

**GLOBAL EXISTENCE OF SOLUTIONS OF INITIAL-BOUNDARY  
VALUE PROBLEM FOR NONLOCAL PARABOLIC EQUATION  
WITH NONLOCAL BOUNDARY CONDITION**

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ABSTRACT. We prove global existence and blow-up of solutions of initial-boundary value problem for nonlinear nonlocal parabolic equation with nonlinear nonlocal boundary condition. Obtained results depend on the behavior of variable coefficients for large values of time.

1. INTRODUCTION

We consider nonlinear nonlocal parabolic equation

$$u_t = \Delta u + a(x, t)u^r \int_{\Omega} u^p(y, t) dy - b(x, t)u^q, \quad x \in \Omega, \quad t > 0, \quad (1.1)$$

with nonlinear nonlocal boundary condition

$$u(x, t) = \int_{\Omega} k(x, y, t)u^l(y, t) dy, \quad x \in \partial\Omega, \quad t > 0, \quad (1.2)$$

and initial datum

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.3)$$

where  $r, p, q, l$  are positive constants,  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  for  $n \geq 1$  with smooth boundary  $\partial\Omega$ .

Throughout this paper we suppose that  $a(x, t)$ ,  $b(x, t)$ ,  $k(x, y, t)$  and  $u_0(x)$  satisfy the following conditions:

$$\begin{aligned} a(x, t), b(x, t) &\in C_{loc}^{\alpha}(\overline{\Omega} \times [0, \infty)), \quad 0 < \alpha < 1, \quad a(x, t) \geq 0, \quad b(x, t) \geq 0; \\ k(x, y, t) &\in C(\partial\Omega \times \overline{\Omega} \times [0, \infty)), \quad k(x, y, t) \geq 0; \\ u_0(x) &\in C(\overline{\Omega}), \quad u_0(x) \geq 0, \quad x \in \overline{\Omega}, \quad u_0(x) = \int_{\Omega} k(x, y, 0)u_0^l(y) dy, \quad x \in \partial\Omega. \end{aligned}$$

For global existence and blow-up of solutions for parabolic equations with nonlocal boundary conditions we refer to [1, 2, 6], [11]–[18], [21, 23, 24, 30, 32, 33] and the references therein. Initial-boundary value problems for nonlocal parabolic equations with nonlocal boundary conditions were considered in many papers also (see, for example, [4, 7, 9, 10, 26, 27, 34]). In particular, blow-up problem for nonlocal parabolic equations with boundary condition (1.2) was investigated in [5, 8, 25, 28, 29, 31, 35, 36]. So, for example, the authors of [5] studied (1.1)–(1.3) with  $b(x, t) \equiv 0$ ,  $a(x, t) \equiv a(x)$  and  $k(x, y, t) \equiv k(x, y)$ , and problem (1.1)–(1.3) with  $r = 0$ ,  $a(x, t) \equiv 1$ ,  $b(x, t) \equiv b > 0$  and  $k(x, y, t) \equiv k(x, y)$  was considered in [31]. The authors of [15] studied (1.1)–(1.3) with  $a(x, t) \equiv 0$ .

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The existence of classical local solutions and comparison principle for (1.1)–(1.3) were proved in [19] and [20].

In this paper we prove global existence and blow-up of solutions of (1.1)–(1.3). Obtained results depend on the behavior of variable coefficients  $a(x, t)$ ,  $b(x, t)$  and  $k(x, y, t)$  as  $t \rightarrow \infty$ .

This paper is organized as follows. Global existence of solutions for any initial data and blow-up in finite time of solutions for large initial data are proved in section 2. In section 3 we present finite time blow-up of all nontrivial solutions as well as the existence of global solutions for small initial data.

## 2. GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS

Let  $Q_T = \Omega \times (0, T)$ ,  $S_T = \partial\Omega \times (0, T)$ ,  $\Gamma_T = S_T \cup \overline{\Omega} \times \{0\}$ ,  $T > 0$ .

**Definition 2.1.** We say that a nonnegative function  $u(x, t) \in C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T)$  is a supersolution of (1.1)–(1.3) in  $Q_T$  if

$$u_t \geq \Delta u + a(x, t)u^r \int_{\Omega} u^p(y, t) dy - b(x, t)u^q, \quad (x, t) \in Q_T, \quad (2.1)$$

$$u(x, t) \geq \int_{\Omega} k(x, y, t)u^l(y, t) dy, \quad (x, t) \in S_T, \quad (2.2)$$

$$u(x, 0) \geq u_0(x), \quad x \in \Omega, \quad (2.3)$$

and  $u(x, t) \in C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T)$  is a subsolution of (1.1)–(1.3) in  $Q_T$  if  $u \geq 0$  and it satisfies (2.1)–(2.3) in the reverse order. We say that  $u(x, t)$  is a solution of (1.1)–(1.3) in  $Q_T$  if  $u(x, t)$  is both a subsolution and a supersolution of (1.1)–(1.3) in  $Q_T$ .

We will repeatedly use the following comparison principle (see [19], [20]).

**Theorem 2.2.** *Let  $\underline{u}(x, t)$  and  $\overline{u}(x, t)$  be a subsolution and a supersolution of problem (1.1)–(1.3) in  $Q_T$ , respectively. Suppose that  $\underline{u}(x, t) > 0$  or  $\overline{u}(x, t) > 0$  in  $Q_T \cup \Gamma_T$  if  $\min(r, p, l) < 1$ . Then  $\overline{u}(x, t) \geq \underline{u}(x, t)$  in  $Q_T \cup \Gamma_T$ .*

To prove global existence of solutions of (1.1)–(1.3) we suppose that

$$b(x, t) > 0 \text{ for } x \in \overline{\Omega} \text{ and } t \geq 0. \quad (2.4)$$

**Theorem 2.3.** *Let  $\max(r + p, l) \leq 1$  or (2.4) hold and either  $l \leq 1$ ,  $1 < r + p < q$  or  $1 < l < (q + 1)/2$ ,  $\max(r + p, 2p + 1) < q$ . Then problem (1.1)–(1.3) has global solutions for any initial data.*

*Proof.* Let  $T$  be any positive constant and

$$M = \max \left( \sup_{Q_T} a(x, t), \sup_{\partial\Omega \times Q_T} k(x, y, t) \right). \quad (2.5)$$

In order to prove global existence of solutions we construct a suitable explicit supersolution of (1.1)–(1.3) in  $Q_T$ .

Suppose at first that  $\max(r + p, l) \leq 1$ . Let  $\lambda_1$  be the first eigenvalue of the following problem

$$\Delta\varphi(x) + \lambda\varphi(x) = 0, \quad x \in \Omega, \quad \varphi(x) = 0, \quad x \in \partial\Omega, \quad (2.6)$$

and  $\varphi(x)$  be the corresponding eigenfunction which is chosen to satisfy that for some  $0 < \varepsilon < 1$

$$M \int_{\Omega} \frac{dy}{(\varphi(y) + \varepsilon)^l} \leq 1.$$

Then it is easy to check that

$$v(x, t) = \frac{\eta \exp(\mu t)}{\varphi(x) + \varepsilon} \quad (2.7)$$

is a supersolution of (1.1)–(1.3) in  $Q_T$  if

$$\eta \geq \max \left( \sup_{\Omega} u_0(x) \sup_{\Omega} (\varphi(x) + \varepsilon), 1 \right),$$

$$\mu \geq \lambda_1 + \sup_{\Omega} \frac{2|\nabla\varphi(x)|^2}{(\varphi(x) + \varepsilon)^2} + M \sup_{\Omega} (\varphi(x) + \varepsilon)^{1-r} \int_{\Omega} \frac{dy}{(\varphi(y) + \varepsilon)^p}.$$

By Theorem 2.2 problem (1.1)–(1.3) has global solutions for any initial data.

