GLOBAL EXISTENCE OF SOLUTIONS OF SEMILINEAR HEAT EQUATION WITH NONLINEAR MEMORY CONDITION

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Abstract. We consider a semilinear parabolic equation with flux at the boundary governed by a nonlinear memory. We give some conditions for this problem which guarantee global existence of solutions as well as blow up in finite time of all nontrivial solutions. The results depend on the behavior of variable coefficients as $t \to \infty$.

1. Introduction

We investigate the global solvability and blow-up in finite time for a semilinear heat equation with a nonlinear memory boundary condition:

$$u_t = \Delta u + c(t)u^p \ \text{ for } \ x \in \Omega, \ \ t > 0, \tag{1.1}$$

$$\frac{\partial u(x,t)}{\partial \nu} = k(t) \int_0^t u^q(x,\tau)\,d\tau \ \text{ for } \ x \in \partial\Omega, \ \ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) \ \text{ for } \ x \in \Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ for $n \geq 1$ with smooth boundary $\partial\Omega$, ν is unit outward normal on $\partial\Omega$, $p > 0$ and $q > 0$. Here $c(t)$ and $k(t)$ are nonnegative continuous functions for $t \geq 0$. The initial datum $u_0(x)$ is a nonnegative $C^1(\overline{\Omega})$ function which satisfies the boundary condition at $t = 0$.

In the literature for parabolic equations, memory terms in the boundary flux appear in many references. For example, in [1] a memory term (1.2) with $k(t) \equiv 1$, $q = 1$ is introduced for the study of Newtonian radiation and calorimetry. A linear memory boundary condition takes into account the hereditary effects on the boundary as those studied in [2], [3]. In the paper [4] similar hereditary boundary conditions have been employed in models of time-dependent electromagnetic fields at dissipative boundaries. A nonlinear memory boundary condition arises in a model of capillary growth in solid tumors as initiated by angiogenic growth factors, for example (see [5]).

Global existence and blow-up in finite time of solutions for variety parabolic problems with memory boundary conditions have been studied in many papers (see, for example, [6, 7, 8, 9, 10, 11, 12, 13, 14] and the references therein).

Let $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $\Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$, $T > 0$.

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Definition 1.1. We say that a nonnegative function $u \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a subsolution of problem (1.1)–(1.3) in Q_T if

$$\begin{cases} u_t \le \Delta u + c(t)u^p \text{ for } (x,t) \in Q_T, \\ \frac{\partial u(x,t)}{\partial \nu} \le k(t) \int_0^t u^q(x,\tau)\, d\tau \text{ for } (x,t) \in S_T, \\ u(x,0) \le u_0(x) \text{ for } x \in \Omega, \end{cases} \tag{1.4}$$

and $u \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T)$ is a supersolution if $u \ge 0$ and it satisfies (1.4) in the reverse order. We say that u is a solution of problem (1.1)–(1.3) in Q_T if it is both a subsolution and a supersolution of (1.1)–(1.3) in Q_T.

Local existence of solutions and comparison principle for (1.1)–(1.3) may be developed using the same techniques as in [8], [15]. We formulate comparison principle which will be used below.

Theorem 1.2. *Let $u(x,t)$ and $v(x,t)$ be a supersolution and a subsolution of problem (1.1)–(1.3) in Q_T, respectively. Suppose that $u(x,t) > 0$ or $v(x,t) > 0$ in $Q_T \cup \Gamma_T$ if $\min(p,q) < 1$. Then $u(x,t) \ge v(x,t)$ in $Q_T \cup \Gamma_T$.*

In this paper we analyze the influence of variable coefficients on global existence and blow-up in finite time of classical solutions of problem (1.1)–(1.3). Our global existence and blow-up results depend on the behavior of the functions $c(t)$ and $k(t)$ as $t \to \infty$.

This paper is organized as follows. In the next section we show that all nonnegative solutions are global for $\max(p,q) \le 1$ and present finite time blow-up of all nontrivial solutions for $\max(p,q) > 1$ as well as the existence of bounded global solutions for small initial data for $\min(p,q) > 1$. In section 3 we investigate the case $p = 1, q > 1$.

