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Data-Driven Optimal Control of Linear Time-Invariant Systems ⋆

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Abstract: We consider an optimal control problem for discrete-time linear time-invariant systems subject to unknown state-space model, constrained inputs and noisy output measurements. Since traditional model-based optimal control problem formulations and methods are not applicable to the problem under consideration, we propose a data-driven robust formulation based on the explicit model description derived from a single measured trajectory of the system. Then we propose an open-loop optimal feedback control scheme and show that its efficient implementation requires solution of a number of optimal estimation problems and a deterministic optimal control problem, all in data-driven formulations. The main contributions of this paper are the separation of the estimation and control processes in the data-driven context and the resulting robust feedback control scheme.

Keywords: data-driven control, optimal control, linear system, constraints, measurements, set-membership uncertainty, feedback

1. INTRODUCTION

Data-driven control methods has received a significant attention recently. This is mainly motivated by growing amount of data that is becoming easier to acquire and rapid development of various learning techniques. The idea of incorporating data directly into a control problem formulation, and thus, avoiding model identification step, is becoming very attractive, especially for complex systems. An overview of data-driven approaches in control theory is given in Hou and Wang (2013).

This paper deals with a finite-time optimal control problem for discrete-time linear time-invariant (LTI) systems subject to constrained inputs and noisy output measurements. For LTI systems a promising approach to obtaining a purely data-driven control problem formulation is provided by results originally obtained in the framework of behavioral systems theory, see Willems et al. (2005). As shown in Willems et al. (2005), under an assumption of persistently exciting input, the space of all LTI system trajectories can be obtained from a single input-output trajectory measurement as a linear combination of its time-shifts. Thus, a system is no longer characterized by its state-space model, but rather by one persistently exciting and long enough trajectory. This result from Willems et al. (2005) has been recently translated in Berberich and Allgöwer (2019) to classical state-space control context. A data-based trajectory characterization from Berberich and Allgöwer (2019) was used in Romer et al. (2019) to verify dissipativity properties, and in Berberich et al. (2019) to propose a data-driven model predictive controller with first results on stability and robustness guarantees. Other approaches in this direction can be found in Yang and Li (2015); Coulson et al. (2019); Kadali et al. (2003); Salvador et al. (2018).

In this note we rely on results from Berberich and Allgöwer (2019) to propose a robust feedback control scheme in an optimal control problem. Our focus is on a data-driven formulation of optimization problems that are solved in a receding horizon manner to obtain the feedback and on robust output constraint satisfaction. Although here the problem is considered on a finite-time interval, the results can be useful in robust model predictive control, as the approach proposed here is close to the simultaneous control and state estimation approach in Copp and Hespanha (2017).

The overall paper is structured as follows. In Section 2 we outline the model-based optimal control problem subject to set-membership uncertainty in the initial state of the system and output measurement errors, and provide some results related to open-loop optimal feedback control, which are needed in the following sections to derive relevant data-driven counterparts of the model-based approach. Section 3 presents required results from Berberich and Allgöwer (2019) on the LTI system's trajectories representation and introduces one possible data-driven optimal control problem formulation. Here, we also discuss the lack of robustness of the proposed formulation and present the idea of robustifying it via the separation of estimation and control problems which is studied in detail in Section 4. In Section 5 we illustrate the proposed approach by a numerical example. Section 6 provides some final conclusion.

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Notations. In the following the discrete-time LTI system is denoted by G. The concatenation of vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ is denoted by (x, y), i.e. $(x, y) = [x^T, y^T]^T$. The Euclidean and ℓ_∞-norm of vectors $x \in \mathbb{R}^n$ are denoted by $||x||$ and $||x||_\infty$, respectively. Further, we denote a vector of ones by $\mathbb{1}$, its dimension follows from the context. Finally, $x(t|x_0, u_t)$ denotes the trajectory of the system G at time t if the initial state is $x(0) = x_0$ and the control is $u_t = (u(0), u(1), \ldots, u(t))$.

2. MODEL-BASED OPTIMAL CONTROL PROBLEM

In this section we consider a model-based problem formulation, i.e. the case when the system G is explicitly defined by known matrices A, B, C, and D:

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1}$$
$$y(t) = Cx(t) + Du(t), \quad t = 0, \ldots, N-1.$$

In (1) $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^q$, denote the state, the input and the output, respectively; (A, B, C, D) is a minimal realization of G.

System (1) is subject to input and output constraints

$$u(t) \in U, \; y(t) \in Y(t), \; t = 0, \ldots, N-1, \tag{2}$$

where $U = \{u \in \mathbb{R}^r : u_{\min} \le u \le u_{\max}\}$, and $Y(t) = \{y \in \mathbb{R}^q : G(t)y \le g(t)\}$, $G(t) \in \mathbb{R}^{m \times q}$, $g(t) \in \mathbb{R}^m$.