From (2.4) we conclude that  $\underline{b} = \inf_{Q_T} b(x, t) > 0$ . Let  $l \leq 1$ ,  $1 < r + p < q$ . Then  $v(x, t)$  in (2.7) is a supersolution of (1.1)–(1.3) in  $Q_T$  if

$$\eta \geq \max \left( \sup_{\Omega} u_0(x) \sup_{\Omega} (\varphi(x) + \varepsilon), \left( \frac{M}{\underline{b}} \sup_{\Omega} (\varphi(x) + \varepsilon)^{q-r} \int_{\Omega} \frac{dy}{(\varphi(y) + \varepsilon)^p} \right)^{\frac{1}{q-r-p}}, 1 \right),$$

$$\mu \geq \lambda_1 + \sup_{\Omega} \frac{2|\nabla\varphi(x)|^2}{(\varphi(x) + \varepsilon)^2}.$$

Let  $1 < l < (q + 1)/2$ ,  $\max(r + p, 2p + 1) < q$ . To construct a supersolution we use the change of variables in a neighborhood of  $\partial\Omega$  as in [3]. Let  $\bar{x} \in \partial\Omega$  and  $\hat{n}(\bar{x})$  be the inner unit normal to  $\partial\Omega$  at the point  $\bar{x}$ . Since  $\partial\Omega$  is smooth it is well known that there exists  $\delta > 0$  such that the mapping  $\psi : \partial\Omega \times [0, \delta] \rightarrow \mathbb{R}^n$  given by  $\psi(\bar{x}, s) = \bar{x} + s\hat{n}(\bar{x})$  defines new coordinates  $(\bar{x}, s)$  in a neighborhood of  $\partial\Omega$  in  $\bar{\Omega}$ . A straightforward computation shows that, in these coordinates,  $\Delta$  applied to a function  $g(\bar{x}, s) = g(s)$ , which is independent of the variable  $\bar{x}$ , evaluated at a point  $(\bar{x}, s)$  is given by

$$\Delta g(\bar{x}, s) = \frac{\partial^2 g}{\partial s^2}(\bar{x}, s) - \sum_{j=1}^{n-1} \frac{H_j(\bar{x})}{1 - sH_j(\bar{x})} \frac{\partial g}{\partial s}(\bar{x}, s), \quad (2.8)$$

where  $H_j(\bar{x})$  for  $j = 1, \dots, n - 1$ , denotes the principal curvatures of  $\partial\Omega$  at  $\bar{x}$ . For  $0 \leq s \leq \delta$  and small  $\delta$  we have

$$\left| \sum_{j=1}^{n-1} \frac{H_j(\bar{x})}{1 - sH_j(\bar{x})} \right| \leq \bar{c}. \quad (2.9)$$

Let  $0 < \varepsilon < \omega < \min(\delta, 1)$ ,  $2/(q - 1) < \beta < \min(1/p, 1/(l - 1))$ ,  $0 < \gamma < \beta/2$ ,  $A \geq \sup_{\Omega} u_0(x)$ . For points in  $Q_{\delta, T} = \partial\Omega \times [0, \delta] \times [0, T]$  of coordinates  $(\bar{x}, s, t)$  define

$$v(x, t) = v(\bar{x}, s, t) = [(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta}{\gamma}} + A, \quad (2.10)$$

where  $s_+ = \max(s, 0)$ . For points in  $\bar{Q}_T \setminus Q_{\delta, T}$  we set  $v(x, t) = A$ . We prove that  $v(x, t)$  is a supersolution of (1.1)–(1.3) in  $Q_T$ . It is not difficult to check that

$$\left| \frac{\partial v}{\partial s} \right| \leq \beta \min \left( [D(s)]^{\frac{\gamma+1}{\gamma}} [(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta+1}{\gamma}}, (s + \varepsilon)^{-(\beta+1)} \right), \quad (2.11)$$

$$\left| \frac{\partial^2 v}{\partial s^2} \right| \leq \beta(\beta+1) \min \left( [D(s)]^{\frac{2(\gamma+1)}{\gamma}} [(s+\varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta+2}{\gamma}}, (s+\varepsilon)^{-(\beta+2)} \right), \quad (2.12)$$

where

$$D(s) = \frac{(s+\varepsilon)^{-\gamma}}{(s+\varepsilon)^{-\gamma} - \omega^{-\gamma}}.$$

Then  $D'(s) > 0$  and for any  $\bar{\varepsilon} > 0$

$$1 \leq D(s) \leq 1 + \bar{\varepsilon}, \quad 0 < s \leq \bar{s}, \quad (2.13)$$

where  $\bar{s} = [\bar{\varepsilon}/(1 + \bar{\varepsilon})]^{1/\gamma} \omega - \varepsilon$ ,  $\varepsilon < [\bar{\varepsilon}/(1 + \bar{\varepsilon})]^{1/\gamma} \omega$ . We denote

$$Lv \equiv v_t - \Delta v - a(x, t)v^r \int_{\Omega} v^p(y, t) dy + b(x, t)v^q \quad (2.14)$$

and

$$\bar{J} = \sup_{0 < s < \delta} \int_{\partial\Omega} |J(\bar{y}, s)| d\bar{y}, \quad (2.15)$$

where  $J(\bar{y}, s)$  is Jacobian of the change of variables in neighborhood of  $\partial\Omega$ . We use the inequality  $(a+b)^p \leq 2^p(a^p + b^p)$ ,  $a \geq 0, b \geq 0, p > 0$  to estimate the integral in (2.14)

$$\begin{aligned} \int_{\Omega} v^p(y, t) dy &\leq 2^p \left( A^p |\Omega| + \int_0^{\omega-\varepsilon} \int_{\partial\Omega} J(\bar{y}, s) [(s+\varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta p}{\gamma}} d\bar{y} ds \right) \\ &\leq 2^p \left( A^p |\Omega| + \frac{\bar{J} \omega^{1-\beta p}}{1-\beta p} \right). \end{aligned} \quad (2.16)$$

Here  $|\Omega|$  is Lebesgue measure of  $\Omega$ . By (2.8)–(2.14), (2.16) we can choose  $\bar{\varepsilon}$  small and  $A$  large so that in  $Q_{\bar{s}, T}$

$$\begin{aligned} Lv &\geq \underline{b} \left( [(s+\varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta}{\gamma}} + A \right)^q - \beta(\beta+1) [D(s)]^{\frac{2(\gamma+1)}{\gamma}} [(s+\varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta+2}{\gamma}} \\ &\quad - \beta \bar{c} [D(s)]^{\frac{\gamma+1}{\gamma}} [(s+\varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta+1}{\gamma}} \\ &\quad - 2^p M \left( [(s+\varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta}{\gamma}} + A \right)^r \left( A^p |\Omega| + \frac{\bar{J} \omega^{1-\beta p}}{1-\beta p} \right) \geq 0. \end{aligned}$$

