = t_3 = t_4 = 0$). Then $2\delta a_{45} = -2(z_1 + z_2)a_{45} - t_5$, so normalize $a_{45} = 0$, and let $\tau_i = T_{ij}\varpi_j$. We have reduced to a 7-dimensional subbundle with:

$$\begin{aligned} \delta a_{42} &= -\left(\tfrac{3z_1+z_2}{2}\right) a_{42} \\ \delta a_{43} &= -\left(\tfrac{z_1+3z_2}{2}\right) a_{43} \\ \delta a_{32} &= -\left(\tfrac{5z_1+3z_2}{2}\right) a_{32}, \\ \delta a_{33} &= -\left(\tfrac{3z_1+5z_2}{2}\right) a_{33} \\ \delta a_{35} &= -2(z_1 + z_2)a_{35} \end{aligned} \qquad \begin{aligned} \delta a_{51} &= \left(\tfrac{z_1+3z_2}{2}\right) a_{51} \\ \delta a_{52} &= \left(\tfrac{z_2-z_1}{2}\right) a_{52} \\ \delta a_{53} &= \left(\tfrac{z_1-z_2}{2}\right) a_{53} \\ \delta a_{54} &= \left(\tfrac{3z_1+z_2}{2}\right) a_{54} \\ \delta a_{55} &= 0 \end{aligned}$$

However, as indicated in Theorem 3.1, there are no type III structures with 7 symmetries.

Now, $0 = \mathrm{d}^2\nu_2 \wedge \varpi_{45} = \mathrm{d}^2\nu_2 \wedge \varpi_{15}$ implies $B_1 = B_5 = 0$, and $0 = \mathrm{d}(B_1) = \mathrm{d}(B_5)$ is equivalent to:

$$\beta_1 = 6B_2\alpha_3, \quad \beta_5 = 6B_6\alpha_3,$$

and further Bianchi identities imply

$$\begin{aligned} &a_{42} = -2B_3, \quad a_{43} = 2B_7, \quad a_{52} = -2B_4, \quad a_{32} = -2B_2, \quad a_{33} = 2B_6, \\ &a_{53} = 2B_8, \quad B_2B_6 = 0. \end{aligned}$$

There is a duality inducing $(B_j, B_{j+4}) \mapsto (-B_{j+4}, B_j)$, where $j = 1, \ldots, 4$, so WLOG, we may assume that one of B_2, B_3, B_4 is nonzero, or $B = 0$. Similar calculations show that for multiply transitive structures, we must have $B_2 = B_3 = 0$ (hence, $B_6 = B_7 = 0$ also). Up to duality, we only have: III.6-1 ($B_4 \neq 0$ branch), and III.6-2 ($B = 0$ branch). Structure equations are given in Table 8.

Table 8 Multiply transitive type III structure equations

Model	SD	Structure equations	Embedding into Cartan bundle
III.6-1	×	$\mathrm{d}\varpi_1 = \varpi_1 \wedge \varpi_4$	$\zeta_2 = -\varpi_1 + 2\varpi_4 + \zeta_1$
		$\mathrm{d}\varpi_2 = (-\frac{5}{4}\varpi_1 + \varpi_4 + 2\zeta_1) \wedge \varpi_2 - \frac{1}{2}\varpi_1 \wedge \varpi_5$	$\nu_1 = \zeta_1 - \frac{1}{2}\varpi_1 + \varpi_4$
		$\mathrm{d}\varpi_3 = -\frac{3}{2}\varpi_1 \wedge \varpi_3 + \frac{3}{16}\varpi_1 \wedge \varpi_5 - \frac{3}{4}\varpi_2 \wedge \varpi_4 - 3\varpi_3 \wedge \varpi_4 - 2\varpi_3 \wedge \zeta_1 + \frac{3}{4}\varpi_4 \wedge \varpi_5$	$\nu_3 = -\frac{3}{4}\varpi_2 - \frac{1}{8}\varpi_5$
		$\mathrm{d}\varpi_4 = -\frac{1}{2}\varpi_1 \wedge \varpi_4$	$\tau_1 = \frac{3}{16}\varpi_1 + \frac{5}{8}\varpi_4$
		$\mathrm{d}\varpi_5 = \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4 - \varpi_1 \wedge \varpi_5 + 2\varpi_4 \wedge \varpi_5 - 2\varpi_5 \wedge \zeta_1$	$\tau_4 = \frac{3}{8}\varpi_1$
		$\mathrm{d}\zeta_1 = \frac{9}{8}\varpi_1 \wedge \varpi_4$	$\nu_2 = \tau_2 = \tau_3 = \tau_5 = 0$
III.6-2	×	$\mathrm{d}\varpi_1 = (-2\zeta_2 + 2\varpi_3) \wedge \varpi_1$	$\zeta_1 = 4\varpi_3 - 3\zeta_2$
		$\mathrm{d}\varpi_2 = (-4\zeta_2 + 6\varpi_3) \wedge \varpi_2 + \frac{1}{4}\varpi_1 \wedge \varpi_5$	$\nu_1 = 2\varpi_3 - \zeta_2$
		$\mathrm{d}\varpi_3 = \frac{3}{8}\varpi_1 \wedge \varpi_4$	$\nu_2 = 0, \nu_3 = \frac{3}{8}\varpi_1 - \frac{1}{8}\varpi_5$
		$\mathrm{d}\varpi_4 = (2\zeta_2 - 2\varpi_3) \wedge \varpi_4,$	$\tau_1 = -\frac{1}{8}\varpi_4$
		$\mathrm{d}\varpi_5 = (-2\zeta_2 + 4\varpi_3) \wedge \varpi_5 + \varpi_1 \wedge \varpi_3 + \varpi_2 \wedge \varpi_4$	$\tau_2 = \tau_3 = \tau_5 = 0$
		$\mathrm{d}\zeta_2 = \frac{5}{8}\varpi_1 \wedge \varpi_4$	$\tau_4 = -\frac{3}{8}\varpi_1$

5 Integration of Structure Equations

In this section, we outline the transition from structure equations found in the previous section to the corresponding systems of 2nd order PDEs. This is done in three steps:

(1) Normalize the algebraic structure of the Lie algebra data defined by the structure equations. This step consists of identifying the type of the Lie algebra $\mathfrak{g}$, the isotropy subalgebra $\mathfrak{k}$ and the subspaces $E, V \subset \mathfrak{g}/\mathfrak{k}$ corresponding to the two Legendrian subbundles. We note that both $E + \mathfrak{k}$ and $V + \mathfrak{k}$ are in fact subalgebras of $\mathfrak{g}$, as we deal only with integrable structures. We also try to find a good basis in $\mathfrak{g}$, adjusting it to the Levi decomposition and the nilradical.
(2) Realize $\mathfrak{g}$ as a transitive Lie algebra of vector fields on $\mathbb{C}^3$ in such a way that its isotropy subalgebra at a certain point is equal exactly to $V + \mathfrak{k}$. This guarantees that the first prolongation of $\mathfrak{g}$ is transitive on $J^1(\mathbb{C}^2, \mathbb{C})$ and has isotropy $\mathfrak{k}$ at a certain point.
(3) Finally, we compute all complete systems of 2nd order PDEs admitting $\mathfrak{g}$ as its symmetry and identify those which correspond to $E + \mathfrak{k}$. In fact, in all cases but one (D.6-3_∞, see Example 5.2 below) there is exactly one such system, and this identification is obtained automatically.

Example 5.1 Consider the structure equations for the model D.7 as given in Table 7. Simple analysis shows the corresponding Lie algebra $\mathfrak{g}$ has radical of dimension 1 if $a \neq \pm\frac{3}{4}$, and of dimension 4 if $a = \pm\frac{3}{4}$. (Note that a and $-a$ yield equivalent models.) Consider first the case $a \neq \frac{3}{4}$. It is clear that $\mathfrak{g}$ has a 6-dimensional Levi subalgebra,

which is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$ (the only complex semisimple Lie algebra in this dimension). As any action of this Levi subalgebra on the 1-dimensional radical is trivial, $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \times \mathbb{C}$. The corresponding basis change from the Cartan reduced basis to the adapted Lie algebra basis in given in Table 12.

Next, analyzing the Cartan basis, we see that the isotropy $\mathfrak{k}$ is 2-dimensional and abelian. Moreover, its projection to each $\mathfrak{sl}_2(\mathbb{C})$-factor is one-dimensional and diagonalizable, while the intersection with each $\mathfrak{sl}_2(\mathbb{C})$-factor is trivial. This implies that $\mathfrak{k}$ is conjugate to the following subalgebra in $\mathfrak{g}$:

$$\mathfrak{k} \sim \langle H_1 - Z, H_2 - \lambda Z \rangle, \quad \lambda \in \mathbb{C} \backslash \{0\},$$

where H_1, H_2 are parts of the standard $\mathfrak{sl}_2(\mathbb{C})$-basis $\{X_i, H_i, Y_i\}$ in each copy of $\mathfrak{sl}_2(\mathbb{C})$, and Z spans the center $\mathfrak{z} = \mathbb{C}$. Also, $\lambda = \frac{3+4a}{3-4a}$, and the redundancy $a \mapsto -a$ induces the redundancy $\lambda \mapsto \frac{1}{\lambda}$.