2. Finite time blow-up and global existence

We begin with the global existence of solutions of (1.1)–(1.3). The proof relies on the continuation principle and the construction of a supersolution.

Theorem 2.1. *If $\max(p,q) \le 1$, then every solution of (1.1)–(1.3) is global.*

Proof. We seek a positive supersolution $\overline{u}$ of (1.1)–(1.3) in Q_T for any positive T. Since $c(t)$ and $k(t)$ are continuous functions there exists a constant $M > 0$ such that $\max(c(t), k(,t)) \le M$ for $t \in [0,T]$. Let λ_1 be the first eigenvalue of the following problem

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, \ x \in \Omega, \\ \varphi(x) = 0, \ x \in \partial\Omega \end{cases}$$

and $\varphi(x)$ be the corresponding eigenfunction with $\sup_\Omega \varphi(x) = 1$. It is well known $\varphi(x) > 0$ in Ω and $\max_{\partial\Omega} \partial\varphi(x)/\partial\nu < 0$. We define

$$\overline{u} = d \exp(bt)[2 - \varphi(x)],$$

where

$$b \ge \max\left(\lambda_1 + 2M, 2M \max_{\partial\Omega}\left(-q\frac{\partial\varphi}{\partial\nu}\right)^{-1}\right), \ d \ge \left\{\sup_\Omega u_0(x), 1\right\}.$$

Then $\overline{u}$ satisfies

$$\begin{array}{ll} \overline{u}_t \geq \Delta\overline{u} + c(t)\overline{u}^p & \text{for } (x,t) \in Q_T, \\ \frac{\partial\overline{u}(x,t)}{\partial\nu} \geq k(t)\int_0^t \overline{u}^q(x,\tau)\,d\tau & \text{for } (x,t) \in S_T, \\ \overline{u}(x,0) \geq u_0(x) & \text{for } x \in \Omega. \end{array}$$

Hence, $\overline{u}$ is the desired supersolution and by Theorem 1.2 problem (1.1)–(1.3) has a global solution for any initial datum. □

We need the following assertion which was proved in [16] for a more general case.

Theorem 2.2. *Let $y(a) \geq 0$, $y'(a) \geq 0$, $y(a) + y'(a) > 0$, $q > 1$, $b(r)$ be a nonnegative continuous function for $r \geq a$. Then for $r > a$ the inequality*

$$y''(r) \geq b(r)y^q(r)$$

has no global solutions if

$$\int_a^\infty r^q b(r)\,dr = \infty$$

and at least one of the following conditions is fulfilled

$$b(r) \leq \frac{B}{r^{q+1}} \text{ for large values of } r,\ B > 0,$$

or

$$b(r) \text{ is nonincreasing for large values of } r.$$

Now we prove blow-up result for $\max(p,q) > 1$.

Theorem 2.3. *There are not nontrivial global solutions of (1.1)–(1.3) if*

$$p > 1 \text{ and } \int_0^\infty c(t)\,dt = \infty \tag{2.1}$$

or

$$q > 1,\ \ k(t) \geq \underline{k}(t) \geq 0 \text{ and } \int_0^\infty t\underline{k}(t)\,dt = \infty \tag{2.2}$$

and at least one of the following conditions is fulfilled

$$\underline{k}(t) \leq \frac{c}{t^2} \text{ for large values of } t,\ c > 0, \tag{2.3}$$

or

$$t^{1-q}\underline{k}(t) \text{ is nonincreasing for large values of } t. \tag{2.4}$$

Proof. Without loss of generality we can suppose that $u_0(x) \not\equiv 0$ in Ω. Then from strong maximum principle and (1.2) we conclude that $u(x,t) > 0$ for $x \in \overline{\Omega}$, $t > 0$ and, moreover, by Theorem 1.2 we have $u(x,t) \geq \min_{\overline{\Omega}} u(x,t_0) > 0$ for $x \in \overline{\Omega}$, $t \geq t_0$ and any $t_0 > 0$.