Let  $s \in [\bar{s}, \delta]$ . From (2.8)–(2.12) we have

$$|\Delta v| \leq \beta(\beta+1) \omega^{-(\beta+2)} \left( \frac{1+\bar{\varepsilon}}{\bar{\varepsilon}} \right)^{\frac{\beta+2}{\gamma}} + \beta \bar{c} \omega^{-(\beta+1)} \left( \frac{1+\bar{\varepsilon}}{\bar{\varepsilon}} \right)^{\frac{\beta+1}{\gamma}}$$

and by (2.16)  $Lv \geq 0$  for large  $A$ . Obviously, in  $\overline{Q_T} \setminus Q_{\delta, T}$

$$Lv \geq -2^p M A^r \left( A^p |\Omega| + \frac{\bar{J} \omega^{1-\beta p}}{1-\beta p} \right) + \underline{b} A^q \geq 0$$

for large  $A$ .

Now we prove the following inequality

$$v((\bar{x}, 0), t) \geq \int_{\Omega} k(x, y, t) v^l(y, t) dy, \quad (x, t) \in S_T \quad (2.17)$$

for a suitable choice of  $\varepsilon$ . To do this we use the change of variables in neighborhood of  $\partial\Omega$ . Estimating the integral  $I$  in the right hand side of (2.17), we get

$$I \leq 2^l M \bar{J} \int_0^{\omega-\varepsilon} [(s+\varepsilon)^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta l}{\gamma}} ds + 2^l M A^l |\Omega| \leq 2^l M \bar{J} C(\varepsilon) + 2^l M A^l |\Omega|,$$

where

$$C(\varepsilon) = \begin{cases} \varepsilon^{-(\beta l - 1)}/(\beta l - 1), & \beta l > 1, \\ \omega^{1 - \beta l}/(1 - \beta l), & \beta l < 1, \\ -\ln \varepsilon, & \beta l = 1. \end{cases}$$

On the other hand, we have

$$v((\bar{x}, 0), t) = [\varepsilon^{-\gamma} - \omega^{-\gamma}]_+^{\frac{\beta}{\gamma}} + A.$$

Hence, (2.17) holds for small values of  $\varepsilon$  and by Theorem 2.2  $u(x, t) \leq v(x, t)$  in  $\overline{Q_T}$ .  $\square$

To prove finite time blow-up result we need lower bound for solutions of (1.1)–(1.3) with large initial data.

**Lemma 2.4.** *Let  $u(x, t)$  be a solution of (1.1)–(1.3) in  $\overline{Q_T}$ . For any  $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$  and any positive constant  $C$  there exists positive constant  $c_1$  such that if  $u_0(x) \geq c_1$  in  $\Omega_1$ , then*

$$u(x, t) \geq C \text{ in } \overline{\Omega_0} \times [0, T]. \quad (2.18)$$

*Proof.* Let  $y(x, t)$  be a solution of the following problem

$$\begin{cases} y_t = \Delta y, & x \in \Omega_1, \quad 0 < t < T, \\ y(x, t) = 0, & x \in \partial\Omega_1, \quad 0 < t < T, \\ y(x, 0) = \chi(x), & x \in \Omega_1, \end{cases} \quad (2.19)$$

where  $\chi(x) \in C_0^\infty(\Omega_1)$ ,  $\chi(x) = 1$  in  $\Omega_0$  and  $0 \leq \chi(x) \leq 1$ . By strong maximum principle

$$\inf_{\Omega_0 \times (0, T)} y(x, t) > 0. \quad (2.20)$$

Suppose that  $q \geq 1$ . We put  $m = \max(\sup_{Q_T} u(x, t), \sup_{Q_T} b(x, t))$  and define function  $v(x, t) = \exp(\rho t)u(x, t)$ . For  $\rho \geq m^q$  we have in  $Q_T$

$$v_t - \Delta v = \exp(\rho t) \left( \rho u + a(x, t)u^r \int_{\Omega} u^p(y, t) dy - b(x, t)u^q \right) \geq v(\rho - b(x, t)u^{q-1}) \geq 0.$$

We assume  $u_0(x) \geq c_1\chi(x)$  in  $\Omega_1$ , where constant  $c_1$  will be chosen below. Then by comparison principle for (2.19) we get  $v(x, t) \geq c_1 y(x, t)$  in  $\overline{\Omega_1} \times [0, T]$ . Taking into account (2.20), we have (2.18) if  $c_1 = C \exp(\rho T) (\inf_{\Omega_0 \times (0, T)} y(x, t))^{-1}$ .

Let  $q < 1$ . We set  $w(x, t) = \exp(mt)(u(x, t) + 1)$ . Since  $u^q \leq u + 1$ , we conclude that

$$w_t - \Delta w = \exp(mt) \left( m(u + 1) + a(x, t)u^r \int_{\Omega} u^p(y, t) dy - b(x, t)u^q \right) \geq 0$$

in  $Q_T$ . Arguing as in previous case, we obtain

$$u(x, t) \geq c_1 \exp(-mt)y(x, t) - 1 \text{ in } \overline{\Omega_1} \times [0, T].$$

Choosing  $c_1 = (C + 1) \exp(mT) (\inf_{\Omega_0 \times (0, T)} y(x, t))^{-1}$ , we have (2.18).  $\square$

Now we prove that problem (1.1)–(1.3) has finite time blow-up solutions if either  $l > \max(1, (q + 1)/2)$  and

$$k(x, y, t) \geq k_0 > 0, \quad x \in \partial\Omega, \quad y \in \Omega, \quad 0 < t < t_0, \quad (2.21)$$

for some positive constants  $k_0$  and  $t_0$  or  $r + p > \max(q, 1)$  and

$$a(x, t) \geq a_0 > 0, \quad x \in \Omega, \quad 0 < t < t_1, \quad (2.22)$$

for some positive constants  $a_0$  and  $t_1$ .

**Theorem 2.5.** *There exist finite time blow-up solutions of (1.1)–(1.3) if either  $l > \max(1, (q + 1)/2)$  and (2.21) holds or  $r + p > \max(q, 1)$  and (2.22) holds.*

*Proof.* We suppose at first that  $l > \max(1, (q + 1)/2)$  and (2.21) holds. Let us consider the following problem

$$\begin{cases} u_t = \Delta u - b(x, t)u^q, & x \in \Omega, \quad t > 0, \\ u(x, t) = \int_{\Omega} k(x, y, t)u^l(y, t) dy, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \quad (2.23)$$

As it is proved in [15], problem (2.23) has positive finite time blow-up solutions. We note that any solution of (2.23) is a subsolution of (1.1)–(1.3). Applying Theorem 2.2, we prove the theorem.

Now we assume that  $r + p > \max(q, 1)$  and (2.22) holds. We put  $\bar{b} = \sup_{Q_{t_1}} b(x, t)$ .