Further, it is easy to check that the projections of both $E + \mathfrak{k}$ and $V + \mathfrak{k}$ to each $\mathfrak{sl}_2(\mathbb{C})$-factor is two-dimensional. Thus, we can assume that:

$$V + \mathfrak{k} = \langle X_1, X_2, H_1 - Z, H_2 - \lambda Z \rangle,$$
$$E + \mathfrak{k} = \langle Y_1, Y_2, H_1 - Z, H_2 - \lambda Z \rangle.$$

Let us now realize $\mathfrak{g}$ as a Lie algebra of vector fields on $\mathbb{C}^3 = J^0(\mathbb{C}^2, \mathbb{C})$ with the isotropy subalgebra equal to $V + \mathfrak{k}$. Note that $\mathfrak{h} = V + \mathfrak{k} + \mathfrak{z}$ is a subalgebra of codimension 2 in $\mathfrak{g}$. However, it is not effective, and the maximal ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$ is exactly $\mathfrak{z}$. So, $\mathfrak{g}/\mathfrak{z}$ can be realized as a Lie algebra of vector fields on $\mathbb{C}^2$ with the isotropy $\mathfrak{h}/\mathfrak{z}$. But $\mathfrak{g}/\mathfrak{z}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$ with $\mathfrak{h}/\mathfrak{z}$ identified with the direct product of two subalgebras of upper-triangular matrices. It is easy to see that it integrates to the global action of $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Locally this leads to the following realization of $\mathfrak{g}/\mathfrak{z}$:

$$\langle \partial_x,\ 2x\partial_x,\ x^2\partial_x,\ \partial_y,\ 2y\partial_y,\ y^2\partial_y \rangle.$$

We can always assume that the realization of $\mathfrak{g}$ is adapted to it. In other words, it can be obtained from the above one by adding terms of the form $f(x, y, u)\partial_u$ to the above vector fields and realizing the center Z as a vector field of the form $g(x, y, u)\partial_u$. Simple computation shows that we can always adapt the coordinates (x, y, u) such that Z becomes equal to ∂_u, and we get the following realization of $\mathfrak{g}$:

$$\left\langle \partial_x,\ 2x\partial_x + \partial_u,\ x^2\partial_x + x\partial_u,\ \partial_y,\ 2y\partial_y + \tfrac{1}{\lambda}\partial_u,\ y^2\partial_y + \tfrac{1}{\lambda}y\partial_u,\ \partial_u \right\rangle.$$

Prolonging this Lie algebra of vector fields to $J^1(\mathbb{C}^2, \mathbb{C})$ and checking which complete systems of 2nd order PDEs are invariant with respect to it, we immediately get that the only such system has the form:

$$u_{11} = p^2, \quad u_{12} = 0, \quad u_{22} = \lambda q^2.$$

Setting now the parameter λ to 0 and computing the symmetry algebra of the above system of PDEs, we obtain exactly the Lie algebra $\mathfrak{g}$, its subalgebra $\mathfrak{k}$ and subspaces $E, V \subset \mathfrak{g}/\mathfrak{k}$, that match the exceptional case $a = \pm\frac{3}{4}$ of the Cartan structure equations in case of D.7.

Example 5.2 Consider now the case D.6-3. We note that in this case the Lie algebra $\mathfrak{g}$ defined by the structure equations is semisimple if $a \neq \pm 3$ and has a 3-dimensional abelian ideal otherwise. First, consider the generic case of $a \neq \pm 3$. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$. As above, denote by $\{X_i, H_i, Y_i\}$, $i = 1, 2$, the standard bases of these two copies of $\mathfrak{sl}_2(\mathbb{C})$. Direct inspection of the Cartan structure equations shows that $\mathfrak{k} = \langle H_1 - H_2 \rangle$ and both subalgebras $E + \mathfrak{k}$ and $V + \mathfrak{k}$ are three-dimensional semisimple. But any simple subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$ has the form:

$$\langle X_1 + \mu Y_2, X_2 + \mu Y_1, H_1 - H_2 \rangle, \quad \mu \neq 0, \tag{5.1}$$

and any two such subalgebras are conjugate to each other by means of inner automorphisms preserving $\mathfrak{k}$. Hence, we can assume that $V + \mathfrak{k}$ corresponds to $\mu = 1$, which is exactly the diagonal of the direct product of $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$. Under the classical isomorphism $\mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$ this subalgebra corresponds to the standard embedding of $\mathfrak{so}(3, \mathbb{C}) \subset \mathfrak{so}(4, \mathbb{C})$. So, we can realize the Lie algebra $\mathfrak{g}$ as a Lie algebra of vector fields corresponding to the action of $SO(4, \mathbb{C})$ on the three-dimensional complex sphere. In an appropriate coordinate system we get the following vector fields:

$$\begin{aligned} X_1 &= \partial_x, & H_1 &= -2x\partial_x - 2u\partial_u, & Y_1 &= -x^2\partial_x - u\partial_y - 2xu\partial_u \\ X_2 &= \partial_y, & H_2 &= -2y\partial_y - 2u\partial_u, & Y_2 &= -y^2\partial_y - u\partial_x - 2yu\partial_u. \end{aligned}$$

Again, prolonging this Lie algebra of vector fields to $J^1(\mathbb{C}^2, \mathbb{C})$ and computing all invariant systems of 2nd order PDEs, we obtain the following family of systems:

$$u_{11} = \lambda p^2 \frac{\sqrt{u - pq}}{u^{3/2}}, \quad u_{12} = 1 + \lambda(pq - 2u)\frac{\sqrt{u - pq}}{u^{3/2}}, \quad u_{22} = \lambda q^2 \frac{\sqrt{u - pq}}{u^{3/2}}.$$

Each such system corresponds to the subalgebra (5.1) with $\mu = \frac{2\lambda - 1}{2\lambda + 1}$.

In the limiting case of $a = \pm 3$ in the structure equations we get $\mathfrak{g} \simeq \mathfrak{so}(3, \mathbb{C}) \leftthreetimes \mathbb{C}^3$ and $\mathfrak{k} + V = \mathbb{C}^3$. This pair corresponds to the group of complex Euclidean transformations of $\mathbb{C}^3$, which preserves the following family of complete systems of 2nd order PDEs:

$$u_{11} = \lambda p^2 \sqrt{1 - 2pq}, \quad u_{12} = \lambda(pq - 1)\sqrt{1 - 2pq}, \quad u_{22} = \lambda q^2 \sqrt{1 - 2pq}.$$

If $\lambda = 0$, this system is flat and has 15-dimensional symmetry algebra. If $\lambda \neq 0$, then we can normalize it to $\lambda = 1$ by means of the transformation $(x, y, u) \mapsto (\lambda x, \lambda y, \lambda u)$. To distinguish this special case from the generic one, we denote it by D.6-3_∞.

Example 5.3 Consider the case N.6-2, which involves two parameters. The Lie algebra $\mathfrak{g}$ is solvable in this case and has a 4-dimensional abelian nilradical $\mathfrak{n}$. Two basis elements complementary to $\mathfrak{n}$ act on $\mathfrak{n}$ by the following two commuting matrices:

$$\begin{pmatrix} 2b & 0 & 0 & 1 \\ 0 & b & 1 & 0 \\ 0 & 1 & 2b & 0 \\ 1 & 0 & 0 & b \end{pmatrix}, \quad \begin{pmatrix} a & 0 & -1 & 0 \\ 0 & 2a & 0 & -1 \\ -1 & 0 & 2a & 0 \\ 0 & -1 & 0 & a \end{pmatrix}.$$

If parameters a, b of the structure equations satisfy $a^2 + 4 \neq 0, b^2 + 4 \neq 0$, then both matrices simultaneously diagonalize in a certain basis $\{N_1, N_2, N_3, N_4\}$ of $\mathfrak{n}$ to become:

$$\tfrac{1}{2}\operatorname{diag}\left(3b - \sqrt{b^2+4}, 3b + \sqrt{b^2+4}, 3b + \sqrt{b^2+4}, 3b - \sqrt{b^2+4}\right),$$
$$\tfrac{1}{2}\operatorname{diag}\left(3a - \sqrt{a^2+4}, 3a - \sqrt{a^2+4}, 3a + \sqrt{a^2+4}, 3a + \sqrt{a^2+4}\right).$$

After rescaling, we can bring them to the form:

$$\operatorname{diag}(\mu - 1, \mu, \mu, \mu - 1), \qquad \mu = \frac{1}{2} + \frac{3b}{2\sqrt{b^4+4}},$$
$$\operatorname{diag}(\kappa + 1, \kappa + 1, \kappa + 2, \kappa + 2), \qquad \kappa = \frac{3}{2} + \frac{3a}{2\sqrt{a^2+4}}.$$

Denote by S_1, S_2 the corresponding elements in $\mathfrak{g}$, which span the complementary subspace to $\mathfrak{n}$. In general, this subspace is not a subalgebra, and $[S_1, S_2] \in \mathfrak{n}$. But if any of these two matrices is invertible (meaning $\mu \neq 0, 1$ or $\kappa \neq -1, -2$) then we can always adjust S_1, S_2 by adding elements from $\mathfrak{n}$ such that we get $[S_1, S_2] = 0$. We note that there are elements $u_1, u_2 \in \mathfrak{n}$ such that $S_1 + u_1 \in V + \mathfrak{k}$, $S_2 + u_2 \in E + \mathfrak{k}$.