In the following we distinguish between the variables used in model (1) and the real system/plant variables. The latter will be denoted by a superscript p, thus,

$$u^p_\tau = (u^p(0), \ldots, u^p(\tau-1)), \quad y^p_\tau = (y^p(0), \ldots y^p(\tau-1))$$

are the input and the output/measurement trajectories which realize in a particular control process by the time τ. We will also refer to them as *past input-otput trajectories.*

The measured output $y^p(t)$ differs from model (1) output $y(t)$ due to two types of uncertainties. First, it is assumed that the initial state x_0 of system (1) is not known. Secondly, the output is measured with an additive bounded error denoted by $\xi(t)$, i.e.

$$y^p(t) = y(t) + \xi(t), \; t = 0, \ldots, N-1.$$

For the uncertain variables set-based estimates in the form $x_0 \in X_0$ and $\xi(t) \in \Xi$ are given. Here, $X_0 = \{x \in \mathbb{R}^n : x_{\min} \le x \le x_{\max}\}$ is the set of possible initial states of system (1), and $\Xi = \{\xi \in \mathbb{R}^p : ||\xi||_\infty \le \varepsilon\}$ is the set of possible realizations of measurement errors, $\varepsilon > 0$.

The control objective is to minimize the quadratic cost of the form

$$\sum_{t=0}^{N-1} ||u(t)||^2, \tag{3}$$

while satisfying input and output constraints (2) robustly, i.e. for all possible initial state and measurement errors realizations.

In the following we discuss the open-loop optimal feedback control strategy for problem (1) – (3). The results are a modification of ideas from Gabasov et al. (2007); Dmitruk et al. (2008), which accounts for constrained outputs in (2) and discrete-time systems (1).

Open-loop (worst-case) optimal feedback control refers to a control strategy that is formed on the base of a repeated solution of the open-loop optimal control problem (1)–(3) at every time instant $\tau = 0, \ldots, N-1$ subject to a shrinking control horizon $\{\tau, \ldots, N\}$ and the past input-output trajectories $\{u^p_\tau, y^p_\tau\}$. If we denote the optimal solution of the problem by $u^0(t|\tau, u^p_\tau, y^p_\tau)$, $t = \tau, \ldots, N$, then the input applied to the plant at time τ is given by

$$u^p(\tau) = u^0(\tau|\tau, u^p_\tau, y^p_\tau).$$

Definition 1. We say that a state $x(\tau)$ is consistent with the past input-output trajectory $\{u^p_\tau, y^p_\tau\}$ if there exist an initial condition $x_0 \in X_0$ and feasible measurement errors $\xi(t)$, $t = 0, \ldots, \tau-1$, such that

$$x(\tau) = x(\tau|x_0, u^p_\tau),$$
$$y^p(t) = Cx(t|x_0, u^p_t) + Du^p(t) + \xi(t), \; t = 0, \ldots, \tau-1.$$

Let $X(\tau, u^p_\tau, y^p_\tau)$ denote the set of all states $x(\tau)$ consistent with $\{u^p_\tau, y^p_\tau\}$.

The open-loop optimal control problem for the time instant τ is formulated as follows

$$\min_u \sum_{t=\tau}^{N-1} ||u(t)||^2, \tag{4}$$
$$\begin{aligned} s.t. \quad & x(t+1) = Ax(t) + Bu(t), \\ & y(t) = Cx(t) + Du(t), \\ & u(t) \in U, \quad y(t) \in Y(t) \quad \forall x(\tau) \in X(\tau, u^p_\tau, y^p_\tau), \\ & t = \tau, \ldots, N-1. \end{aligned}$$

The solution of problem (4) is a feasible input $u^0(t|\tau, u^p_\tau, y^p_\tau)$, $t = \tau, \ldots, N$, that for every realization of the initial state $x(\tau) \in X(\tau, u^p_\tau, y^p_\tau)$ steers the output inside the set $Y(t)$ and minimizes the cost-to-go $\sum_{t=\tau}^{N-1} ||u(t)||^2$. Thus, (4) guarantees robust constraint satisfaction for all states $x(\tau)$ consistent with $\{u^p_\tau, y^p_\tau\}$.

Due to the separation principle for linear systems (see e.g. Kurzhanskii and Vályi (1997)), the optimal input $u^0(t|\tau, u^p_\tau, y^p_\tau)$, $t = \tau, \ldots, N$, of problem (4) can be obtained by the solution of several optimal estimation problems and one deterministic optimal control problem.