Suppose at first that (2.1) holds. Let us introduce an auxiliary function

$$w(t) = \int_\Omega u(x,t)\,dx.$$

Integrating (1.1) over Ω and using Green's identity, Jensen's inequality and boundary condition (1.2), we have

$$\begin{aligned} w'(t) &= \int_\Omega (\Delta u(x,t) + c(t)u^p(x,t))\,dx = k(t)\int_{\partial\Omega}\int_0^t u^q(x,\tau)d\tau\,dS \\ &+ c(t)\int_\Omega u^p(x,t)\,dx \geq |\Omega|^{1-p}c(t)w^p(t). \end{aligned} \tag{2.5}$$

From (2.1) and (2.5) we obtain blow-up of all nontrivial solutions.

Suppose now that either (2.2), (2.3) or (2.2), (2.4) hold. Let $G(x,y;t-\tau)$ be the Green function of the heat equation with homogeneous Neumann boundary condition. We note that $G(x,y;t-\tau)$ has the following properties (see, for example, [17]):

$$G(x,y;t-\tau)\geq 0,\ x,y\in\Omega,\ 0\leq\tau<t, \tag{2.6}$$

$$\int_{\partial\Omega} G(x,y;t-\tau)\,dS_x\geq c_1,\ y\in\partial\Omega,\ 0\leq\tau<t. \tag{2.7}$$

Here and subsequently by c_i $(i\in\mathbb{N})$ we denote positive constants. It is well known that problem (1.1)–(1.3) is equivalent to the equation

$$\begin{aligned} u(x,t) &= \int_\Omega G(x,y;t)u_0(y)\,dy+\int_0^t\int_\Omega G(x,y;t-\tau)c(\tau)u^p(y,\tau)\,dy\,d\tau \\ &+ \int_0^t\int_{\partial\Omega} G(x,y;t-\tau)k(\tau)\int_0^\tau u^q(y,\sigma)\,d\sigma\,dS_y\,d\tau. \end{aligned} \tag{2.8}$$

Integrating (2.8) over $\partial\Omega$ and applying (2.6), (2.7) and Jensen's inequality, we obtain

$$\begin{aligned} \int_{\partial\Omega} u(x,t)\,dS_x &\geq c_1\int_0^t k(\tau)\int_0^\tau\int_{\partial\Omega} u^q(y,\sigma)\,dS_y\,d\sigma\,d\tau \\ &\geq c_1|\partial\Omega|^{1-q}\int_0^t k(\tau)\tau^{1-q}\left(\int_0^\tau\int_{\partial\Omega} u(y,\sigma)dS_y d\sigma\right)^q d\tau. \end{aligned} \tag{2.9}$$

Let us define

$$f(t)=\int_0^t\int_{\partial\Omega} u(x,\sigma)\,dS_x d\sigma.$$

Then from (2.9) we have

$$f'(t)\geq c_2\int_0^t \tau^{1-q}k(\tau)f^q(\tau)d\tau. \tag{2.10}$$

After integration of (2.10) over $[0,t]$ we obtain

$$f(t)\geq c_2\int_0^t (t-\tau)\tau^{1-q}k(\tau)f^q(\tau)d\tau.$$

Now we denote

$$g(t)=c_2\int_0^t (t-\tau)\tau^{1-q}k(\tau)f^q(\tau)d\tau.$$

Then

$$g''(t)=c_2t^{1-q}k(t)f^q(t)\geq c_2t^{1-q}\underline{k}(t)g^q(t). \tag{2.11}$$

Applying Theorem 2.2 to (2.11), we complete the proof. □

To formulate global existence result for problem (1.1)–(1.3) we suppose that

$$\int_0^\infty (c(t)+tk(t))\,dt<\infty \tag{2.12}$$

and there exist positive constants α, t_0 and K such that $\alpha>t_0$ and

$$\int_{t-t_0}^t \frac{\tau k(\tau)}{\sqrt{t-\tau}}\,d\tau\leq K\ \text{ for }\ t\geq\alpha. \tag{2.13}$$

Theorem 2.4. *Let* $\min(p,q) > 1$ *and (2.12), (2.13) hold. Then problem (1.1)–(1.3) has bounded global solutions for small initial data.*