It is easy to check that the intersection of $V + \mathfrak{k}$ with $\mathfrak{n}$ is two-dimensional and can be made equal to $\langle N_1 - N_4, N_2 - N_3\rangle$ after suitable rescaling to basis vectors $\{N_i\}$. Hence, in any realization of $\mathfrak{g}$ as a transitive Lie algebra of vector fields on $\mathbb{C}^3$ having $V + \mathfrak{k}$ as a stabilizer, $\mathfrak{n}$ will be a 4-dimensional abelian Lie algebra with 2-dimensional orbits. In particular, we can always choose a local coordinate system (x, y, u) in such a way that $N_3 = \partial_u$, $N_4 = \partial_y$, and two other basis vectors N_1, N_2 will be of the form $f(x)\partial_y + g(x)\partial_u$. As S_1, S_2 act by scalings on any of N_i, $i = 1, \ldots, 4$, it is natural to assume that they are represented as linear combinations of vector fields $x\partial_x$, $y\partial_y$, $u\partial_u$. Using this ansatz, we immediately get the following representation of $\mathfrak{g}$:

$$S_1 = -(\mu - 1)y\partial_y - \mu u\partial_u,$$
$$S_2 = -x\partial_x - (\kappa + 2)y\partial_y - (\kappa + 2)u\partial_u,$$
$$N_1 = x\partial_y, \quad N_2 = x\partial_u, \quad N_3 = \partial_u, \quad N_4 = \partial_y.$$

Prolonging this Lie algebra of vector fields to $J^1(\mathbb{C}^2, \mathbb{C})$ and computing all invariant complete systems of 2nd order PDEs, we arrive at the following system:

$$u_{11} = q^\mu x^\kappa, \quad u_{12} = 0, \quad u_{22} = 0.$$

The special values of parameters we omitted on the way can be treated in a similar way and lead to the following systems of PDEs:

- $a^2 + 4 = 0, b^2 + 4 \neq 0$ (or equivalently, $a^2 + 4 \neq 0, b^2 + 4 = 0$):

$$u_{11} = e^q x^\kappa, \quad u_{12} = 0, \quad u_{22} = 0.$$

- $a^2 + 4 = b^2 + 4 = 0$:

$$u_{11} = e^q e^x, \quad u_{12} = 0, \quad u_{22} = 0.$$

- $\mu = 0, 1$ (or equivalently, $\kappa = -1, -2$):

$$u_{11} = \ln(q) x^\kappa, \quad u_{12} = 0, \quad u_{22} = 0.$$

More details on restrictions on parameters and realizations of $\mathfrak{g}$ in terms of vector fields for these special values of parameters are given in Table 9.

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Appendix: Classification Tables

See Tables 9, 10, 11, 12, 13 and 14

Table 9 Classification of type N cases

	Model	Parameters	Symmetries	Lie algebra structure	Abstract ILC structure
N.8	$u_{11} = q^2$ $u_{12} = 0$ $u_{22} = 0$		$S_1 = -y\partial_y - 2u\partial_u - 2p\partial_p - q\partial_q$ $S_2 = -x\partial_x - 2y\partial_y - 2u\partial_u - p\partial_p$ $N_1 = \frac{x^2}{2}\partial_y - \frac{y}{2}\partial_u - qx\partial_p - \frac{1}{2}\partial_q$ $N_2 = \partial_x$ $N_3 = -x\partial_y + q\partial_p$ $N_4 = -\frac{x}{2}\partial_u - \frac{1}{2}\partial_p$ $N_5 = -\frac{1}{2}\partial_u$ $N_6 = -\partial_y$ (General pt: $x = y = u = p = q = 0$)	$\begin{array}{c\|cccccccc} & S_1 & S_2 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 \\ \hline S_1 & \cdot & \cdot & N_1 & \cdot & N_3 & 2N_4 & 2N_5 & N_6 \\ S_2 & & \cdot & \cdot & N_2 & N_3 & N_4 & 2N_5 & 2N_6 \\ N_1 & & & \cdot & N_3 & N_4 & \cdot & \cdot & N_5 \\ N_2 & & & & \cdot & N_6 & N_5 & \cdot & \cdot \\ N_3 & & & & & \cdot & \cdot & \cdot & \cdot \\ N_4 & & & & & & \cdot & \cdot & \cdot \\ N_5 & & & & & & & \cdot & \cdot \\ N_6 & & & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:$ N_2, N_6 $V/\mathfrak{k}:$ N_1, N_4 $\mathfrak{k}:$ N_3, S_1, S_2
N.7-1	$u_{11} = q^2 x^\kappa$ $u_{12} = 0$ $u_{22} = 0$	$\kappa \neq -1, -2, 0, -3, \infty$ $(a^2 \neq \frac{1}{2}, -2)$	$S_1 = -y\partial_y - 2u\partial_u - 2p\partial_p - q\partial_q$ $S_2 = -x\partial_x - (\kappa+2)y\partial_y - (\kappa+2)u\partial_u$ $\quad -(\kappa+1)p\partial_p$ $N_1 = \frac{x^{\kappa+2}}{\kappa+2}\partial_y - \frac{\kappa+1}{2}y\partial_u - x^{\kappa+1}q\partial_p$ $\quad -\frac{\kappa+1}{2}\partial_q$ $N_2 = x\partial_y - q\partial_p$ $N_3 = \frac{\kappa+1}{2}(x\partial_u + \partial_p)$ $N_4 = \frac{\kappa+1}{2}\partial_u$ $N_5 = \partial_y$ (General pt: $y = u = p = q = 0, x = 1$)	$\begin{array}{c\|ccccccc} & S_1 & S_2 & N_1 & N_2 & N_3 & N_4 & N_5 \\ \hline S_1 & \cdot & \cdot & N_1 & N_2 & 2N_3 & 2N_4 & N_5 \\ S_2 & & \cdot & \cdot & (\kappa+1)N_2 & (\kappa+1)N_3 & (\kappa+2)N_4 & (\kappa+2)N_5 \\ N_1 & & & \cdot & N_3 & \cdot & \cdot & N_4 \\ N_2 & & & & \cdot & \cdot & \cdot & \cdot \\ N_3 & & & & & \cdot & \cdot & \cdot \\ N_4 & & & & & & \cdot & \cdot \\ N_5 & & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:$ S_2, N_2 $V/\mathfrak{k}:$ $N_3 - N_4,$ $\quad -N_1 + \frac{1}{\kappa+2}N_2$ $\mathfrak{k}:$ $S_1, N_2 - N_5$
	$u_{11} = q^2 x^{-1}$ $u_{12} = 0$ $u_{22} = 0$	$\kappa = -1$ $(a = \frac{1}{\sqrt{2}})$	$S_1 = -y\partial_y - 2u\partial_u - 2p\partial_p - q\partial_q$ $S_2 = -x\partial_x - y\partial_y - u\partial_u$ $N_1 = 2x(\ln(x) - 1)\partial_y - y\partial_u$ $\quad -2\ln(x)q\partial_p - \partial_q$ $N_2 = x\partial_y - q\partial_p$ $N_3 = x\partial_u + \partial_p$ $N_4 = \partial_u$ $N_5 = \partial_y$ (General pt: $y = u = p = q = 0, x = 1$)	$\begin{array}{c\|ccccccc} & S_1 & S_2 & N_1 & N_2 & N_3 & N_4 & N_5 \\ \hline S_1 & \cdot & \cdot & N_1 & N_2 & 2N_3 & 2N_4 & N_5 \\ S_2 & & \cdot & -2N_2 & \cdot & \cdot & N_4 & N_5 \\ N_1 & & & \cdot & N_3 & \cdot & \cdot & N_4 \\ N_2 & & & & \cdot & \cdot & \cdot & \cdot \\ N_3 & & & & & \cdot & \cdot & \cdot \\ N_4 & & & & & & \cdot & \cdot \\ N_5 & & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:$ S_2, N_2 $V/\mathfrak{k}:$ $N_3 - N_4,$ $\quad N_1 + 2N_2$ $\mathfrak{k}:$ $S_1, N_2 - N_5$
	$u_{11} = q^2 e^x$ $u_{12} = 0$ $u_{22} = 0$	$\kappa = \infty$ $(a = \pm 2i)$	$S_1 = -y\partial_y - 2u\partial_u - 2p\partial_p - q\partial_q$ $S_2 = -\partial_x - y\partial_y - u\partial_u - p\partial_p$ $N_1 = 2e^x\partial_y - y\partial_u$ $\quad -2e^x q\partial_p - \partial_q$ $N_2 = x\partial_y - q\partial_p$ $N_3 = x\partial_u + \partial_p$ $N_4 = \partial_u$ $N_5 = \partial_y$ (General pt: $x = y = u = p = q = 0,$)	$\begin{array}{c\|ccccccc} & S_1 & S_2 & N_1 & N_2 & N_3 & N_4 & N_5 \\ \hline S_1 & \cdot & \cdot & N_1 & N_2 & 2N_3 & 2N_4 & N_5 \\ S_2 & & \cdot & \cdot & N_2 - N_5 & N_3 - N_4 & N_4 & N_5 \\ N_1 & & & \cdot & N_3 & \cdot & \cdot & N_4 \\ N_2 & & & & \cdot & \cdot & \cdot & \cdot \\ N_3 & & & & & \cdot & \cdot & \cdot \\ N_4 & & & & & & \cdot & \cdot \\ N_5 & & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:$ S_2, N_5 $V/\mathfrak{k}:$ $N_3, -N_1 + 2N_5$ $\mathfrak{k}:$ S_1, N_2
N.7-2	$u_{11} = q^{-1}$ $u_{12} = 1$ $u_{22} = 0$		$X = -\partial_x + \partial_u$ $H = -2x\partial_x + \partial_y - 2u\partial_u - 2q\partial_q$ $Y = x^2\partial_x + u\partial_y + (2ux + x^2)\partial_u$ $\quad +(2x + 2u - qp)\partial_p + q(2x - q)\partial_q$ $N_1 = x\partial_y + x^2\partial_u - (q - 2x)\partial_p$ $N_2 = -\partial_y - 2x\partial_u - 2\partial_p$ $N_3 = 2\partial_u$ $N_4 = \partial_y$ (General pt: $y = u = p = q = 0, x = 1$)	$\begin{array}{c\|ccccccc} & X & H & Y & N_1 & N_2 & N_3 & N_4 \\ \hline X & \cdot & -2X & H & N_2 & N_3 & \cdot & \cdot \\ H & & \cdot & -2Y & -2N_1 & \cdot & 2N_3 & \cdot \\ Y & & & \cdot & \cdot & 2N_1 & 2N_2 & \cdot \\ N_1 & & & & \cdot & \cdot & \cdot & \cdot \\ N_2 & & & & & \cdot & \cdot & \cdot \\ N_3 & & & & & & \cdot & \cdot \\ N_4 & & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:$ $X + Y + N_1 - \frac{1}{2}N_3,$ $\quad -N_1 + \frac{1}{2}N_3 + N_4$ $V/\mathfrak{k}:$ Y, N_1 $\mathfrak{k}:$ $H - 2Y - N_4,$ $\quad -2N_1 + N_2 + N_4$