To obtain these optimization problems we separate the state $x(t)$ and the output $y(t)$, $t = \tau, \ldots, N-1$, in (4) into the sums

$$x(t) = x_0(t) + \hat{x}(t), \quad y(t) = y_0(t) + \hat{y}(t), \tag{5}$$

where $x_0(t)$, $y_0(t)$ denote the state and the output of the nominal system with the trivial initial state

$$x_0(t+1) = Ax_0(t) + Bu(t), \quad x_0(\tau) = 0,$$
$$y_0(t) = Cx_0(t) + Du(t), \; t = \tau, \ldots, N-1,$$

and $\hat{x}(t)$, $\hat{y}(t)$ denote the state and the output of the uncontrolled system with the uncertain initial state

$$\hat{x}_0(t+1) = A\hat{x}(t), \quad \hat{x}(\tau) \in X(\tau, u^p_\tau, y^p_\tau),$$
$$\hat{y}(t) = C\hat{x}(t), \; t = \tau, \ldots, N-1.$$

Now consider the output constraint at the time instant t of problem (4) in the form

$$G(t)(y_0(t) + \hat{y}(t)) \le g(t) \quad \forall x(\tau) \in X(\tau, u^p_\tau, y^p_\tau).$$

Obviously, this constraint is satisfied robustly, if the nominal output $y_0(t)$ satisfies the tightened constraint of the form

$$y_0(t) \in Y_0(t|\tau) = \{y \in \mathbb{R}^q : G(t)y \le g(t) - \chi(t|\tau)\}.$$

Here $\chi(t|\tau) = \chi(t|\tau, u^p_\tau, y^p_\tau) = (\chi_i(t|\tau), i = 1, \ldots, m)$, and each $\chi_i(t|\tau)$ corresponds to the worst-case realization of the state $x(\tau)$ as obtained by the solution of the optimal estimation problem

$$\begin{aligned} \chi_i(t|\tau) = & \max_z \ G_i(t)\hat{y}(t), \\ s.t. \ \ & \hat{x}(s+1) = A\hat{x}(s), \ \hat{x}(\tau) = z, \\ & \hat{y}(s) = C\hat{x}(s), \quad s = \tau, \ldots, t-1, \\ & z \in X(\tau, u^p_\tau, y^p_\tau), \end{aligned} \tag{6}$$

where $G_i(t)$ denotes the i-th row of the matrix $G(t)$.

After calculating all estimates $\chi(t|\tau)$, $t = \tau, \ldots, N-1$, the optimal input $u^0(t|\tau, u^p_\tau, y^p_\tau)$, $t = \tau, \ldots, N$, is now obtained by the solution of the following deterministic optimal control problem

$$\begin{aligned} \min_u \ & \sum_{t=\tau}^{N-1} ||u(t)||^2, \\ s.t. \ \ & x_0(t+1) = Ax_0(t) + Bu(t), \quad x_0(\tau) = 0, \\ & y_0(t) = Cx_0(t) + Du(t), \\ & u(t) \in U, \quad y_0(t) \in Y_0(t|\tau), \\ & t = \tau, \ldots, N-1. \end{aligned} \tag{7}$$

As outlined, vectors $\chi(t|\tau)$, $t = \tau, \ldots, N-1$, contain all information required to solve the optimal control problem (7). To conclude this section we show how to obtain the estimates $\chi(t|\tau)$ in an efficient way, without calculating the set $X(\tau, u^p_\tau, y^p_\tau)$.

Following Gabasov et al. (2007), this can be achieved on the base of "purified measurements" $y^p_0(s) = y^p(s) - Cx(s|0, u^p_s) - Du^p(s)$, $s = 0, \ldots, \tau-1$, by the solution of linear programs (for all $i = 1, \ldots, m$, and $t = \tau \ldots, N-1$)

$$\begin{aligned} \gamma_i(t|\tau) = & \max_{x_0} \ G_i(t)CA^t x_0, \\ s.t. \quad & -\varepsilon\mathbb{1} \le y^p_0(s) - CA^s x_0 \le \varepsilon\mathbb{1}, \ s = 0, \ldots, \tau-1, \\ & x_{\min} \le x_0 \le x_{\max}. \end{aligned}$$

Then $\chi(t) = G(t)CA^{t-\tau}x(\tau|0, u^p_\tau) + \gamma(t|\tau)$.

3. DATA-DRIVEN OPTIMAL CONTROL PROBLEM

In this section we assume that matrices A, B, C, and D of the system G are not known, therefore the mathematical model (1) is not available for formulating the optimal control problem (1)–(3). Moreover, the state dimension n is also not known, but an upper estimate for it is given (see, Berberich et al. (2019)).