Proof. Let $y(x,t)$ be a solution of the following problem

$$\begin{cases} y_t = \Delta y, \ x \in \Omega, \ t > 0 \\ \frac{\partial y(x,t)}{\partial \nu} = tk(t), \ x \in \partial\Omega, \ t > 0, \\ y(x,0) = 1, \ x \in \Omega. \end{cases}$$

According to Lemma 3.3 of [18] there exists a positive constant Y such that

$$1 \le y(x,t) \le Y, \, x \in \Omega, \ t > 0.$$

Next, for any $T > 0$ we construct a positive supersolution of (1.1)–(1.3) in Q_T in such a form that

$$\overline{u}(x,t) = \alpha z(t) y(x,t),$$

where $\alpha > 0$ and

$$z(t) = \left(1 + (p-1)(\alpha Y)^{p-1} \int_t^\infty c(\tau)\, d\tau \right)^{-\frac{1}{p-1}}.$$

It is easy to check that $z(t)$ is the solution of the equation

$$z'(t) - (\alpha Y)^{p-1} c(t) z^p(t) = 0$$

and satisfies the inequality $z(t) \le 1$. After simple computations it follows that

$$\begin{aligned} \overline{u}_t - \Delta \overline{u} - c(t)\overline{u}^p &= \alpha z' y + \alpha z y_t - \alpha z \Delta y - \alpha^p c(t) z^p y^p \\ &\ge \alpha y (z' - \alpha^{p-1} Y^{p-1} c(t) z^p) = 0, \ x \in \Omega, \ t > 0, \end{aligned}$$

and

$$\frac{\partial \overline{u}}{\partial \nu} - k(t) \int_0^t \overline{u}^q(x,\tau)\, d\tau \ge \alpha t k(t) z(t) (1 - \alpha^{q-1} Y^q) \ge 0, \ x \in \partial\Omega, \ t > 0,$$

if $\alpha \le Y^{q/(q-1)}$. Thus, by Theorem 1.2 there exist bounded global solutions of (1.1)–(1.3) for any initial data satisfying the inequality

$$u_0(x) \le \alpha \left(1 + (p-1)(\alpha Y)^{p-1} \int_0^\infty c(\tau)\, d\tau \right)^{-\frac{1}{p-1}}.$$

□

Let us introduce the following notations:

$$\ln_1 t = \ln t, \, \ln_{j+1} t = \ln(\ln_j t), \, l_j(t) = \prod_{i=1}^{j} \ln_i t, \, l_{j,\gamma}(t) = l_j(t) \ln_j^\gamma t, \, j \in \mathbb{N}, \, \gamma > 0. \tag{2.14}$$

Remark 2.5. Arguing in the same way as in [18] it is easy to show that (2.13) is a necessary condition for the boundedness of global solutions for (1.1)–(1.3). It follows from Theorem 2.3 and Theorem 2.4 that the condition (2.1) is optimal for blow-up in finite time of all nontrivial solutions of (1.1)–(1.3). Furthermore, from Theorem 1.2 and Theorem 2.3 we conclude that problem (1.1)–(1.3) has no nontrivial global solutions if $q > 1$ and

$$k(t) \ge \frac{c_3}{t^2 l_j(t)} \ \text{ for } \ j \in \mathbb{N} \ \text{ and large values of } \ t.$$

On the other hand, from Theorem 2.4 we obtain the existence of nontrivial bounded global solutions of (1.1)–(1.3) if $\min(p,q) > 1$,

$$\int_0^\infty c(t)\,dt < \infty \text{ and } k(t) \le \frac{c_4}{t^2 l_{j,\gamma}(t)} \text{ for } j \in \mathbb{N}, \gamma > 0 \text{ and large values of } t.$$

3. Global existence and blow-up for $p = 1$, $q > 1$

In this section we obtain sufficient conditions for the existence and nonexistence of global solutions of problem (1.1)–(1.3) for $p = 1, q > 1$.