Table 9 continued

	Model	Parameters	Symmetries	Lie algebra structure	Abstract ILC structure
N.6-1	$u_{11} = q^\mu$ $u_{12} = 1$ $u_{22} = 0$	$\mu \neq -1, 2, 0, 1, \infty$ $(a^2 \neq 0, \frac{1}{2}, 2, -1)$	$S = -x\partial_x - (\mu+1)y\partial_y$ $\quad -(\mu+2)u\partial_u - (\mu+1)p\partial_p - q\partial_q$ $N_1 = \partial_x$ $N_2 = x\partial_y + x^2\partial_u + (2x-q)\partial_p$ $N_3 = \partial_y + 2x\partial_u + 2\partial_p$ $N_4 = 2\partial_u$ $N_5 = \partial_y$ (General pt: $x = y = u = p = 0, q = 1$)	$\begin{array}{c\|ccccc} & S & N_1 & N_2 & N_3 & N_4 & N_5 \\ \hline S & \cdot & N_1 & \mu N_2 & (\mu+1)N_3 & (\mu+2)N_4 & (\mu+1)N_5 \\ N_1 & & \cdot & N_3 & N_4 & \cdot & \cdot \\ N_2 & & & \cdot & \cdot & \cdot & \cdot \\ N_3 & & & & \cdot & \cdot & \cdot \\ N_4 & & & & & \cdot & \cdot \\ N_5 & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:\ -S+N_1-N_2,$ $\quad N_2+N_3+\frac{1}{2}N_4$ $V/\mathfrak{k}:\ S, N_2$ $\mathfrak{k}:\ 2N_2+N_3-N_5$
	$u_{11} = \ln(q)$ $u_{12} = 1$ $u_{22} = 0$	$\mu = 0\,(a^2 = \frac{1}{2})$	$S = -x\partial_x + (\frac{y}{2} - y)\partial_y$ $\quad -2u\partial_u - (\frac{q}{2} + p)\partial_p - q\partial_q$ $N_1 = \partial_x$ $N_2 = x\partial_y + x^2\partial_u + (2x-q)\partial_p$ $N_3 = \partial_y + 2x\partial_u + 2\partial_p$ $N_4 = 2\partial_u$ $N_5 = -\frac{1}{2}\partial_y$ (General pt: $x = y = u = p = 0, q = 1$)	$\begin{array}{c\|ccccc} & S & N_1 & N_2 & N_3 & N_4 & N_5 \\ \hline S & \cdot & N_1+N_5 & \cdot & N_3 & 2N_4 & N_5 \\ N_1 & & \cdot & N_3 & N_4 & \cdot & \cdot \\ N_2 & & & \cdot & \cdot & \cdot & \cdot \\ N_3 & & & & \cdot & \cdot & \cdot \\ N_4 & & & & & \cdot & \cdot \\ N_5 & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:\ -S+N_1+\frac{1}{2}N_2,$ $\quad N_2+N_3+\frac{1}{2}N_4$ $V/\mathfrak{k}:\ N_2, -S+\frac{1}{2}N_2$ $\mathfrak{k}:\ 2N_2+N_3+2N_5$
	$u_{11} = q\ln(q)$ $u_{12} = 1$ $u_{22} = 0$	$\mu = 1\,(a^2 = 2)$	$S = -x\partial_x + (\frac{x^2}{2} - 2y)\partial_y - (3u - \frac{x^3}{3})\partial_u$ $\quad -(2p + qx - x^2)\partial_p - q\partial_q$ $N_1 = \partial_x$ $N_2 = x\partial_y + x^2\partial_u + (2x-q)\partial_p$ $N_3 = \partial_y + 2x\partial_u + 2\partial_p$ $N_4 = 2\partial_u$ $N_5 = \partial_y$ (General pt: $x = y = u = p = 0, q = 1$)	$\begin{array}{c\|ccccc} & S & N_1 & N_2 & N_3 & N_4 & N_5 \\ \hline S & \cdot & N_1-N_2 & N_2 & 2N_3 & 3N_4 & 2N_5 \\ N_1 & & \cdot & N_3 & N_4 & \cdot & \cdot \\ N_2 & & & \cdot & \cdot & \cdot & \cdot \\ N_3 & & & & \cdot & \cdot & \cdot \\ N_4 & & & & & \cdot & \cdot \\ N_5 & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:\ -S+N_1,$ $\quad N_2+N_3+\frac{1}{2}N_4$ $V/\mathfrak{k}:\ S, N_2$ $\mathfrak{k}:\ 2N_2+N_3-N_5$
N.6-2	$u_{11} = q^\mu x^\kappa$ $u_{12} = 0$ $u_{22} = 0$	$\mu \neq -1, 2, 0, 1, \infty$ $(b \neq \pm\frac{1}{\sqrt{2}}, \pm 2i),$ $\kappa \neq 0, -3, \infty$ $(a \neq \pm 2i)$	$S_1 = -(\mu-1)y\partial_y - \mu u\partial_u - \mu p\partial_p - q\partial_q$ $S_2 = -x\partial_x - (\kappa+2)y\partial_y - (\kappa+2)u\partial_u$ $\quad -(\kappa+1)p\partial_p$ $N_1 = x\partial_y - q\partial_p$ $N_2 = x\partial_u + \partial_p$ $N_3 = \partial_u$ $N_4 = \partial_y$ (General pt: $y = u = p = 0, q = x = 1$)	$\begin{array}{c\|cccccc} & S_1 & S_2 & N_1 & N_2 & N_3 & N_4 \\ \hline S_1 & \cdot & \cdot & (\mu-1)N_1 & \mu N_2 & \mu N_3 & (\mu-1)N_4 \\ S_2 & & \cdot & (\kappa+1)N_1 & (\kappa+1)N_2 & (\kappa+2)N_3 & (\kappa+2)N_4 \\ N_1 & & & \cdot & \cdot & \cdot & \cdot \\ N_2 & & & & \cdot & \cdot & \cdot \\ N_3 & & & & & \cdot & \cdot \\ N_4 & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:\ -S_2-N_1+N_4,$ $\quad N_1+N_2$ $V/\mathfrak{k}:\ -N_1+N_4, S_1$ $\mathfrak{k}:\ N_1+N_2-N_3-N_4$
	$u_{11} = e^q x^\kappa$ $u_{12} = 0$ $u_{22} = 0$	$\mu = \infty$ $(b = \pm 2i)$ $\kappa \neq 0, -3, \infty$ $(a \neq \pm 2i)$	$S_1 = -y\partial_y - (u+y)\partial_u - p\partial_p - \partial_q$ $S_2 = -x\partial_x - (\kappa+2)y\partial_y - (\kappa+2)u\partial_u$ $\quad -(\kappa+1)p\partial_p$ $N_1 = x\partial_y - q\partial_p$ $N_2 = x\partial_u + \partial_p$ $N_3 = \partial_u$ $N_4 = \partial_y$ (General pt: $y = u = p = x = 0, q = 1$)	$\begin{array}{c\|cccccc} & S_1 & S_2 & N_1 & N_2 & N_3 & N_4 \\ \hline S_1 & \cdot & \cdot & N_1+N_2 & N_2 & N_3 & N_3+N_4 \\ S_2 & & \cdot & (\kappa+1)N_1 & (\kappa+1)N_2 & (\kappa+2)N_3 & (\kappa+2)N_4 \\ N_1 & & & \cdot & \cdot & \cdot & \cdot \\ N_2 & & & & \cdot & \cdot & \cdot \\ N_3 & & & & & \cdot & \cdot \\ N_4 & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:\ S_2-N_2+N_3,$ $\quad N_1$ $V/\mathfrak{k}:\ N_2-N_3, S_1$ $\mathfrak{k}:\ N_1-N_4$
	$u_{11} = e^q e^x$ $u_{12} = 0$ $u_{22} = 0$	$\mu = \infty$ $(b = \pm 2i)$ $\kappa = \infty$ $(a = \pm 2i)$	$S_1 = -y\partial_y - (u+y)\partial_u - p\partial_p - \partial_q$ $S_2 = -\partial_x - y\partial_y - u\partial_u - p\partial_p$ $N_1 = x\partial_y - q\partial_p$ $N_2 = x\partial_u + \partial_p$ $N_3 = \partial_u$ $N_4 = \partial_y$ (General pt: $y = u = p = x = q = 0$)	$\begin{array}{c\|cccccc} & S_1 & S_2 & N_1 & N_2 & N_3 & N_4 \\ \hline S_1 & \cdot & \cdot & N_1+N_2 & N_2 & N_3 & N_3+N_4 \\ S_2 & & \cdot & N_1-N_4 & N_2-N_3 & N_3 & N_4 \\ N_1 & & & \cdot & \cdot & \cdot & \cdot \\ N_2 & & & & \cdot & \cdot & \cdot \\ N_3 & & & & & \cdot & \cdot \\ N_4 & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:\ S_2+N_1-N_2,$ $\quad N_4,$ $V/\mathfrak{k}:\ N_2, S_1$ $\mathfrak{k}:\ N_1$