Our knowledge about the system G is limited to a single a priory measured input-output trajectory

$$u^d = \left(u^d(0), \ldots, u^d(N^d-1)\right), \tag{8}$$

$$y^d = \left(y^d(0), \ldots y^d(N^d-1)\right). \tag{9}$$

Data (8),(9) can be obtained via experiments or simulations and satisfies relations (1) with some initial state x_0. Note that here the output y^d is not affected by any measurement noise. This is a restrictive assumption that will be relaxed in the future work.

3.1 Trajectory-based representation of LTI systems

Recent results, obtained in Berberich and Allgöwer (2019), provide characterization of the trajectory space of the LTI system G using only its single input-output trajectory (8),(9). This is done via Hankel matrices that consist of the time-shifts of trajectory (8),(9).

For u^d and y^d define the Hankel matrices, respectively

$$H_u = H_N(u^d) = \begin{bmatrix} u^d(0) & u^d(1) & \ldots & u^d(N^d-N) \\ u^d(1) & u^d(2) & \ldots & u^d(N^d-N+1) \\ \vdots & \vdots & \ddots & \vdots \\ u^d(N-1) & u^d(N) & \ldots & u^d(N^d-1) \end{bmatrix},$$

$$H_y = H_N(y^d) = \begin{bmatrix} y^d(0) & y^d(1) & \ldots & y^d(N^d-N) \\ y^d(1) & y^d(2) & \ldots & y^d(N^d-N+1) \\ \vdots & \vdots & \ddots & \vdots \\ y^d(N-1) & y^d(N) & \ldots & y^d(N^d-1) \end{bmatrix}.$$

The following definition and theorem are given according to Berberich and Allgöwer (2019); Berberich et al. (2019).

Definition 2. We say that a signal u^d defined by (8) with $u(t) \in \mathbb{R}^r$ is persistently exciting of order L if rank $H_L(u^d) = rL$.

Theorem 1. Suppose that (8),(9) is a trajectory of the LTI system G, where u^d is persistently exciting of order $N+n$. Then,

$$u = (u(0), \ldots, u(N-1)), \ y = (y(0), \ldots y(N-1))$$

is a trajectory of G if and only if there exists $\alpha \in \mathbb{R}^{N^d-N+1}$ such that

$$\begin{bmatrix} H_u \\ H_y \end{bmatrix} \alpha = \begin{bmatrix} u \\ y \end{bmatrix}. \tag{10}$$

Theorem 1 gives a useful representation of the input-output trajectories of the system G. The LTI system is no longer characterized by its state-space model, but rather by a single persistently exciting and long enough trajectory. It also allows us to skip identification of model (1) and use relation (10) directly in the optimization problem. This is done in the next section.

3.2 Data-driven problem formulation

In the following we consider the LTI system G with its trajectory-based representation (10). Since there is no state information or state-space model of the system G, there is no information about the initial state or set X_0 available. As shown in Berberich and Allgöwer (2019), the trajectory of the system, and hence, its initial state, is uniquely defined by n first (for $t = 0, \ldots, n-1$) input-output measurements. Therefore, we assume that the control of the real plant starts at the time instant $\tau = n$.

Consider an arbitrary τ, $n \le \tau \le N-1$. As in Section **??**, the triple $(\tau, u^p_\tau, y^p_\tau)$ is the current position of the control process.

We are interested in representing the trajectory $\{(u^p_\tau, u), (\hat{y}^p_\tau, y)\}$ with a noiseless past output $\hat{y}^p = (\hat{y}^p(0), \hat{y}^p(1), \ldots \hat{y}^p(\tau-1))$ and *future input-output trajectories*

$$u = (u(\tau), \ldots, u(N-1)), y = (y(\tau), \ldots y(N-1)).$$

The noiseless output $\hat{y}^p$ relates to the measurements y^p, via the constraint

$$||\hat{y}^p(t) - y^p(t)||_\infty \le \varepsilon, \quad t = 0, \ldots, \tau-1.$$