Theorem 3.1. *Let $p = 1, q > 1$. Then there are not nontrivial global solutions of (1.1)–(1.3) if $k(t) \ge \underline{k}(t) \ge 0$,*

$$\int_0^\infty t^{1-q} \exp\left(-\int_0^t c(s)\,ds\right)\left(\int_0^t \exp\left(\int_0^\tau c(s)\,ds\right)d\tau\right)^q \underline{k}(t)\,dt = \infty \qquad (3.1)$$

and at least one of the following conditions is fulfilled

$$t^{1-q}\exp\left(-2\int_0^t c(s)ds\right)\left(\int_0^t \exp\left(\int_0^\tau c(s)\,ds\right)d\tau\right)^{q+1}\underline{k}(t) \le C,\ C > 0, \quad (3.2)$$

for large values of t, or

$$t^{1-q}\exp\left(-2\int_0^t c(s)\,ds\right)\underline{k}(t) \text{ is nonincreasing for large values of } t. \qquad (3.3)$$

Proof. We can suppose that $u_0(x) \not\equiv 0$, since otherwise $u(x,t) \equiv 0$. Let us change unknown function in the following way

$$u(x,t) = v(x,t)\exp\int_0^t c(\tau)\,d\tau. \qquad (3.4)$$

Then $v(x,t)$ is a solution to the problem

$$\begin{cases} v_t = \Delta v, & x \in \Omega,\ t > 0,\\ \frac{\partial v(x,t)}{\partial \nu} = k(t)\int_0^t \exp\left(q\int_0^\tau c(s)\,ds - \int_0^t c(s)\,ds\right)v^q(x,\tau)\,d\tau, & x \in \partial\Omega,\ t > 0,\\ v(x,0) = u_0(x), & x \in \Omega. \end{cases} \quad (3.5)$$

It is well known that problem (3.5) is equivalent to the equation

$$\begin{aligned} v(x,t) &= \int_\Omega G(x,y;t)u_0(y)\,dy \\ &+ \int_0^t\int_{\partial\Omega} G(x,y;t-\tau)k(\tau)\int_0^\tau \exp\left(q\int_0^\sigma c(s)ds - \int_0^\tau c(s)ds\right)v^q(y,\sigma)d\sigma dS_y d\tau. \end{aligned} \qquad (3.6)$$

Integrating (3.6) over $\partial\Omega$ and applying (2.6), (2.7) and Jensen's inequality, we obtain

$$\begin{aligned} &\int_{\partial\Omega} v(x,t)\,dS_x \\ \ge\ & c_1|\partial\Omega|^{1-q}\int_0^t \tau^{1-q}k(\tau)\exp\left(-\int_0^\tau c(s)ds\right)\left(\int_0^\tau\int_{\partial\Omega}\exp\left(\int_0^\sigma c(s)ds\right)v(y,\sigma)dS_y d\sigma\right)^q d\tau. \end{aligned} \qquad (3.7)$$

We set

$$f(t) = \int_0^t \int_{\partial\Omega} \exp\left(\int_0^\sigma c(s)ds\right) v(y,\sigma)\, dS_y d\sigma. \tag{3.8}$$

Then from (3.7) and (3.8) we deduce that $f'(t) > 0$ for $t > 0$ and

$$f'(t) \geq c_5 \exp\left(\int_0^t c(s)ds\right) \int_0^t \tau^{1-q} \exp\left(-\int_0^\tau c(s)ds\right) k(\tau) f^q(\tau) d\tau. \tag{3.9}$$

After integration of (3.9) over $[0,t]$ we obtain

$$f(t) \geq c_5 \int_0^t \exp\left(\int_0^\sigma c(s)ds\right) \int_0^\sigma \tau^{1-q} \exp\left(-\int_0^\tau c(s)ds\right) k(\tau) f^q(\tau) d\tau d\sigma.$$

Defining

$$g(t) = c_5 \int_0^t \exp\left(\int_0^\sigma c(s)ds\right) \int_0^\sigma \tau^{1-q} \exp\left(-\int_0^\tau c(s)ds\right) k(\tau) f^q(\tau) d\tau d\sigma$$

we have $f(t) \geq g(t)$. Moreover,

$$g''(t) \geq c(t) g'(t) + c_5 t^{1-q} \underline{k}(t) g^q(t). \tag{3.10}$$

Multiplying (3.10) by $\exp\left(-\int_0^t c(s)ds\right)$, we obtain

$$\left(\exp\left(-\int_0^t c(s)ds\right) g'(t)\right)' \geq c_5 t^{1-q} \exp\left(-\int_0^t c(s)ds\right) \underline{k}(t) g^q(t). \tag{3.11}$$