Table 9 continued

Model	Parameters	Symmetries	Lie algebra structure	Abstract ILC structure
$u_{11} = \ln(q)x^{\kappa}$ $u_{12} = 0$ $u_{22} = 0$	$\mu = 0$ $(b = \pm\frac{1}{\sqrt{2}})$ $\kappa \neq -1, -2, 0, -3$ $(a \neq \pm\frac{1}{\sqrt{2}})$	$S_1 = y\partial_y - \frac{x^{\kappa+2}}{(\kappa+1)(\kappa+2)}\partial_u - \frac{x^{\kappa+1}}{\kappa+1}\partial_p - q\partial_q$ $S_2 = -x\partial_x - (\kappa+2)y\partial_y - (\kappa+2)u\partial_u$ $\quad -(\kappa+1)p\partial_p$ $N_1 = x\partial_y - q\partial_p$ $N_2 = x\partial_u + \partial_p$ $N_3 = \partial_u$ $N_4 = \partial_y$ (General pt: $y = u = p = 0, q = x = 1$)	$\begin{array}{c\|cccccc} & S_1 & S_2 & N_1 & N_2 & N_3 & N_4 \\ \hline S_1 & \cdot & \cdot & -N_1 & \cdot & \cdot & -N_4 \\ S_2 & & \cdot & (\kappa+1)N_1 & (\kappa+1)N_2 & (\kappa+2)N_3 & (\kappa+2)N_4 \\ N_1 & & & \cdot & \cdot & \cdot & \cdot \\ N_2 & & & & \cdot & \cdot & \cdot \\ N_3 & & & & & \cdot & \cdot \\ N_4 & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:$ $S_2, N_1 + N_2$ $V/\mathfrak{k}:$ $-N_1 + N_4,$ $S_1 + \frac{1}{\kappa+1}N_2 - \frac{1}{\kappa+2}N_3$ $\mathfrak{k}:$ $N_1 + N_2 - N_3 - N_4$
$u_{11} = \ln(q)x^{-2}$ $u_{12} = 0$ $u_{22} = 0$	$\mu = 0$ $(b = \pm\frac{1}{\sqrt{2}})$ $\kappa = -2$ $(a = \pm\frac{1}{\sqrt{2}})$	$S_1 = y\partial_y + \ln(x)\partial_u + \frac{1}{x}\partial_p - q\partial_q$ $S_2 = -x\partial_x + p\partial_p$ $N_1 = x\partial_y - q\partial_p$ $N_2 = x\partial_u + \partial_p$ $N_3 = \partial_u$ $N_4 = \partial_y$ (General pt: $y = u = p = 0, q = x = 1$)	$\begin{array}{c\|cccccc} & S_1 & S_2 & N_1 & N_2 & N_3 & N_4 \\ \hline S_1 & \cdot & N_3 & -N_1 & \cdot & \cdot & -N_4 \\ S_2 & & \cdot & -N_1 & -N_2 & \cdot & \cdot \\ N_1 & & & \cdot & \cdot & \cdot & \cdot \\ N_2 & & & & \cdot & \cdot & \cdot \\ N_3 & & & & & \cdot & \cdot \\ N_4 & & & & & & \cdot \end{array}$	$E/\mathfrak{k}:$ $S_2, N_1 + N_2$ $V/\mathfrak{k}:$ $N_1 - N_4,$ $S_1 - N_2 + N_3$ $\mathfrak{k}:$ $N_1 + N_2 - N_3 - N_4$

Table 10 Classification of type D cases

	Model	Parameters	Symmetries	Lie algebra structure	Abstract ILC structure
D.7	$u_{11}=p^2$ $u_{12}=0$ $u_{22}=\lambda q^2$	$\lambda\neq 0,-1$ $(a\neq\pm 3/4)$	$X_1=\partial_x$ $H_1=-2x\partial_x+\partial_u+2p\partial_p$ $Y_1=-x^2\partial_x+x\partial_u+(1+2xp)\partial_p$ $X_2=\partial_y$ $H_2=-2y\partial_y+\frac{1}{\lambda}\partial_u+2q\partial_q$ $Y_2=-y^2\partial_y+\frac{1}{\lambda}y\partial_u+(\frac{1}{\lambda}+2yq)\partial_q$ $Z=\partial_u$ (General pt: $x=y=u=p=q=0$)	$\begin{array}{c\|ccccccc} & X_1 & H_1 & Y_1 & X_2 & H_2 & Y_2 & Z\\ \hline X_1 & \cdot & -2X_1 & H_1 & \cdot & \cdot & \cdot & \cdot\\ H_1 & & \cdot & -2Y_1 & \cdot & \cdot & \cdot & \cdot\\ Y_1 & & & \cdot & \cdot & \cdot & \cdot & \cdot\\ X_2 & & & & \cdot & -2X_2 & H_2 & \cdot\\ H_2 & & & & & \cdot & -2H_2 & \cdot\\ Y_2 & & & & & & \cdot & \cdot\\ Z & & & & & & & \cdot \end{array}$	$\mathfrak{k}:\ H_1-Z,\lambda H_2-Z$ $E/\mathfrak{k}:\ X_1,X_2$ $V/\mathfrak{k}:\ Y_1,Y_2$
	$u_{11}=p^2$ $u_{12}=0$ $u_{22}=0$	$\lambda=0$ $(a=\pm 3/4)$	$X_1=\partial_x$ $H_1=-2x\partial_x+\partial_u+2p\partial_p$ $Y_1=-x^2\partial_x+x\partial_u+(1+2xp)\partial_p$ $S=-y\partial_y+q\partial_q$ $X_2=\partial_y$ $Y_2=y\partial_u+\partial_q$ $Z=\partial_u$ (General pt: $x=y=u=p=q=0$)	$\begin{array}{c\|ccccccc} & X_1 & H_1 & Y_1 & S & X_2 & Y_2 & Z\\ \hline X_1 & \cdot & -2X_1 & H_1 & \cdot & \cdot & \cdot & \cdot\\ H_1 & & \cdot & -2Y_1 & \cdot & \cdot & \cdot & \cdot\\ Y_1 & & & \cdot & \cdot & \cdot & \cdot & \cdot\\ S & & & & \cdot & X_2 & -Y_2 & \cdot\\ X_2 & & & & & \cdot & Z & \cdot\\ Y_2 & & & & & & \cdot & \cdot\\ Z & & & & & & & \cdot \end{array}$	$\mathfrak{k}:\ H_1-Z,S$ $E/\mathfrak{k}:\ X_1,X_2$ $V/\mathfrak{k}:\ Y_1,Y_2$
D.6-1	$u_{11}=p^2-q^4/4$ $u_{12}=q(p-q^2/2)$ $u_{22}=p-q^2/2$		$X_1=\partial_x$ $H_1=-2x\partial_x-y\partial_y+\partial_u+2p\partial_p+q\partial_q$ $Y_1=-x^2\partial_x-xy\partial_y+(x+y^2/2)\partial_u$ $+(1+2xp+yq)\partial_p+(xq+y)\partial_q$ $X_2=\partial_y$ $Y_2=x\partial_y-y\partial_u-q\partial_p-\partial_q$ $Z=-\partial_u$ (General pt: $x=y=u=p=q=0$)	$\begin{array}{c\|cccccc} & X_1 & H_1 & Y_1 & X_2 & Y_2 & Z\\ \hline X_1 & \cdot & -2X_1 & H_1 & \cdot & X_2 & \cdot\\ H_1 & & \cdot & -2Y_1 & X_2 & -Y_2 & \cdot\\ Y_1 & & & \cdot & Y_2 & \cdot & \cdot\\ X_2 & & & & \cdot & Z & \cdot\\ Y_2 & & & & & \cdot & \cdot\\ Z & & & & & & \cdot \end{array}$	$\mathfrak{k}:\ H_1+Z$ $E/\mathfrak{k}:\ X_1,X_2$ $V/\mathfrak{k}:\ Y_1,Y_2$
D.6-2	$u_{11}=p^\mu$ $u_{12}=0$ $u_{22}=0$	$\mu\neq 0,1,2,\infty$ $(a\neq 1,2/3,4/3)$	$S_1=-y\partial_y+q\partial_q$ $S_2=-\frac{\mu-1}{\mu-2}x\partial_x-y\partial_y-u\partial_u+\frac{1}{\mu-2}p\partial_p$ $X=\partial_y$ $Y=y\partial_u+\partial_q$ $Z=\partial_u$ $Z_1=\partial_x$ (General pt: $x=y=u=q=0,p=1$)	$\begin{array}{c\|cccccc} & S_1 & S_2 & X & Y & Z & Z_1\\ \hline S_1 & \cdot & \cdot & X & -Y & \cdot & \cdot\\ S_2 & & \cdot & X & \cdot & Z & \frac{\mu-1}{\mu-2}Z_1\\ X & & & \cdot & Z & \cdot & \cdot\\ Y & & & & \cdot & \cdot & \cdot\\ Z & & & & & \cdot & \cdot\\ Z_1 & & & & & & \cdot \end{array}$	$\mathfrak{k}:\ S_1$ $E/\mathfrak{k}:\ (\mu-2)S_2+Z+Z_1,X$ $V/\mathfrak{k}:\ S_2,Y$
	$u_{11}=\exp(p)$ $u_{12}=0$ $u_{22}=0$	$\mu=\infty$ $(a=4/3)$	$S_1=-y\partial_y+q\partial_q$ $S_2=-x\partial_x-y\partial_y-(u-x)\partial_u+\partial_p$ $X=\partial_y$ $Y=y\partial_u+\partial_q$ $Z=\partial_u$ $Z_1=\partial_x$ (General pt: $x=y=u=p=q=0$)	$\begin{array}{c\|cccccc} & S_1 & S_2 & X & Y & Z & Z_1\\ \hline S_1 & \cdot & \cdot & X & -Y & \cdot & \cdot\\ S_2 & & \cdot & X & \cdot & Z & -Z+Z_1\\ X & & & \cdot & Z & \cdot & \cdot\\ Y & & & & \cdot & \cdot & \cdot\\ Z & & & & & \cdot & \cdot\\ Z_1 & & & & & & \cdot \end{array}$	$\mathfrak{k}:\ S_1$ $E/\mathfrak{k}:\ S_2+Z_1,X$ $V/\mathfrak{k}:\ S_2,Y$