According to (10), the trajectory $\{(u^p_\tau, u), (\hat{y}^p_\tau, y)\}$ is represented by the relation

$$\begin{bmatrix} H_u \\ H_y \end{bmatrix} \alpha(\tau) = \begin{bmatrix} u_\tau^p \\ u \\ \hat{y}_\tau^p \\ y \end{bmatrix} \Leftrightarrow \begin{array}{l} H_u^p(\tau)\alpha(\tau) = u_\tau^p, \\ H_u^f(\tau)\alpha(\tau) = u, \\ H_y^p(\tau)\alpha(\tau) = \hat{y}_\tau^p, \\ H_y^f(\tau)\alpha(\tau) = y, \end{array} \tag{11}$$

where $\alpha(\tau) \in \mathbb{R}^{N^d - N + 1}$ is a parameter, corresponding to the current time instant τ, the Hankel matrices are splitted as follows

$$H_u^p(\tau) = \begin{bmatrix} u^d(0) & \dots & u^d(N^d - N) \\ \vdots & \ddots & \vdots \\ u^d(\tau - 1) & \dots & u^d(N^d - N + \tau - 1) \end{bmatrix},$$

$$H_u^f(\tau) = \begin{bmatrix} u^d(\tau) & \dots & u^d(N^d - N + \tau) \\ \vdots & \ddots & \vdots \\ u^d(N - 1) & \dots & u^d(N^d - 1) \end{bmatrix},$$

$$H_y^p(\tau) = \begin{bmatrix} y^d(0) & \dots & y^d(N^d - N) \\ \vdots & \ddots & \vdots \\ y^d(\tau - 1) & \dots & y^d(N^d - N + \tau - 1) \end{bmatrix},$$

$$H_y^f(\tau) = \begin{bmatrix} y^d(\tau) & \dots & y^d(N^d - N + \tau) \\ \vdots & \ddots & \vdots \\ y^d(N - 1) & \dots & y^d(N^d - 1) \end{bmatrix}.$$

The simplest way to formulate a data-driven optimal control problem is to replace the dynamical system (1) by the equality constraints (11). This yields the following formulation of the optimal control problem

$$\begin{aligned} \min_{\alpha(\tau), u, y, \hat{y}^p} \quad & ||u||^2, \\ s.t. \quad & H_u^p(\tau)\alpha(\tau) = u_\tau^p, \\ & H_u^f(\tau)\alpha(\tau) = u, \\ & H_y^p(\tau)\alpha(\tau) = \hat{y}_\tau^p, \\ & H_y^f(\tau)\alpha(\tau) = y, \\ & ||\hat{y}^p(t) - y^p(t)||_\infty \le \varepsilon, \quad t = 0, \dots, \tau - 1, \\ & u(t) \in U, \quad y(t) \in Y(t), \quad t = \tau, \dots, N - 1. \end{aligned} \tag{12}$$

Problem (12), however, is not a data-driven counterpart of the robust model-based problem (4). Formulation (12) fits a single parameter $\alpha(\tau)$ to both past data $\{u_\tau^p, y_\tau^p\}$ and predicted input-output trajectory $\{u, y\}$, and, thus, has no robustness. In the real process realization y could be such that output constraints are violated.

To overcome the inconsistency of formulation (12) we aim to find a separation between fitting past and future trajectories. In the next section we propose one approach to tackle this idea.

4. DATA-DRIVEN ROBUST OPTIMAL CONTROL PROBLEM WITH SEPARATION

The goal of this section is to incorporate separation of estimation and control processes, as in Section 2, in the data-based trajectory representation (11).

4.1 Robust problem formulation

We follow the ideas of Section 2, namely decomposition (5), and represent $\alpha(\tau)$ as a sum

$$\alpha(\tau) = \alpha_0(\tau) + \hat{\alpha}(\tau), \tag{13}$$

where $\alpha_0(\tau)$ will be used to characterize future (predicted) optimal input-output trajectory $\{u, y\}$ and $\hat{\alpha}(\tau)$ will define possible past trajectories $\{u^p, \hat{y}^p\}$.

Accordingly, the future output will be decomposed into two parts:

$$y = y_0 + \hat{y}, \tag{14}$$

where $y_0 = (y_0(\tau), \dots, y_0(N-1))$ is the nominal output, corresponding to $\alpha_0(\tau)$ and optimized future inputs u, and $\hat{y} = (\hat{y}(\tau), \dots \hat{y}(N-1))$ corresponds to $\hat{\alpha}(\tau)$ and feasible future outputs, consistent with past trajectories $\{u^p, \hat{y}^p\}$.

Definition 3. We say that the parameter $\hat{\alpha}(\tau)$ is consistent with the past input-output trajectory $\{u^p, y^p\}$ if

$$\begin{bmatrix} H_u \\ H_y^p(\tau) \end{bmatrix} \hat{\alpha}(\tau) = \begin{bmatrix} u_\tau^p \\ 0 \\ \hat{y}_\tau^p \end{bmatrix}, \tag{15}$$

$$||\hat{y}^p(t) - y^p(t)||_\infty \le \varepsilon, \quad t = 0, \dots, \tau - 1.$$

Each $\hat{\alpha}(\tau)$ satisfying constraints (15) defines a trajectory $\{u^p, \hat{y}^p\}$ of the system G, i.e. there exists an initial state x_0 such that $\hat{y}^p(t) = Cx(t|x_0, u_t^p) + Du^p(t)$, $t = 0, \dots, \tau - 1$, and the internal state $x(\tau) = x(\tau|x_0, u_\tau^p)$ is consistent with the past input-output trajectory $\{u_\tau^p, y_\tau^p\}$ as in Definition 1.