Let  $r \geq q > 1$ . We denote

$$J(t) = \exp(\lambda_1 t) \int_{\Omega} u(x, t)\varphi(x) dx, \quad (2.24)$$

where  $\varphi(x)$  is the solution of (2.6) satisfying

$$\int_{\Omega} \varphi(x) dx = 1. \quad (2.25)$$

Then using (1.1), (2.6), Green's identity and the inequality

$$\frac{\partial \varphi(x)}{\partial n} \leq 0, \quad x \in \partial\Omega, \quad (2.26)$$

where  $\nu$  is unit outward normal on  $\partial\Omega$ , we get for  $t < t_1$

$$\begin{aligned} J'(t) &\geq \exp(\lambda_1 t) \int_{\Omega} \left( a(x, t)u^r \int_{\Omega} u^p(y, t) dy - b(x, t)u^q \right) \varphi(x) dx \\ &\geq \exp(\lambda_1 t) \left( a_0 \int_{\Omega} u^p(y, t) dy \int_{\Omega} u^r(x, t)\varphi(x) dx - \bar{b} \int_{\Omega} u^q(x, t)\varphi(x) dx \right). \end{aligned} \quad (2.27)$$

By Lemma 2.4

$$a_0 \int_{\Omega} u^p(y, t) dy \left[ \int_{\Omega} u^q(x, t)\varphi(x) dx \right]^{\frac{r-q}{q}} - \bar{b} \geq c_2 > 0 \quad (2.28)$$

for  $t < t_1$  and large initial data. Taking into account (2.27), (2.28), Hölder's and Jensen's inequalities, we have for  $t < t_1$

$$\begin{aligned} J'(t) &\geq \exp(\lambda_1 t) \int_{\Omega} u^q(x, t)\varphi(x) dx \left( a_0 \int_{\Omega} u^p(y, t) dy \left[ \int_{\Omega} u^q(x, t)\varphi(x) dx \right]^{\frac{r-q}{q}} - \bar{b} \right) \\ &\geq c_2 \exp[\lambda_1(1-q)t] J^q(t). \end{aligned}$$

Hence,  $J(t)$  blows up in finite time  $T$  ( $T < t_1$ ) for large initial data.

Let  $r > 1 \geq q$ . By Lemma 2.4

$$\int_{\Omega} u^p(y, t) dy \geq c_3 > 0 \quad (2.29)$$

for  $t < t_1$  and large initial data. Then using (2.27), (2.29),  $u^q \leq u + 1$  and Jensen's inequalities, we obtain for  $t < t_1$

$$J'(t) \geq a_0 c_3 \exp[\lambda_1(1-r)t] J^r(t) - \bar{b} J(t) - \bar{b} \exp(\lambda_1 t)$$

and again  $J(t)$  blows up in finite time  $T$  ( $T < t_1$ ) for large initial data.

Let  $r < q$ . Without loss of generality, we may assume that  $\Omega$  contains the origin. We introduce designations

$$G = \{(x, t) : 0 \leq t < T_1, |x| \leq A\sqrt{T-t}\} \quad (T_1 < T), \quad G_\tau = G \cap \{t = \tau\} \quad (0 < \tau < T_1)$$

and consider the auxiliary problem

$$\begin{cases} u_t = \Delta u + a(x, t)u^r \int_{G_t} u^p(y, t) dy - b(x, t)u^q, & (x, t) \in G, \\ u(x, t) = 0, & |x| = A\sqrt{T-t}, \quad 0 < t < T_1, \\ u(x, 0) = A^2 - |x|^2/T, & |x| < A\sqrt{T}, \end{cases} \quad (2.30)$$

where  $A > 0$  will be determined below and  $T < \min\{1, t_1\}$  we choose in such a way that points  $x$  with  $|x| \leq A\sqrt{T}$  belong to  $\Omega$ . Let  $u(x, t)$  be a positive in  $\bar{G}$  solution of (1.1)–(1.3) such that  $u_0(x) \geq (A^2 - |x|^2/T)_+$ . Obviously,  $u(x, t)$  is a supersolution of (2.30). We construct a subsolution of (2.30) in the following form

$$\underline{u}(x, t) = (T-t)^{-\gamma} V\left(\frac{|x|}{\sqrt{T-t}}\right), \quad (2.31)$$

where  $V(\xi) = (A^2 - \xi^2)_+$ ,  $\xi = |x|/\sqrt{T-t}$  and  $\gamma > 0$  will be chosen below. It is easy to see  $\underline{u}(0, t) \rightarrow \infty$  as  $t \rightarrow T_1$  and  $T_1 \rightarrow T$ . We show that

$$\Lambda \underline{u} \leq 0 \quad (2.32)$$

in  $G$ , where

$$\Lambda u \equiv u_t - \Delta u - a(x, t)u^r \int_{G_t} u^p(y, t) dy + b(x, t)u^q.$$

Note that

$$\int_{G_t} V^p(\xi) dx = (T-t)^{\frac{n}{2}} \int_{|z| \leq A} (A^2 - |z|^2)_+^p dz = C(A)(T-t)^{\frac{n}{2}}. \quad (2.33)$$

By (2.33) we obtain

$$\begin{aligned} \Lambda \underline{u} &\leq \gamma(T-t)^{-\gamma-1} V(\xi) - (T-t)^{-\gamma-1} \xi^2 + 2n(T-t)^{-\gamma-1} \\ &\quad - a_0 C(A)(T-t)^{\frac{n}{2}-\gamma(r+p)} V^r(\xi) + \bar{b}(T-t)^{-\gamma q} V^q(\xi) \end{aligned} \quad (2.34)$$

for points of  $G$ . Further we distinguish the two zones  $0 \leq \xi < \theta A$  and  $\theta A \leq \xi < A$ , where  $\theta \in (0, 1)$  will be chosen below.

For  $\theta A \leq \xi < A$  we have

$$V(\xi) \leq (1 - \theta^2) A^2. \quad (2.35)$$

From (2.34), (2.35) it follows that

$$\begin{aligned} \Lambda \underline{u} &\leq (\gamma(1 - \theta^2) A^2 - \theta^2 A^2 + 2n)(T-t)^{-\gamma-1} - a_0 C(A)(T-t)^{\frac{n}{2}-\gamma(r+p)} V^r(\xi) \\ &\quad + \bar{b}(T-t)^{-\gamma q} V^q(\xi). \end{aligned} \quad (2.36)$$

We put

$$A = 3\sqrt{n}, \quad \theta^2 = \frac{\gamma + 1/2}{\gamma + 1} \quad (2.37)$$

and estimate the first term on the right hand side of (2.36)

$$(\gamma(1 - \theta^2)A^2 - \theta^2 A^2 + 2n)(T - t)^{-\gamma-1} = -\frac{5n}{2}(T - t)^{-\gamma-1} < 0. \quad (2.38)$$

By (2.35)

$$a_0 C(A)(T - t)^{\frac{\alpha}{2} - \gamma(r+p)} V^r(\xi) \geq \bar{b}(T - t)^{-\gamma q} V^q(\xi) \quad (2.39)$$

for small values of  $T$  and  $\gamma > n/[2(r + p - q)]$ . From (2.36), (2.38), (2.39) it follows (2.32) for  $\xi \in [\theta A, A)$ .