Table 10 continued

Model	Parameters	Symmetries	Lie algebra structure	Abstract ILC structure
D.6-3 $u_{11} = \lambda p^2 \frac{\sqrt{u-pq}}{u^{3/2}}$ $u_{12} = 1 + \lambda(pq-2u)\frac{\sqrt{u-pq}}{u^{3/2}}$ $u_{22} = \lambda q^2 \frac{\sqrt{u-pq}}{u^{3/2}}$	$\lambda \neq 0, \pm 1/2$ $(a \neq 0, \pm 3)$	$X_1 = \partial_x$ $H_1 = -2x\partial_x - 2u\partial_u - 2q\partial_q$ $Y_1 = -x^2\partial_x - u\partial_y - 2xu\partial_u +$ $(pq-2u)\partial_p + (q^2-2xq)\partial_q$ $X_2 = \partial_y$ $H_2 = -2y\partial_y - 2u\partial_u - 2p\partial_p$ $Y_2 = -u\partial_x - y^2\partial_y - 2yu\partial_u +$ $(p^2-2py)\partial_p + (pq-2u)\partial_q$ (General pt: $x=y=p=q=0, u=1$)	$\begin{array}{c\|cccccc} & X_1 & H_1 & Y_1 & X_2 & H_2 & Y_2 \\ \hline X_1 & \cdot & -2X_1 & H_1 & \cdot & \cdot & \cdot \\ H_1 & & \cdot & -2Y_1 & \cdot & \cdot & \cdot \\ Y_1 & & & \cdot & \cdot & \cdot & \cdot \\ X_2 & & & & \cdot & -2X_2 & H_2 \\ H_2 & & & & & \cdot & -2Y_2 \\ Y_2 & & & & & & \cdot \end{array}$	$\mathfrak{k}: \quad H_1 - H_2$ $E/\mathfrak{k}: \quad X_1 + \frac{2\lambda-1}{2\lambda+1}Y_2,$ $\qquad\qquad X_2 + \frac{2\lambda-1}{2\lambda+1}Y_1$ $V/\mathfrak{k}: \quad X_1+Y_2, X_2+Y_1$
$u_{11} = p^2\sqrt{1-2pq}$ $u_{12} = (pq-1)\sqrt{1-2pq}$ $u_{22} = q^2\sqrt{1-2pq}$	$(a = \pm 3)$	$X = -u\partial_x - y\partial_u + p^2\partial_p + (pq-1)\partial_q$ $H = -x\partial_x + y\partial_y + p\partial_p - q\partial_q$ $Y = -u\partial_y - x\partial_u + (pq-1)\partial_p + q^2\partial_q$ $E_1 = \partial_x$ $E_2 = \partial_u$ $E_3 = \partial_y$ (General pt: $x=y=u=0, p=q=1$)	$\begin{array}{c\|cccccc} & X & H & Y & E_1 & E_2 & E_3 \\ \hline X & \cdot & -X & H & \cdot & E_1 & E_2 \\ H & & \cdot & -Y & E_1 & \cdot & -E_3 \\ Y & & & \cdot & E_2 & E_3 & \cdot \\ E_1 & & & & \cdot & \cdot & \cdot \\ E_2 & & & & & \cdot & \cdot \\ E_3 & & & & & & \cdot \end{array}$	$\mathfrak{k}: \quad H$ $E/\mathfrak{k}: \quad X+E_1, Y+E_2$ $V/\mathfrak{k}: \quad X, Y$
D.6-4 $u_{11} = 0$ $u_{12} = \frac{1+pq}{u}$ $u_{22} = 0$		$X_1 = \partial_x$ $H_1 = -2x\partial_x - u\partial_u + p\partial_p - q\partial_q$ $Y_1 = -x^2\partial_x - xu\partial_u + (xp-u)\partial_p - xq\partial_q$ $X_2 = \partial_y$ $H_2 = -2y\partial_y - u\partial_u - p\partial_p + q\partial_q$ $Y_2 = -y^2\partial_y - yu\partial_u - yp\partial_p + (yq-u)\partial_q$ (General pt: $x=y=p=q=0, u=1$)	$\begin{array}{c\|cccccc} & X_1 & H_1 & Y_1 & X_2 & H_2 & Y_2 \\ \hline X_1 & \cdot & -2X_1 & H_1 & \cdot & \cdot & \cdot \\ H_1 & & \cdot & -2Y_1 & \cdot & \cdot & \cdot \\ Y_1 & & & \cdot & \cdot & \cdot & \cdot \\ X_2 & & & & \cdot & -2X_2 & H_2 \\ H_2 & & & & & \cdot & -2Y_2 \\ Y_2 & & & & & & \cdot \end{array}$	$\mathfrak{k}: \quad H_1 - H_2$ $E/\mathfrak{k}: \quad X_1 - Y_2, X_2 - Y_1$ $V/\mathfrak{k}: \quad Y_1, Y_2$

Table 11 Classification of type III cases

Model	Symmetries	Lie algebra structure	Abstract ILC structure
III.6-1 $u_{11} = p/(x-q)$, $u_{12} = 0$, $u_{22} = 0$	$S_1 = -x\partial_x - u\partial_u - q\partial_q$ $S_2 = -y\partial_y - u\partial_u - p\partial_p$ $N_1 = \partial_x + y\partial_u + \partial_q$ $N_2 = x\partial_y + \frac{x^2}{2}\partial_u + (x-q)\partial_p$ $N_3 = \partial_y$ $N_4 = -\partial_u$ (General pt: $y = u = p = q = 0, x = 1$)	$[S_1,S_2]=0$, $[S_1,N_1]=N_1$, $[S_1,N_2]=-N_2$, $[S_1,N_3]=0$, $[S_1,N_4]=N_4$ $[S_2,N_1]=0$, $[S_2,N_2]=N_2$, $[S_2,N_3]=N_3$, $[S_2,N_4]=N_4$ $[N_1,N_2]=N_3$, $[N_1,N_3]=N_4$, $[N_1,N_4]=0$ $[N_2,N_3]=0$, $[N_2,N_4]=0$ $[N_3,N_4]=0$	$\mathfrak{k}:\ S_2$ $E/\mathfrak{k}:\ S_1, N_3$ $V/\mathfrak{k}:\ S_1+N_1, N_2-N_3+\frac{1}{2}N_4$
III.6-2 $u_{11} = 2q(2p-qu)$, $u_{12} = q^2$, $u_{22} = 0$	$X = \partial_x$ $H = -2x\partial_x - y\partial_y + u\partial_u + 3p\partial_p + 2q\partial_q$ $Y = -x^2\partial_x - xy\partial_y + (ux+y)\partial_u$ $+(3px+u+qy)\partial_p + (2qx+1)\partial_q$ $S = -y\partial_y - u\partial_u - p\partial_p$ $N_1 = \partial_y$ $N_2 = x\partial_y - \partial_u - q\partial_p$ (General pt: $x = y = u = q = 0, p = 1$)	$[X,H]=-2X$, $[X,Y]=H$, $[X,S]=0$, $[X,N_1]=0$, $[X,N_2]=N_1$ $[H,Y]=-2Y$, $[H,S]=0$, $[H,N_1]=N_1$, $[H,N_2]=-N_2$ $[Y,S]=0$, $[Y,N_1]=N_2$, $[Y,N_2]=0$ $[S,N_1]=N_1$, $[S,N_2]=N_2$ $[N_1,N_2]=0$	$\mathfrak{k}:\ H+3S$ $E/\mathfrak{k}:\ X-N_2, N_1$ $V/\mathfrak{k}:\ S, Y$

Table 12 Basis change from Cartan reduced basis to adapted Lie algebra basis

	Parameters change	Basis change
N.8		$S_1 = -\frac{1}{2}e_7 - \frac{3}{2}e_8$, $S_1 = -\frac{1}{2}e_8 - \frac{3}{2}e_7$, $N_1 = \frac{1}{2}e_4$, $N_2 = -2e_1$, $N_3 = -e_6$, $N_4 = -\frac{1}{2}e_3$, $N_5 = -e_5$, $N_6 = -2e_2$
N.7-1	$\kappa = -\frac{3}{2} + \frac{3a}{2\sqrt{a^2+4}}$, $\kappa \neq -1, -2, \infty$	$S_1 = -\frac{1}{2}e_6$, $S_2 = \frac{1}{\sqrt{a^2+4}}e_1 - \frac{3a}{2\sqrt{a^2+4}}e_6$, $N_1 = -\frac{1}{\sqrt{a^2+4}(3a+\sqrt{a^2+4})}\left(e_2 + (2a^2-1)e_4 + 2ae_7\right)$, $N_2 = -\frac{1}{2(2a^2-1)}\left(e_2 + \frac{a+\sqrt{a^2+4}}{2}e_7\right)$,
	$a = \frac{2\kappa+3}{\sqrt{-\kappa^2-3\kappa}}$, $a \neq \pm\frac{1}{\sqrt{2}}, \pm 2i$	$N_3 = \frac{3a-\sqrt{a^2+4}}{8(a^2+4)^{\frac{3}{2}}}\left(\frac{a+\sqrt{a^2+4}}{2}e_3 + e_5\right)$, $N_4 = \frac{3a-\sqrt{a^2+4}}{8(a^2+4)^{\frac{3}{2}}}\left(\frac{a-\sqrt{a^2+4}}{2}e_3 + e_5\right)$, $N_5 = -\frac{1}{2(2a^2-1)}\left(e_2 + \frac{a-\sqrt{a^2+4}}{2}e_7\right)$
N.7-1	$\kappa = -1$	$S_1 = -\frac{1}{2}e_6$, $S_2 = \frac{\sqrt{2}}{3}e_1 - \frac{1}{2}e_6$, $N_1 = \frac{2}{9}e_2 - e_4 - \frac{\sqrt{2}}{9}e_7$, $N_2 = -\frac{1}{9}e_2 - \frac{\sqrt{2}}{9}e_7$,
	$a = \frac{1}{\sqrt{2}}$	$N_3 = \frac{\sqrt{2}}{9}e_3 + \frac{1}{9}e_5$, $N_4 = -\frac{1}{9\sqrt{2}}e_3 + \frac{1}{9}e_5$, $N_5 = -\frac{1}{9}e_2 + \frac{1}{9\sqrt{2}}e_7$
N.7-1	$\kappa = \infty$	$S_1 = -\frac{1}{2}e_6$, $S_2 = \frac{i}{3}e_1 - e_6$, $N_1 = -\frac{i}{3}e_2 + 3ie_4 - \frac{4}{3}e_7$, $N_2 = -\frac{1}{2}e_7$,
	$a = 2i$	$N_3 = -\frac{3i}{2}e_3$, $N_4 = -\frac{i}{2}e_3 + \frac{1}{2}e_5$, $N_5 = -\frac{i}{6}e_2 - \frac{1}{6}e_7$
N.7-2		$X = -e_1 - \frac{1}{2}e_2 + e_4 + \frac{1}{2}e_5 + 2e_6 + \frac{1}{2}e_7$, $H = -2e_4 - \frac{1}{2}e_5 - 2e_6 - \frac{1}{2}e_7$, $Y = -e_4$, $N_1 = \frac{1}{2}e_3$, $N_2 = e_3 + \frac{1}{2}e_5 - \frac{1}{2}e_7$, $N_3 = -e_2 + e_3 + e_5 - e_7$, $N_4 = -\frac{1}{2}e_5 - \frac{1}{2}e_7$