The set of all $\hat{\alpha}(\tau)$ consistent with the past input-output trajectory $\{u^p, y^p\}$ is denoted by $\mathcal{A}(\tau, u_\tau^p, y_\tau^p)$, i.e.

$$\mathcal{A}(\tau, u_\tau^p, y_\tau^p) = \Big\{ \hat{\alpha}(\tau) \in \mathbb{R}^{N^d - N + 1} : \; H_u^p(\tau)\hat{\alpha}(\tau) = u_\tau^p,$$
$$H_u^f(\tau)\hat{\alpha}(\tau) = 0, \; -\varepsilon \mathbb{1} \le H_y^p(\tau)\hat{\alpha}(\tau) - y^p \le \varepsilon \mathbb{1} \Big\},$$

where $\mathbb{1}$ is a $q(N - \tau)$-vector of ones.

From the previous discussion it follows that the set $\mathcal{A}(\tau, u_\tau^p, y_\tau^p)$ is a data-driven counterpart of the set $X(\tau, u_\tau^p, y_\tau^p)$ of states $x(\tau)$ consistent with the past trajectory $\{u_\tau^p, y_\tau^p\}$ in Section **??**.

The future input-output nominal trajectory $\{u, y_0\}$ is defined by the parameter $\alpha_0(\tau)$ via the following representation

$$\begin{bmatrix} H_u \\ H_y \end{bmatrix} \alpha_0(\tau) = \begin{bmatrix} 0 \\ u \\ 0 \\ y_0 \end{bmatrix}. \tag{16}$$

Obviously, the sum of (15), (16) results in (11), where $\alpha(\tau)$, y are defined by decompositions (13), (14) and $\hat{y} = H_y^f(\tau)\hat{\alpha}(\tau)$:

$$\begin{bmatrix} H_u \\ H_y \end{bmatrix} \alpha(\tau) = \begin{bmatrix} H_u \\ H_y \end{bmatrix} (\alpha_0(\tau) + \hat{\alpha}(\tau)) = \begin{bmatrix} u^p \\ u \\ \hat{y}^p \\ y_0 + \hat{y} \end{bmatrix}.$$

The future output y is subject to constraints

$$H_y(t)(\alpha_0(\tau) + \hat{\alpha}(\tau)) \in Y(t), \quad t = \tau, \dots, N - 1, \tag{17}$$

that should be satisfied robustly, i.e. for all $\hat{\alpha}(\tau) \in \mathcal{A}(\tau, u_\tau^p, y_\tau^p)$. Here, $H_y(t) = \big(y^d(t), \dots, y^d(N^d - N + t)\big)$ is the block of the matrix H_y, corresponding to the time instant t.

Summarizing, the optimal control problem for the current position $(\tau, u_\tau^p, y_\tau^p)$ can be formulated as

$$\min_{\alpha_0(\tau),u} \quad ||u||^2, \tag{18}$$
$$\begin{aligned} s.t. \quad & H_u^p(\tau)\alpha_0(\tau) = 0, \\ & H_y^p(\tau)\alpha_0(\tau) = 0, \\ & H_u^f(\tau)\alpha_0(\tau) = u, \\ & H_y(t)(\alpha_0(\tau) + \hat{\alpha}(\tau)) \in Y(t), \ \forall \hat{\alpha}(\tau) \in \mathcal{A}(\tau, u_\tau^p, y_\tau^p), \\ & u(t) \in U, \quad t = \tau, \dots, N-1. \end{aligned}$$

In contrast to (12), problem (18) is a robust formulation of the data-driven optimal control problem. Its solution, denoted by $\alpha_0^0(\tau)$, guarantees constraint satisfaction for every output y consistent with the past trajectory $\{u_\tau^p, y_\tau^p\}$.

In the rest of this section we show that problem (18) corresponds to model-based optimal control problem (4) in Section 2, i.e. both problems yield same results for the same open-loop optimal feedback control process if $X_0 = \mathbb{R}^n$ in the model-based approach and the control starts at the same time $\tau = n$ as in the data-driven approach.