For  $0 \leq \xi < \theta A$  we have

$$V(\xi) \geq (1 - \theta^2)A^2 = \frac{9n}{2(\gamma + 1)}.$$

Then by (2.34) inequality (2.32) still holds for  $0 \leq \xi < \theta A$  if  $T$  is small and

$$\gamma > \max\left(\frac{n}{2(r + p - q)}, \frac{n + 2}{2(r + p - 1)}\right).$$

Applying comparison principle for (2.30), we obtain  $u(x, t) \geq \underline{u}(x, t)$  in  $G$ . Hence,  $u(x, t)$  blows up in finite time.

In the case  $q \leq r \leq 1$  we have  $\gamma q < \gamma + 1$ . Then the function in (2.31) satisfies (2.32) for  $0 \leq \xi < A$ . Indeed, by virtue of (2.36)–(2.38) we have

$$\Lambda \underline{u} \leq \left(-\frac{5n}{2}(T - t)^{-\gamma-1} + \bar{b}(T - t)^{-\gamma q} V^q(\xi)\right) - a_0 C(A)(T - t)^{\frac{\alpha}{2} - \gamma(r+p)} V^r(\xi) \leq 0$$

for  $\theta A \leq \xi < A$  and small values of  $T$ . For  $0 \leq \xi < \theta A$  inequality (2.32) holds if  $\gamma > (n + 2)/[2(r + p - 1)]$  and  $T$  is small. Arguing as in the previous case, we complete the proof.  $\square$

### 3. BLOW-UP OF ALL NONTRIVIAL SOLUTIONS AND GLOBAL EXISTENCE OF SOLUTIONS FOR SMALL INITIAL DATA

In this section we find conditions which guarantee blow-up in finite time of all nontrivial solutions and prove global existence of solutions for small initial data.

First we show that for  $q < \min(r + p, 1)$  under some conditions problem (1.1)–(1.3) has nontrivial global solutions for any  $a(x, t)$  and  $k(x, y, t)$ . Suppose that

$$\inf_{\Omega} b(x, 0) > 0. \quad (3.1)$$

**Theorem 3.1.** *Let (3.1) hold and either  $q < \min(r + p, 1)$ ,  $l > 1$  or  $r \geq q$ ,  $(q + 1)/2 < l \leq 1$ . Then problem (1.1)–(1.3) has global solutions for small initial data.*

*Proof.* We put  $b_0 = \inf_{Q_T} b(x, t)$  and choose  $T$  so that  $b_0 > 0$ .

Suppose that  $q < \min(r + p, 1)$ ,  $l > 1$ . A straightforward computation shows that for small  $\beta$ ,  $\varepsilon$  and  $u_0(x)$

$$g(t) = \beta[T - t]_+^{\frac{1}{1-q}} + \varepsilon$$

is a supersolution of (1.1)–(1.3) in  $Q_T$ . Applying Theorem 2.2, we have  $u(x, t) \leq g(t)$  in  $Q_T$ . Passing to the limit as  $\varepsilon \rightarrow 0$ , we obtain

$$u(x, t) \leq \beta[T - t]_+^{\frac{1}{1-q}}, \quad (x, t) \in Q_T.$$

Now we put  $u(x, t) \equiv 0$  for  $t \geq T$ .

For  $r \geq q$ ,  $(q+1)/2 < l \leq 1$  to construct a supersolution we use the change of variables as in Theorem 2.3. For points of  $Q_{\delta, t_0}$  define

$$v(x, t) = v(\bar{x}, s), t) = (\delta - s - t)_+^\gamma + \varepsilon,$$

where  $\delta > 0$ ,  $\varepsilon > 0$ ,  $t_0 < \min(\delta, T)$ ,  $2/(1-q) < \gamma < 1/(1-l)$  for  $l < 1$  and  $2/(1-q) < \gamma$  for  $l = 1$ . In  $\overline{Q_{t_0}} \setminus Q_{\delta, t_0}$  we put  $v(x, t) = \varepsilon$ . Then  $v(x, t)$  is a supersolution of (1.1)–(1.3) in  $Q_{t_0}$  if  $u_0(x) \leq (\delta - s)_+^\gamma$ . Indeed, by (2.5), (2.8), (2.9), (2.14)–(2.16) for small  $\delta$  and  $\varepsilon$  we have  $Lv \geq 0$  in  $Q_{t_0} \setminus Q_{\delta, t_0}$  and

$$Lv \geq -\gamma(\delta - s - t)_+^{\gamma-1} - \gamma(\gamma-1)(\delta - s - t)_+^{\gamma-2} - \gamma\bar{c}(\delta - s - t)_+^{\gamma-1}$$

$$-2^p M ((\delta - s - t)_+^\gamma + \varepsilon)^r (\delta^{\gamma p+1} \bar{J} + \varepsilon^p |\Omega|) + b_0 ((\delta - s - t)_+^\gamma + \varepsilon)^q \geq 0 \text{ in } Q_{\delta, t_0}.$$

Estimating the integral  $I$  in right hand side of (2.17), we obtain

$$I \leq M \left( \int_{\Omega} (\delta - s - t)_+^{\gamma l} dy + |\Omega| \varepsilon^l \right) \leq M \left( \bar{J} \frac{(\delta - t)_+^{\gamma l+1}}{\gamma l + 1} + |\Omega| \varepsilon^l \right).$$

On the other hand, we have  $v(\bar{x}, 0), t) = (\delta - t)_+^\gamma + \varepsilon$  and (2.17) holds for

$$\delta < \left( \frac{\gamma l + 1}{2M \bar{J}} \right)^{\frac{1}{\gamma(l-1)+1}}, \quad \varepsilon < \left\{ \frac{(\delta - t_0)^\gamma}{2M |\Omega|} \right\}^{\frac{1}{l}}.$$

By Theorem 2.2  $u(x, t) \leq v(x, t)$  in  $\overline{Q_{t_0}}$  and passing to the limit as  $t_0 \rightarrow \delta$  and  $\varepsilon \rightarrow 0$ , we deduce

$$u(x, t) \leq (\delta - s - t)_+^\gamma \text{ in } Q_\delta.$$

We put  $u(x, t) \equiv 0$  for  $t \geq \delta$  and complete the proof.  $\square$

Now suppose that  $q = 1$ . We set

$$\bar{a}(t) = \sup_{\Omega} a(x, t), \underline{a}(t) = \inf_{\Omega} a(x, t), \bar{b}(t) = \sup_{\Omega} b(x, t), \underline{b}(t) = \inf_{\Omega} b(x, t), \underline{k}(t) = \inf_{\partial\Omega \times \Omega} k(x, y, t). \quad (3.2)$$