Table 12 continued

	Parameters change	Basis change
N.6-1	$\mu=\frac{2a^2-1}{a^2+1}\neq 0,1,$	$S=\frac{1}{a}e_4$, $N_1=-\frac{a}{a^2+1}e_1-\frac{a^2+1}{a(2a^2-1)(a^2-2)}e_3+\frac{1}{a}e_4+\frac{2(a^2-1)}{(2a^2-1)(a^2-2)}e_6$, $N_2=-\frac{1}{(2a^2-1)(a^2-2)}\left(\frac{a^2+1}{a}e_3+a^2e_6\right)$,
	$a^2=\frac{1+\mu}{2-\mu}\neq\frac{1}{2},2$	$N_3=\frac{1}{(2a^2-1)(a^2-2)}\left(\frac{a^3}{a^2+1}e_2+\frac{a^2+2}{a}e_3-e_5+\frac{1}{a^2+1}e_6\right)$, $N_4=\frac{1}{(2a^2-1)(a^2-2)}\left(\frac{2a}{a^2+1}e_2-\frac{2}{a}e_3+2e_5-\frac{2}{a^2+1}e_6\right)$, $N_5=\frac{1}{(2a^2-1)(a^2-2)}\left(\frac{a^3}{a^2+1}e_2-ae_3-e_5-\frac{2a^2+1}{a^2+1}e_6\right)$
	$\mu=0,\ a^2=\frac{1}{2}$	$S=\frac{\sqrt{2}}{3}e_3+\sqrt{2}e_4-\frac{1}{9}e_6$, $N_1=-\frac{\sqrt{2}}{3}e_1+\sqrt{2}e_4-\frac{2}{3}e_6$, $N_2=\frac{2\sqrt{2}}{3}e_3-\frac{2}{9}e_6$, $N_3=-\frac{2\sqrt{2}}{27}e_2-\frac{10\sqrt{2}}{9}e_3+\frac{4}{9}e_5-\frac{8}{27}e_6$, $N_4=-\frac{8\sqrt{2}}{27}e_2+\frac{8\sqrt{2}}{9}e_3-\frac{8}{9}e_5+\frac{16}{27}e_6$, $N_5=\frac{\sqrt{2}}{27}e_2-\frac{\sqrt{2}}{9}e_3-\frac{2}{9}e_5-\frac{8}{27}e_6$
	$\mu=1,\ a^2=2$	$S=\frac{1}{\sqrt{2}}e_4$, $N_1=-\frac{\sqrt{2}}{3}e_1+\frac{1}{\sqrt{2}}e_4+\frac{1}{3}e_6$, $N_2=-\frac{1}{3\sqrt{2}}e_3+\frac{2}{9}e_6$, $N_3=\frac{2\sqrt{2}}{27}e_2+\frac{2\sqrt{2}}{9}e_3-\frac{1}{9}e_5+\frac{1}{27}e_6$, $N_4=\frac{2\sqrt{2}}{27}e_2-\frac{\sqrt{2}}{9}e_2+\frac{2}{9}e_5-\frac{2}{27}e_6$, $N_5=\frac{2\sqrt{2}}{27}e_2-\frac{\sqrt{2}}{9}e_3-\frac{1}{9}e_5-\frac{5}{27}e_6$
N.6-2	$\mu=\frac{1}{2}+\frac{3b}{2\sqrt{b^2+4}}$, $b=\frac{2\mu-1}{\sqrt{-\mu^2+\mu+2}}$, $\kappa=-\frac{3}{2}+\frac{3a}{2\sqrt{a^2+4}}$, $a=\frac{2\kappa+3}{\sqrt{-\kappa^2-3\kappa}}$, $b\neq\pm\frac{1}{\sqrt{2}},\pm 2i$, $\mu\neq 0,1,\infty$;	$S_1=\frac{-1}{\sqrt{b^2+4}}e_4$, $S_2=\frac{-1}{\sqrt{a^2+4}}e_1+\frac{1}{(1-2b^2)\sqrt{a^2+4}}e_3+\frac{2b}{(1-2b^2)\sqrt{a^2+4}}e_6$, $N_1=\frac{1}{(2b^2-1)(a^2+4)}\left(\frac{-b-\sqrt{b^2+4}}{2}e_2+\frac{a+\sqrt{a^2+4}}{2}e_3-e_5+\frac{(a+\sqrt{a^2+4})(\sqrt{b^2+4}+b)}{4}e_6\right)$, $N_2=\frac{1}{(2b^2-1)(a^2+4)}\left(\frac{b-\sqrt{b^2+4}}{2}e_2-\frac{a+\sqrt{a^2+4}}{2}e_3+e_5+\frac{(a+\sqrt{a^2+4})(\sqrt{b^2+4}-b)}{4}e_6\right)$,

Table 12 continued

Parameters change	Basis change
$a \neq \pm 2i, \kappa \neq \infty$	$N_3 = \frac{1}{(2b^2-1)(a^2+4)}\left(\frac{b-\sqrt{b^2+4}}{2}e_2 - \frac{a-\sqrt{a^2+4}}{2}e_3 + e_5 + \frac{(a-\sqrt{a^2+4})(\sqrt{b^2+4}-b)}{4}e_6\right)$, $N_4 = \frac{1}{(2b^2-1)(a^2+4)}\left(\frac{-b-\sqrt{b^2+4}}{2}e_2 + \frac{a-\sqrt{a^2+4}}{2}e_3 - e_5 + \frac{(a-\sqrt{a^2+4})(\sqrt{b^2+4}+b)}{4}e_6\right)$,
$\mu = \infty,\ b = 2i,\ \kappa = -\frac{3}{2} + \frac{3a}{2\sqrt{a^2+4}}$	$S_1 = \frac{i}{3}e_4$, $S_2 = \frac{-1}{\sqrt{a^2+4}}\left(e_1 - \frac{1}{9}e_3 - \frac{4i}{9}e_6\right)$, $N_1 = -\frac{i}{3(a^2+4)}\left(e_2 - \frac{a+\sqrt{a^2+4}}{2}e_6\right)$, $N_2 = \frac{1}{9(a^2+4)}\left(ie_2 - \frac{a+\sqrt{a^2+4}}{2}e_3 + e_5 - \frac{i(a+\sqrt{a^2+4})}{2}e_6\right)$
$a = \frac{2\kappa+3}{\sqrt{-\kappa^2-3\kappa}}, a \neq \pm 2i, \kappa \neq \infty$	$N_3 = \frac{1}{9(a^2+4)}\left(ie_2 + \frac{\sqrt{a^2+4}-a}{2}e_3 + e_5 + \frac{i(\sqrt{a^2+4}-a)}{2}e_6\right)$, $N_4 = -\frac{i}{3(a^2+4)}\left(e_2 + \frac{\sqrt{a^2+4}-a}{2}e_6\right)$
$\mu = \infty,\ b = \pm 2i,$	$S_1 = -\frac{i}{3}e_4$, $S_2 = -\frac{i}{3}e_1 + \frac{i}{27}e_3 + \frac{4}{27}e_6$, $N_1 = -\frac{1}{9}e_6$, $N_2 = \frac{i}{27}e_3 + \frac{1}{27}e_6$, $N_3 = -\frac{i}{81}e_2 + \frac{i}{81}e_3 + \frac{1}{81}e_5 + \frac{1}{81}e_6$, $N_4 = \frac{i}{27}e_2 - \frac{1}{27}e_6$
$\kappa = \infty,\ a = \pm 2i$	
$\mu = 0,\ b = \frac{1}{\sqrt{2}},\ \kappa = -\frac{3}{2} + \frac{3a}{2\sqrt{a^2+4}}$	$S_1 = \frac{\sqrt{2}}{3}e_4 + \frac{2}{9(2a^2-1)}\left(\sqrt{2}e_2 - 2ae_3 + e_5 - 2\sqrt{2}ae_6\right)$, $S_2 = -\frac{1}{\sqrt{a^2+4}}\left(e_1 + \frac{\sqrt{2}}{3}e_6\right)$,
$a = \frac{2\kappa+3}{\sqrt{-\kappa^2-3\kappa}}, a \neq \pm 2i, \kappa \neq \infty$	$N_1 = -\frac{1}{9(a^2+4)}\left(\sqrt{2}e_2 - 2e_5 + \left(a + \sqrt{a^2+4}\right)\left(e_3 - \frac{1}{\sqrt{2}}e_6\right)\right)$, $N_2 = -\frac{1}{9(a^2+4)}\left(2\sqrt{2}e_2 + 2e_5 + \left(\sqrt{a^2+4}+a\right)\left(e_3 + \sqrt{2}e_6\right)\right)$, $N_3 = -\frac{1}{9(a^2+4)}\left(2\sqrt{2}e_2 + 2e_5 + \left(\sqrt{a^2+4}-a\right)\left(e_3 + \sqrt{2}e_6\right)\right)$, $N_4 = -\frac{1}{9(a^2+4)}\left(\sqrt{2}e_2 - 2e_5 + \left(a - \sqrt{a^2+4}\right)\left(e_3 - \frac{1}{\sqrt{2}}e_6\right)\right)$
$\mu = 0,\ b = \frac{1}{\sqrt{2}},$	$S_1 = \frac{2\sqrt{2}}{27}e_3 + \frac{\sqrt{2}}{3}e_4 + \frac{4}{27}e_6$, $S_2 = -\frac{\sqrt{2}}{3}e_1 - \frac{2}{9}e_6$, $N_1 = -\frac{2\sqrt{2}}{81}e_2 - \frac{2\sqrt{2}}{81}e_3 + \frac{4}{81}e_5 + \frac{2}{81}e_6$, $N_2 = -\frac{4\sqrt{2}}{81}e_2 + \frac{2\sqrt{2}}{81}e_3 - \frac{4}{81}e_5 + \frac{4}{81}e_6$,
$\kappa = -2, a = -\frac{1}{\sqrt{2}}$	$N_3 = -\frac{4}{81}\left(\sqrt{2}e_2 + \sqrt{2}e_3 + e_5 + 2e_6\right)$, $N_4 = -\frac{2\sqrt{2}}{81}\left(e_2 - 2e_3 - \sqrt{2}e_5 + \sqrt{2}e_6\right)$,