4.2 Optimal estimation and control problems

Here we discuss how problem (18) is solved for a given time instant τ. Our focus is on constraint (17) that can be rewritten using the definition of the set $Y(t)$ as

$$G(t)H_y(t)(\alpha_0(\tau) + \hat{\alpha}(\tau)) \le g(t) \ \ \forall \hat{\alpha}(\tau) \in \mathcal{A}(\tau, u_\tau^p, y_\tau^p),$$
$$t = \tau, \dots, N-1.$$

This is equivalent to the following constraint

$$G(t)H_y(t)\alpha_0(\tau) + \chi^\alpha(t|\tau) \le g(t),$$

where $\chi^\alpha(t|\tau) = \chi^\alpha(t|\tau, u_\tau^p, t_\tau^p)$ corresponds to the estimates of the set $\mathcal{A}(\tau, u_\tau^p, y_\tau^p)$ in the directions defined by the rows of the matrix $G(t)H_y(t)$ and can be found as the solutions of m maximization problems of the form

$$\chi_i^\alpha(t|\tau) = \max_{\hat{\alpha}(\tau) \in \mathcal{A}(\tau, u_\tau^p, y_\tau^p)} G_i(t)H_y(t)\hat{\alpha}(\tau), \quad i = 1, \dots, m.$$

Thus, as in Section 2, problem (18) is separated into $m(N-\tau)$ optimal estimation problems

$$\chi_i^\alpha(t|\tau) = \max_{\hat{\alpha}(\tau)} G_i(t)H_y(t)\hat{\alpha}(\tau), \tag{19}$$
$$\begin{aligned} s.t. \quad & H_u^p(\tau)\hat{\alpha}(\tau) = u_\tau^p, \\ & H_u^f(\tau)\hat{\alpha}(\tau) = 0, \\ & -\varepsilon \mathbb{1} \le H_y^p(\tau)\hat{\alpha}(\tau) - y_\tau^p \le \varepsilon \mathbb{1}, \end{aligned}$$

and the optimal control problem

$$\min_{\alpha_0(\tau),u} \quad ||u||^2, \tag{20}$$
$$\begin{aligned} s.t. \quad & H_u^p(\tau)\alpha_0(\tau) = 0, \\ & H_y^p(\tau)\alpha_0(\tau) = 0, \\ & H_u^f(\tau)\alpha_0(\tau) = u, \\ & G(t)H_y(t)\alpha_0(\tau) \le g(t) - \chi^\alpha(t|\tau), \\ & u(t) \in U, \quad t = \tau, \dots, N-1. \end{aligned}$$

Let's consider problems (6) and (19) for a fixed $\tau \ge n$ and assume that $X_0 = \mathbb{R}^n$. Based on the Theorem 1, to each $\hat{\alpha}(\tau) \in \mathcal{A}(\tau, u_\tau^p, y_\tau^p)$ there corresponds a state $z \in X(\tau, u_\tau^p, y_\tau^p)$ such that $\hat{y}(t) = H_y(t)\hat{\alpha}(\tau) = CA^{t-\tau}z$, and vise versa. Obviously,

$$\chi(t|\tau) = \chi^\alpha(t|\tau) \ \forall \ t = \tau, \dots, N-1, \ \forall \ \tau = n, \dots, N-1.$$

It immediately follows that problems (7) and (20) also yield same results, i.e.

$$u^0(t|\tau, u_\tau^p, y_\tau^p) = H_u(t)\alpha_0^0(\tau), \quad t = \tau, \dots, N-1.$$

4.3 Data-driven open-loop optimal feedback control

Open-loop optimal feedback control in data-driven context presented in this section is now a control strategy formed on the base of a repeated solution of $m(N-\tau)$ optimal estimation problems (19) and one optimal control problem (20) at every time instant $\tau = n, \dots, N-1$. The first value of the optimal input $u^0(t|\tau, u_\tau^p, y_\tau^p) = H_u(t)\alpha_0^0(\tau)$, $t = \tau, \dots, N$, is applied to the plant at time τ. Therefore, the applied data-driven open-loop optimal feedback is given by

$$u^p(\tau) = u^0(\tau|\tau, u_\tau^p, y_\tau^p) = H_u(\tau)\alpha_0^0(\tau), \quad \tau = n, \dots, N-1.$$

Given this feedback we can derive:

Proposition 1. Assume that a solution of problem (20) at the time instant $\tau = n$ exists. Then (20) is feasible for all $\tau = n, \dots, N-1$, and the overall cost at the time instant τ given by

$$J(\tau) = \sum_{t=0}^{\tau-1} ||u^p(t)||^2 + \sum_{t=\tau}^{N-1} ||u^0(t|\tau, u_\tau^p, y_\tau^p)||^2$$

is non-increasing as a function of τ.