Problem (1.1)–(1.3) has global solutions for small initial data if  $q = 1$ ,  $\min(r+p, l) > 1$  and

$$\int_0^\infty \bar{a}(t) \exp \left[ -(r+p-1) \left( \sigma t + \int_0^t \underline{b}(\tau) d\tau \right) \right] dt < \infty, \quad \sigma < \lambda_1, \quad (3.3)$$

$$\int_{\Omega} k(x, y, t) dy \leq K \exp \left[ (l-1) \left( \gamma t + \int_0^t \underline{b}(\tau) d\tau \right) \right], \quad x \in \partial\Omega, \quad t > 0, \quad K > 0, \quad \gamma < \lambda_1 \quad (3.4)$$

and conversely (1.1)–(1.3) has no global nontrivial solutions if either  $q = 1$ ,  $\min(r, p) \geq 1$  and

$$\int_0^\infty \underline{a}(t) \exp \left[ -(r+p-1) \left( \lambda_1 t + \int_0^t \bar{b}(\tau) d\tau \right) \right] dt = \infty, \quad (3.5)$$

or  $q = 1$ ,  $l > 1$  and

$$\int_0^\infty \underline{k}(t) \exp \left[ -(l-1) \left( \lambda_1 t + \int_0^t \bar{b}(\tau) d\tau \right) \right] dt = \infty. \quad (3.6)$$

**Theorem 3.2.** *Let  $q = 1$ ,  $\min(r + p, l) > 1$  and (3.3), (3.4) hold. Then there exist global solutions of (1.1)–(1.3) for small initial data. If either  $q = 1$ ,  $\min(r, p) \geq 1$  and (3.5) holds or  $q = 1$ ,  $l > 1$  and (3.6) holds, then any nontrivial solution of (1.1)–(1.3) blows up in finite time.*

*Proof.* Assume that  $T$  is any positive constant,  $q = 1$ ,  $\min(r + p, l) > 1$  and (3.3), (3.4) hold. We choose  $\tilde{\lambda}$  in such a way that

$$\max(\sigma, \gamma) < \tilde{\lambda} < \lambda_1.$$

Let  $\tilde{\Omega}$  be bounded domain in  $\mathbb{R}^n$  with smooth boundary such that  $\Omega \subset \subset \tilde{\Omega}$  and  $\tilde{\lambda}$  be the first eigenvalue of (2.6) in  $\tilde{\Omega}$ . Then correspondent eigenfunction  $\tilde{\varphi}(x)$  satisfies

$$\frac{\sup_{\tilde{\Omega}} \tilde{\varphi}(x)}{\inf_{\tilde{\Omega}} \tilde{\varphi}(x)} < d$$

for some  $d > 0$ . Choosing

$$0 < \varepsilon \leq (Kd^l)^{-\frac{1}{r-1}}, \quad \sup_{\tilde{\Omega}} \tilde{\varphi}(x) = d\varepsilon,$$

we have  $\inf_{\partial\tilde{\Omega}} \tilde{\varphi}(x) > \varepsilon$ . We put  $N = \sup_{\tilde{\Omega}} \tilde{\varphi}^{r-1}(x) \int_{\tilde{\Omega}} \tilde{\varphi}^p(y) dy$  and

$$f(t) = \exp(-\tilde{\lambda}t) \left( B - (r + p - 1)N \int_0^t \bar{a}(\tau) \exp \left[ -(r + p - 1) \left( \tilde{\lambda}\tau + \int_0^\tau \underline{b}(s) ds \right) \right] d\tau \right)^{-\frac{1}{r+p-1}},$$

where

$$B = 1 + (r + p - 1)N \int_0^\infty \bar{a}(\tau) \exp \left[ -(r + p - 1) \left( \tilde{\lambda}\tau + \int_0^\tau \underline{b}(s) ds \right) \right] d\tau.$$

It is easy to check that

$$v(x, t) = \tilde{\varphi}(x) f(t) \exp \left( - \int_0^t \underline{b}(\tau) d\tau \right)$$

is a supersolution of (1.1)–(1.3) in  $Q_T$  for  $u_0(x) \leq B^{-\frac{1}{r+p-1}} \tilde{\varphi}(x)$ . By Theorem 2.2 there exist global solutions of (1.1)–(1.3).

Now suppose that  $q = 1$ ,  $\min(r, p) \geq 1$  and (3.5) holds. Multiplying (1.1) by  $\varphi(x) \exp(\lambda_1 t)$ , where  $\varphi(x)$  is defined in (2.6) and (2.25), and integrating the obtained equation over  $\Omega$ , from (2.24), (2.26), Green's identity and Jensen's inequality, we obtain

$$J'(t) \geq [\sup_{\Omega} \varphi(x)]^{-1} \underline{a}(t) \exp[\lambda_1(1 - r - p)t] J^{r+p}(t) - \bar{b}(t) J(t).$$

Now (3.5) guarantees blow-up of  $J(t)$  in finite time. The case  $q = 1$ ,  $l > 1$  is treated similarly.  $\square$

*Remark 3.3.* The conclusion of Theorem 3.2 is not true if  $\sigma > \lambda_1$  in (3.3), or  $\gamma = \lambda_1$  in (3.4), or  $\lambda_1$  is replaced by a smaller value in (3.5) or (3.6).

Further we consider the case  $q > 1$ . To prove blow-up of all nontrivial solutions we need universal lower bound for solutions of (1.1)–(1.3). Assume that

$$b(x, t) \leq \varepsilon(t) \exp[\lambda_1(q - 1)t], \quad x \in \Omega, \quad t > 0, \quad (3.7)$$

where

$$\varepsilon(t) \in C([0, \infty)), \varepsilon(t) \geq 0, \int_0^\infty \varepsilon(t) dt < \infty. \quad (3.8)$$

**Lemma 3.4.** *Let  $u(x, t)$  be a solution of (1.1)–(1.3) in  $Q_T$ ,  $q > 1$ ,  $u_0(x) \not\equiv 0$  and (3.7), (3.8) hold. Then for any  $t_0 \in (0, T)$  there exists  $d > 0$ , which does not depend on  $T$ , such that*

$$u(x, t) \geq d\varphi(x) \exp(-\lambda_1 t), \quad x \in \Omega, \quad t \in (t_0, T), \quad (3.9)$$

where  $\varphi(x)$  is defined in (2.6) and (2.25).

*Proof.* For  $T_0 \in (0, T)$  denote  $m_0 = \max \left( \sup_{Q_{T_0}} u(x, t), \sup_{Q_{T_0}} b(x, t) \right)$ . Let  $v(x, t)$  be a solution of the problem

$$\begin{cases} v_t = \Delta v - b(x, t)v^q, & (x, t) \in Q_{T_0}, \\ v(x, t) = 0, & (x, t) \in S_{T_0}, \\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (3.10)$$

where  $v_0(x) \in C_0^\infty(\Omega)$ ,  $0 \leq v_0(x) \leq u_0(x)$  and  $v_0(x) \not\equiv 0$ . Obviously,  $u(x, t)$  is a supersolution of (3.10). By comparison principle for (3.10) we obtain

$$u(x, t) \geq v(x, t), \quad (x, t) \in Q_{T_0}. \quad (3.11)$$

We put  $h(x, t) = \exp(\mu t)v(x, t)$ , where  $\mu \geq m_0^q$ . Then in  $Q_{T_0}$

$$h_t - \Delta h \geq \exp(\mu t)v(\mu - b(x, t)v^{q-1}) \geq 0.$$

Since  $h(x, 0) = v_0(x)$  and  $v_0(x)$  is nontrivial nonnegative function in  $\Omega$ , by strong maximum principle

$$h(x, t) > 0, \quad (x, t) \in Q_{T_0}. \quad (3.12)$$

By virtue of Theorem 3.6 in [22]

$$\max_{\partial\Omega} \frac{\partial h(x, t_0)}{\partial n} < 0, \quad (3.13)$$

where  $t_0 \in (0, T_0)$ . From (3.12) and (3.13) it follows that

$$v(x, t) > 0 \text{ in } Q_{T_0} \text{ and } \max_{\partial\Omega} \frac{\partial v(x, t_0)}{\partial n} < 0.$$