Let us change variable and unknown function in the following way

$$s = \int_0^t \exp\left(\int_0^\tau c(\sigma)d\sigma\right) d\tau, \quad \phi(s) = g(t)$$

and rewrite (3.11) as

$$\phi''(s) \geq c_5 t^{1-q} \underline{k}(t) \exp\left(-2\int_0^t c(\sigma)d\sigma\right) \phi^q(s). \tag{3.12}$$

Applying Theorem 2.2 to (3.12), we complete the proof. □

Theorem 3.2. *Let $p = 1$, $q > 1$,*

$$\int_0^\infty k(t) \exp\left(-\int_0^t c(s)\, ds\right) \int_0^t \exp\left(q \int_0^\tau c(s)\, ds\right) d\tau\, dt < \infty \tag{3.13}$$

and there exist positive constants α, t_0 and K such that $\alpha > t_0$ and

$$\int_{t-t_0}^t \frac{k(\tau)\exp\left(-\int_0^\tau c(s)\, ds\right) \int_0^\tau \exp\left(q\int_0^\sigma c(s)\, ds\right) d\sigma\, d\tau}{\sqrt{t-\tau}} \leq K \ \text{ for } \ t \geq \alpha. \tag{3.14}$$

Then problem (1.1)–(1.3) has global solutions for small initial data. If, in addition,

$$\int_0^\infty c(t)\, dt < \infty \tag{3.15}$$

then problem (1.1)–(1.3) has bounded global solutions for small initial data.

Proof. To prove the theorem we construct a positive supersolution of (1.1)–(1.3) in such a form that

$$\overline{u}(x,t) = \alpha \exp\left(\int_0^t c(s)\,ds\right) h(x,t),$$

where $h(x,t)$ is a solution to the following problem

$$\begin{cases} h_t = \Delta h, \ x \in \Omega, \ t > 0, \\ \frac{\partial h(x,t)}{\partial \nu} = k(t) \exp\left(-\int_0^t c(s)\,ds\right) \int_0^t \exp\left(q \int_0^\tau c(s)\,ds\right)\,d\tau, \ x \in \partial\Omega, \ t > 0, \\ h(x,0) = 1, \ x \in \Omega. \end{cases} \tag{3.16}$$

As it is proved in [18] the solution of (3.16) satisfies the inequalities

$$1 \le h(x,t) \le H, \, x \in \Omega, \ t > 0$$

for some $H > 0$. It is easy to check that $\overline{u}(x,t)$ is the supersolution of (1.1)–(1.3) if $\alpha \le H^{-q/(q-1)}$ and $u_0(x) \le \alpha$. Moreover, $\overline{u}(x,t)$ is bounded function under the condition (3.15). □

Remark 3.3. Let $p = 1, q > 1$. Arguing in the same way as in [18] it is easy to prove from (3.4) and (3.5) that both conditions (3.14) and (3.15) are necessary for the boundedness of global solutions of (1.1)–(1.3). Furthermore, we conclude from Theorem 1.2, Theorem 3.1 and Theorem 3.2 that problem (1.1)–(1.3) has no nontrivial global solutions if

$$c(t) \ge \frac{\beta}{t} \ \text{ for large values of } t \text{ and some } \beta > 0$$

and

$$k(t) \ge \frac{c_6}{t^{\beta(q-1)+2} l_j(t)} \ \text{ for } j \in \mathbb{N} \text{ and large values of } t$$

and problem (1.1)–(1.3) has nontrivial bounded global solutions if

$$c(t) \le \frac{\omega}{t} \ \text{ for large values of } t \text{ and some } \omega > 0$$

and

$$k(t) \le \frac{c_7}{t^{\omega(q-1)+2} l_{j,\gamma}(t)} \ \text{ for } j \in \mathbb{N}, \gamma > 0 \text{ and large values of } t,$$

where $l_j(t)$ and $l_{j,\gamma}(t)$ were introduced in (2.14).

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