This proposition implies that if the optimal solution exists for $\tau = n$ then it exists for all times (i.e. recursive feasibility property holds), and thus, the data-driven open-loop optimal feedback control can indeed be implemented. Note that the proof is trivial due to equivalence of data-driven problem (18) and model-based problem (4) for which the result as in Proposition 1 follows from Bellman's principle of optimality (see also Gabasov et al. (2007) and Gabasov et al. (2004)).

5. EXAMPLE

In this section, we apply the data-driven open-loop optimal feedback control to the following discrete-time LTI system:

$$x(t+1) = \begin{bmatrix} 0.9950 & 0.0998 \\ -0.0998 & 0.9950 \end{bmatrix} x(t) + \begin{bmatrix} 0.0050 \\ 0.0998 \end{bmatrix} u(t),$$
$$y(t) = [1 \ 0]\, x(t), \quad t = 0, 1, \dots, N-1.$$

The control objective is to minimize the overall input (3) over the period of $N = 100$.

Input and output of the system are subject to constraints of the form

$$|u(t)| \le 0.7, \ t = 0, \dots, N-1,$$
$$|y(t)| \le 0.2, \ t = N-10, \dots, N-1,$$

where output constraints are enforced only at the last ten time instants of the control interval.

The output is measured with an additive bounded error, with a bound $\varepsilon = 0.02$.

For the purposes of application of the data-driven approach, the system matrices are unknown, only measured input-output trajectory $\{u_\tau^p, y_\tau^p\}$ realized in a particular control process is available. We also assume that exact estimate of the state dimension is known, i.e. $n = 2$ is

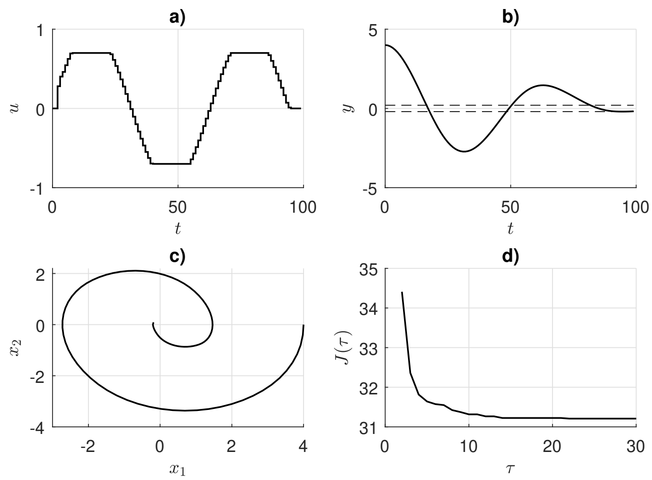

Fig. 1. Optimal feedback realization (a), the resulting output (b), internal trajectory (c) and cost decrease at the beginning of process (d).

used and the feedback control starts at $\tau = 2$. The process considered in this numerical example is generated by the initial state $x(0) = (4, 0)$, the initial input $u(0) = u(1) = 0$, and a random sequence of measurement errors within the interval $[-0.02, 0.02]$.

In an experiment, in which data for Hankel matrices is acquired, an input-output trajectory (8),(9) of length $N^d = 250$ is measured. The input $u^d(t)$ is chosen randomly lying within the input constraint interval. The output $y^d(t)$ corresponds to the initial condition $x(0) = (0, 0)$ and is not affected by the measurement noise.

The results of the application of open-loop optimal feedback control scheme, described in Section 4.3, are illustrated in Figure 1. It can be observed that the applied input u^p satisfies the constraints that become active on a part of the control interval, and the output y^p is within the dash line corresponding to the constraints at times from $\tau = 90$ to $\tau = 99$. To illustrate Proposition 1 a fragment of the cost $J(\tau)$ is shown for $\tau = 0, \ldots, 30$. After $\tau = 30$ the cost change is insignificant and its final value is $J(N-1) = 31.2056$.

Application of the model-based approach of Section 2 to this example gives the same results.

6. CONCLUSIONS

This paper presents a finite-time optimal feedback control scheme in the data-driven context for discrete-time LTI systems subject to constrained inputs and constrained noisy output measurements. In particular, the feedback is obtained in a receding horizon control fashion based on a repeated solution of the robust optimal control problem. We present the efficient data-driven optimal control problem formulation that allows to separate estimation and control problems and guarantees output constraints satisfaction despite uncertainty generated by noisy measurements. The proposed approach is a data-driven counterpart of the model-based approach from Dmitruk et al. (2008) and Gabasov et al. (2007). The results of this paper can be useful for data-driven model predictive control schemes that utilize the simultaneous control and state estimation approach as in Copp and Hespanha (2017).

Future research will investigate the case of noisy output measurements in experimental data (9) for Hankel matrices construction.

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