Then there exists positive constant  $d_0$  such that

$$v(x, t_0) \geq d_0\varphi(x) \exp(-\lambda_1 t_0), \quad x \in \Omega. \quad (3.14)$$

By (3.11) and (3.14)

$$u(x, t_0) \geq d_0\varphi(x) \exp(-\lambda_1 t_0), \quad x \in \Omega.$$

A straightforward computation shows that for large  $f_0$

$$\underline{u}(x, t) = \varphi(x) \exp(-\lambda_1 t) \left\{ f_0 + (q-1) [\sup_{\Omega} \varphi(x)]^{q-1} \int_{t_0}^t \varepsilon(\tau) d\tau \right\}^{-\frac{1}{q-1}}$$

is a subsolution of (3.10) in  $Q_{T_0} \setminus \overline{Q_{t_0}}$  with initial datum  $v(x, t_0) = u(x, t_0)$ . Application of comparison principle for (3.10) completes the proof.  $\square$

Next we assume that

$$\int_{\Omega} k(x, y, t) dy \leq A \exp(\sigma t), \quad x \in \partial\Omega, \quad t > 0, \quad A > 0, \quad \sigma < \lambda_1(l-1) \quad (3.15)$$

and

$$b(x, t) \geq Ba(x, t) \exp(-\omega t), \quad x \in \Omega, \quad t > 0, \quad B > 0, \quad \omega < \lambda_1(r+p-q) \quad (3.16)$$

or  $b(x, t)$  satisfies (3.7), (3.8), where

$$\lim_{t \rightarrow \infty} \varepsilon(t) = 0, \quad (3.17)$$

and

$$k(x, y, t) \geq D \exp[\lambda_1(l-1)t], \quad x \in \partial\Omega, \quad y \in \Omega, \quad D > 0 \quad (3.18)$$

for large values of  $t$ .

**Theorem 3.5.** *If  $l > 1$ ,  $1 < q < r + p$  and (3.15), (3.16) hold, then there exist global solutions of (1.1)–(1.3) for small initial data. If  $l \geq q > 1$  and (3.7), (3.8), (3.17), (3.18) hold, then any nontrivial solution of (1.1)–(1.3) blows up in finite time.*

*Proof.* Let  $l > 1$ ,  $1 < q < r + p$  and (3.15), (3.16) hold. We choose  $\tilde{\lambda}_1$  in the following way

$$\max(\omega/(r+p-q), \sigma/(l-1)) < \tilde{\lambda}_1 < \lambda_1.$$

Let  $\tilde{\Omega}$  be bounded domain in  $\mathbb{R}^n$  with smooth boundary such that  $\Omega \subset\subset \tilde{\Omega}$  and  $\tilde{\lambda}_1$  be the first eigenvalue of (2.6) in  $\tilde{\Omega}$ . The correspondent eigenfunction  $\tilde{\varphi}(x)$  is chosen to satisfy that  $\sup_{\tilde{\Omega}} \tilde{\varphi}(x) = 1$ . Obviously,  $\inf_{\Omega} \tilde{\varphi}(x) > d$  for some  $d > 0$ . Then  $v(x, t) = \beta \exp(-\tilde{\lambda}_1 t) \tilde{\varphi}(x)$  is a supersolution of (1.1)–(1.3) in  $Q_T$  for any  $T > 0$  if

$$\beta \leq \min \left( \left( \frac{B}{\sup_{\Omega} \tilde{\varphi}^{r-q}(x) \int_{\Omega} \tilde{\varphi}^p(y) dy} \right)^{\frac{1}{r+p-q}}, \left( \frac{d}{A} \right)^{\frac{1}{l-1}} \right), \quad u_0(x) \leq \beta \tilde{\varphi}(x).$$

By Theorem 2.2 there exist global solutions of (1.1)–(1.3).

Let  $u(x, t)$  be nontrivial global solution of (1.1)–(1.3),  $l \geq q > 1$  and (3.7), (3.8), (3.17), (3.18) hold. Then by (1.2), (3.9) and (3.18) there exist positive constants  $t_1$  and  $d_1$  such that

$$u(x, t) \geq d_1 \exp(-\lambda_1 t), \quad x \in \partial\Omega, \quad t \geq t_1. \quad (3.19)$$

Let us consider auxiliary problem

$$\begin{cases} v_t = \Delta v - b(x, t)v^q, & x \in \Omega, \quad t > t_2, \\ v(x, t) = u(x, t), & x \in \partial\Omega, \quad t > t_2, \\ v(x, t_2) = u(x, t_2), & x \in \Omega, \end{cases} \quad (3.20)$$

where  $t_2 \geq t_1$ . Using (3.7), (3.17), (3.19), we check that  $\underline{u}(x, t) = d_2 \exp(-\lambda_1 t)$  is a subsolution of (3.20) under suitable choice of  $t_2$  and  $d_2 > 0$ . Comparison principle for (3.20) gives

$$u(x, t) \geq d_2 \exp(-\lambda_1 t), \quad x \in \Omega, \quad t \geq t_2. \quad (3.21)$$

Let  $\varphi(x)$  satisfy (2.6) and (2.25). Multiplying (1.1) by  $\varphi(x) \exp(\lambda_1 t)$ , integrating over  $\Omega$  and using

$$\int_{\partial\Omega} \frac{\partial \varphi(x)}{\partial n} ds = -\lambda_1,$$

Green's identity, Jensen's inequality and (2.24), (2.26), (3.7), (3.17)–(3.19), (3.21), we obtain

$$J'(t) \geq \int_{\Omega} \left( \lambda_1 [\sup_{\Omega} \varphi(x)]^{-1} D \exp[\lambda_1 l t] u^{l-q} - \varepsilon(t) \exp[\lambda_1 q t] \right) u^q \varphi(x) dx \geq d_3 J^q(t)$$

for some  $d_3 > 0$  and large values of  $t > 0$ . Integrating differential inequality, we prove the theorem.  $\square$

*Remark 3.6.* Theorem 3.5 does not hold if  $\sigma = \lambda_1(l-1)$  in (3.15) or  $\lambda_1$  is replaced by a smaller value in (3.18). Furthermore, we can not  $\varepsilon(t)$  replace by any positive constant in (3.7). Indeed, let  $a(x, t) \equiv 0$ ,  $b(x, t) = b \exp[\lambda_1(q-1)t]$ ,  $k(x, y, t) = k \exp[\lambda_1(l-1)t]$ , where  $b$  and  $k$  are positive constants. Then

$$\bar{u}(x, t) = \left\{ \frac{\lambda_1}{b} \right\}^{\frac{1}{q-1}} \exp(-\lambda_1 t)$$

is a supersolution of (1.1)–(1.3) if  $\min(q, l) > 1$ ,

$$k \leq \frac{1}{|\Omega|} \left\{ \frac{b}{\lambda_1} \right\}^{\frac{l-1}{q-1}} \quad \text{and} \quad u_0(x) \leq \left\{ \frac{\lambda_1}{b} \right\}^{\frac{1}{q-1}}.$$

By Theorem 2.2 there exist global solutions of (1.1)–(1.3).