Table 12 continued

	Parameters change	Basis change
D.7	$\lambda = \frac{3+4a}{3-4a}, a \neq \pm\frac{3}{4},$	$X_1 = \frac{2}{4a-3}e_1, H_1 = -\frac{1}{4a-3}(2e_5 + 2(2a-1)e_6 - (2a-3)e_7), Y_1 = e_3, \quad X_2 = -e_2, H_2 = \frac{1}{4a+3}(2e_5 - 2(2a+1)e_6 - (2a+3)e_7), Y_2 = \frac{2}{4a+3}e_4,$
	$a = \frac{3}{4}\frac{\lambda-1}{\lambda+1}, \lambda \neq 0, -1$	$Z = -\frac{1}{4a-3}(2e_5 + e_6 + 2ae_7)$
	$a = \pm\frac{3}{4}, \lambda = 0$	$X_1 = -\frac{1}{3}e_4, H_1 = -\frac{1}{3}e_5 + \frac{5}{6}e_6 + \frac{3}{4}e_7, Y_1 = e_2, \quad S = \frac{1}{2}(e_6 - e_7), X_2 = -\frac{1}{3}e_3, Y_2 = e_1, \quad Z = -\frac{1}{3}e_5 - \frac{1}{6}e_6 - \frac{1}{4}e_7$
D.6-1		$X_1 = \frac{1}{\sqrt{2}}e_1, H_1 = -\frac{1}{2}e_5 + \frac{1}{8}e_6, Y_1 = \frac{1}{\sqrt{2}}e_3, \quad X_2 = \sqrt{2}e_2, Y_2 = -\sqrt{2}e_4, Z = \frac{1}{2}e_5 + \frac{3}{8}e_6$
D.6-2	$\mu = \frac{6(a-1)}{3a-4}, a \neq 1, 2/3, 4/3$	$S_1 = \frac{1}{2}e_6, S_2 = -\frac{3a-2}{12}(2(a-1)e_4 + 3e_6), X = e_1, Y = -\frac{(a-1)(3a-2)}{(3a-4)^2}e_3, Z = -\frac{1}{(3a-4)^2}\left(6e_2 + \frac{2(a-1)(3a-2)}{3}e_4 - (a-1)(3a-2)e_5 - \frac{9a^2-15a-2}{4}e_6\right),$
	$a = \frac{2}{3}\frac{2\mu-3}{\mu-2}, \mu \neq 0, 1, 2, \infty$	$Z_1 = \frac{1}{(3a-4)^2}\left(3(3a-2)e_2 + \frac{(a-1)(3a-2)^2}{3}e_4 - (a-1)(3a-2)e_5 + \frac{9(a-1)(3a-2)}{4}e_6\right)$
	$a = 4/3, \mu = \infty$	$S_1 = -\frac{1}{2}e_6, S_2 = -\frac{1}{9}e_4 - \frac{1}{2}e_6, X = -\frac{1}{6}e_3, Y = e_1, Z = \frac{3}{2}e_2 + \frac{1}{9}e_4 - \frac{1}{6}e_5 + \frac{3}{8}e_6, Z_1 = \frac{3}{2}e_2 + \frac{1}{9}e_4 + \frac{3}{2}e_6$
D.6-3	$\lambda = \frac{3}{2\sqrt{9-a^2}}, a \neq 0, \pm 3$	$X_1 = \lambda e_1 + \frac{\lambda(2\lambda-1)}{\sqrt{4\lambda^2-1}}e_4, H_1 = \frac{4\lambda}{3}e_5 + e_6, Y_1 = -\frac{2}{3}\sqrt{4\lambda^2-1}e_2 - \frac{2}{3}(2\lambda+1)e_3,$
	$a = \frac{3\sqrt{4\lambda^2-1}}{2\lambda}, \lambda \neq 0, \pm 1/2$	$X_2 = \frac{2}{3}\sqrt{4\lambda^2-1}e_2 + \frac{2}{3}(2\lambda+1)e_3, H_2 = \frac{4\lambda}{3}e_5 - e_6, Y_2 = -\lambda e_1 - \frac{\lambda(2\lambda+1)}{\sqrt{4\lambda^2-1}}e_4$
	$a = \pm 3$	$X = -e_4, H = e_6, Y = -\frac{2}{3}e_3, \quad E_1 = e_1 + e_4, E_2 = -\frac{2}{3}e_5, E_3 = \frac{2}{3}(e_2 + e_3)$
D.6-4		$X_1 = -2e_1 + \frac{1}{3}e_4, H_1 = \frac{2}{3}e_5 + e_6, Y_1 = \frac{1}{3}e_3, \quad X_2 = -2e_2 + \frac{1}{3}e_3, H_2 = \frac{2}{3}e_5 - e_6, Y_2 = \frac{1}{3}e_4$
III.6-1		$S_1 = 2e_1 + \frac{5}{4}e_6, S_2 = -\frac{1}{2}e_6, N_1 = -2e_1 + e_4 - \frac{9}{4}e_6, N_2 = -\frac{1}{2}e_2 + \frac{1}{8}e_3 + \frac{1}{2}e_5, N_3 = -e_2, N_4 = -e_2 + \frac{3}{4}e_3 - e_5$
III.6-2		$X = 2e_1 + e_5, H = \frac{3}{2}e_3 + \frac{5}{2}e_6, Y = -2e_4, S = -\frac{1}{2}(e_3 + e_6), N_1 = -\frac{1}{2}e_2, N_2 = e_5$

Table 13 Basis change which reflects redundancy in parameters

	Parameters change	Basis change
N.7-1	$a \to -a, (\kappa \to -\kappa - 3),$	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (-e_1, e_2, -e_3, e_4, e_5, -e_6)$
N.6-1	$a \to -a$	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (-e_1, -e_2, -e_3, -e_4, e_5, e_6)$
N.6-2	$a \to -a, (\kappa \to -\kappa - 3),$	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (-e_1, e_2, -e_3, e_4, e_5, -e_6)$
	$b \to -b, (\mu \to -\mu + 1)$	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (e_1, -e_2, e_3, -e_4, e_5, -e_6)$
D.7	$a \to -a, (\lambda \to 1/\lambda)$	$(e_1, e_2, e_3, e_4, e_5, e_6, e_7) \to (e_2, e_1, e_4, e_3, e_5, e_6, -e_7)$
D.6-3	$a \to -a, (\lambda \to -\lambda)$	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (e_2, -e_1, e_4, -e_3, e_5, -e_6)$

Table 14 Duality

	Parameters change	Basis change
N.8	(self-dual)	$(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8) \to (e_4, e_3, e_2, e_1, -e_5, -e_6, e_8, e_7)$
N.7-2	(self-dual)	$(e_1, e_2, e_3, e_4, e_5, e_6, e_7) \to (e_4, e_3, e_2, e_1, -e_5, -e_6, -e_7)$
N.6-1	(self-dual)	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (\frac{a^2+1}{a^2} e_4, \frac{a^2+1}{a^2} e_3, \frac{a^2}{a^2+1} e_2, \frac{a^2}{a^2+1} e_1, -e_5, -e_6)$
N.6-2	$(a, b) \to (b, a),$ $(\mu, \kappa) \to (\kappa + 2, \mu - 2)$	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (e_4, e_3, e_2, e_1, -e_5, -e_6)$
D.7	(self-dual)	$(e_1, e_2, e_3, e_4, e_5, e_6, e_7) \to (e_3, e_4, e_1, e_2, -e_5, -e_6, -e_7)$
D.6-1	(self-dual)	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (e_3, e_4, e_1, e_2, -e_5, -e_6)$
D.6-2	(self-dual)	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (e_3, -\frac{(a-1)(3a-2)}{9} e_4, e_1, -\frac{9}{(a-1)(3a-2)} e_2, -e_5, -e_6)$
D.6-3	(self-dual)	$(e_1, e_2, e_3, e_4, e_5, e_6) \to (e_3, e_4, e_1, e_2, -e_5, -e_6)$

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