To prove blow-up of all nontrivial solutions of (1.1)–(1.3) for  $\max(r, p) \geq q > 1$  we assume that

$$\underline{a}(t) = \gamma(t) \exp[\lambda_1(r+p-q)t] \bar{b}(t), \quad (3.22)$$

$$\int_0^{\infty} \underline{a}(t) \exp[-\lambda_1(r+p-1)t] dt = \infty, \quad (3.23)$$

where  $\lim_{t \rightarrow \infty} \gamma(t) = \infty$ ,  $\underline{a}(t)$  and  $\bar{b}(t)$  are defined in (3.2).

**Theorem 3.7.** *Let  $\max(r, p) \geq q > 1$  and (3.7), (3.8), (3.22), (3.23) hold. Then any nontrivial solution of (1.1)–(1.3) blows up in finite time.*

*Proof.* Denote

$$I(t) = \int_{\Omega} \left\{ \frac{1}{2} \underline{a}(t) u^r(x, t) \int_{\Omega} u^p(y, t) dy - \bar{b}(t) u^q(x, t) \right\} \varphi(x) dx, \quad (3.24)$$

where  $\varphi(x)$  is defined in (2.6) and (2.25). Suppose at first that  $p \geq q$ . By (3.9), (3.22), (3.24) and Hölder's inequality it follows

$$\begin{aligned} I(t) &\geq \int_{\Omega} \left\{ \frac{1}{2} \underline{a}(t) u^r [\sup_{\Omega} \varphi(x)]^{-1} \int_{\Omega} u^p(y, t) \varphi(y) dy - \bar{b}(t) u^q \right\} \varphi(x) dx \\ &\geq \bar{b}(t) \left[ \int_{\Omega} u^p \varphi dx \right]^{\frac{q}{p}} \left\{ \frac{d^r}{2} [\sup_{\Omega} \varphi(x)]^{-1} \gamma(t) \exp[\lambda_1(p-q)t] \int_{\Omega} \varphi^{r+1} dx \left[ \int_{\Omega} u^p \varphi dy \right]^{\frac{p-q}{p}} - 1 \right\} \\ &\geq \bar{b}(t) \left[ \int_{\Omega} u^p \varphi dx \right]^{\frac{q}{p}} \left\{ \frac{d^{r+p-q}}{2} [\sup_{\Omega} \varphi(x)]^{-1} \gamma(t) \int_{\Omega} \varphi^{r+1} dx \left[ \int_{\Omega} \varphi^{p+1} dy \right]^{\frac{p-q}{p}} - 1 \right\} \geq 0 \end{aligned} \quad (3.25)$$

for large values of  $t$ . Using Hölder's inequality again and (2.27), (3.9), (3.25), we obtain

$$J'(t) \geq \exp(\lambda_1 t) \int_{\Omega} \left\{ \underline{a}(t) u^r [\sup_{\Omega} \varphi(x)]^{-1} \int_{\Omega} u^p(y, t) \varphi(y) dy - \bar{b}(t) u^q \right\} \varphi(x) dx$$

$$\begin{aligned}
&\geq \frac{1}{2} [\sup_{\Omega} \varphi(x)]^{-1} \exp(\lambda_1 t) \underline{a}(t) \int_{\Omega} u^r(x, t) \varphi(x) dx \int_{\Omega} u^p(y, t) \varphi(y) dy \\
&\geq \frac{d^r}{2} [\sup_{\Omega} \varphi(x)]^{-1} \int_{\Omega} \varphi^{r+1} dx \exp[-\lambda_1(r+p-1)t] \underline{a}(t) J^p(t) \tag{3.26}
\end{aligned}$$

for large values of  $t$ . By (3.23), (3.26)  $J(t)$  blows up in finite time.

Suppose that  $r \geq q$ . Arguing as in previous case, we obtain

$$J'(t) \geq \frac{d^p}{2} \int_{\Omega} \varphi^p dx \exp[-\lambda_1(r+p-1)t] \underline{a}(t) J^r(t) \tag{3.27}$$

for large values of  $t$  and  $J(t)$  blows up in finite time again.  $\square$

*Remark 3.8.* Theorem 3.7 does not hold if  $\gamma(t)$  is a bounded function in (3.22). Indeed, suppose that  $p \geq 1$ ,  $r \geq q > 1$ ,  $a(x, t) \equiv \underline{a}(t)$ ,  $b(x, t) \equiv \bar{b}(t)$  and  $k(x, y, t) \equiv 0$ . Let  $\varphi(x)$  be defined in (2.6) and (2.25). Then  $\bar{u}(x, t) = \beta \varphi(x) \exp(-\lambda_1 t)$  is a supersolution of (1.1)–(1.3) for  $u_0(x) \leq \beta \varphi(x)$ ,  $x \in \Omega$  and small  $\beta > 0$ . By Theorem 2.2 there exist global solutions of (1.1)–(1.3).

*Remark 3.9.* We note here an importance of divergence of the integral in (3.23) for blow-up of all nontrivial solutions of (1.1)–(1.3). Suppose that  $a(x, t) \equiv \underline{a}(t)$  and  $k(x, y, t) \equiv 0$ . Let  $\varphi(x)$  be the solution of (2.6) with  $\sup_{\Omega} \varphi(x) = 1$  and  $f(t)$  be a solution of the following differential equation

$$f'(t) + \lambda_1 f(t) - |\Omega| \underline{a}(t) f^{r+p}(t) = 0.$$

Since the integral in (3.23) converges,  $f(t)$  exists for any  $t \geq 0$  if  $f(0)$  is small enough. Then  $\bar{u}(x, t) = \varphi(x) f(t)$  is a supersolution of (1.1)–(1.3) with  $u_0(x) \leq \varphi(x) f(0)$ . By Theorem 2.2 there exist global solutions of (1.1)–(1.3).

*Remark 3.10.* Theorem 3.7 is not true if  $\lambda_1$  is replaced by a larger value in (3.7). Indeed, let  $\sigma > 0$ . We put

$$a(x, t) = \frac{\lambda_1(q-1) + \sigma}{(q-1)|\Omega|} \exp\left[(r+p-1)\left(\lambda_1 + \frac{\sigma}{q-1}\right)t\right],$$

$$b(x, t) = 2\left(\lambda_1 + \frac{\sigma}{q-1}\right) \exp\{[\lambda_1(q-1) + \sigma]t\},$$

$$k(x, y, t) = \frac{1}{|\Omega|} \exp\left[(l-1)\left(\lambda_1 + \frac{\sigma}{q-1}\right)t\right].$$

It is easy to see that

$$u(x, t) = \exp\left[-\left(\lambda_1 + \frac{\sigma}{q-1}\right)t\right]$$

is the solution of (1.1)–(1.3) with  $u_0(x) \equiv 1$ ,  $x \in \Omega$ .

*Remark 3.11.* From Theorem 3.7 it follows that Theorem 3.5 does not hold for  $\omega > \lambda_1(r+p-q)$  in (3.16).

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