УДК 517.9

МЕТОД ОСЛАБЛЕНИЯ ФАЗОВЫХ ОГРАНИЧЕНИЙ В НЕГЛАДКИХ ЗАДАЧАХ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ

М. П. ДЫМКОВ[1]***, С. М. ДЫМКОВ***[2]

[1]*Белорусский государственный экономический университет,*
пр. Партизанский, 26, 220070, г. Минск, Беларусь
[2]*Независимый исследователь, г. Минск, Беларусь*

Рассматривается задача оптимального управления, описываемая системой обыкновенных дифференциальных уравнений при наличии фазовых ограничений. Получены теоретические результаты, касающиеся аппроксимации этой задачи последовательностью новых задач оптимального управления с модифицированной правой частью системы управления и без фазовых ограничений. Обсуждаются также вопросы аппроксимации непрерывных систем управления их дискретными версиями.

Ключевые слова: оптимальное управление; фазовые ограничения; негладкая оптимизация; аппроксимация.

A METHOD FOR RELAXING STATE CONSTRAINTS IN NONSMOOTH OPTIMAL CONTROL PROBLEMS

M. P. DYMKOV[a]***, S. M. DYMKOU***[b]

[a]*Belarus State Economic University, 26 Partyzanski Avenue, Minsk 220070, Belarus*
[b]*Independent researcher, Minsk, Belarus*
Corresponding author: M. P. Dymkov (dymkov_m@bseu.by)

In this paper, we consider the optimal control problem described by a system of ordinary differential equations in the presence of state constraints. Theoretical results are obtained concerning the approximation of this problem by a sequence of new optimal control problems with a modified right-hand side of the control system and no state constraints. The issues of the approximation of continuous control systems by their discrete versions are also discussed.

Keywords: optimal control; state constraints; nonsmooth optimisation; approximation.

Образец цитирования:
Дымков МП, Дымков СМ. Метод ослабления фазовых ограничений в негладких задачах оптимального управления. *Журнал Белорусского государственного университета. Математика. Информатика.* 2022;2:107–114 (на англ.).
https://doi.org/10.33581/2520-6508-2022-2-107-114

For citation:
Dymkov MP, Dymkou SM. A method for relaxing state constraints in nonsmooth optimal control problems. *Journal of the Belarusian State University. Mathematics and Informatics.* 2022; 2:107–114.
https://doi.org/10.33581/2520-6508-2022-2-107-114

Авторы:
Михаил Пахомович Дымков – доктор физико-математических наук, профессор; профессор кафедры высшей математики факультета цифровой экономики.
Сергей Михайлович Дымков – кандидат физико-математических наук; независимый исследователь.

Authors:
Michael P. Dymkov, doctor of science (physics and mathematics), full professor; professor at the department of higher mathematics, faculty of digital economy.
dymkov_m@bseu.by
Siarhey M. Dymkou, PhD (physics and mathematics); independent researcher.
dymkov78@mail.ru

Introduction

This paper is concerned with the optimal control problem described by the ordinary differential equations in the presence of the trajectory constraints of min-max kind which often arise during the mathematical modelling of real technical processes. The research of the extremal problems with nonsmooth state restriction (min-max constraints are typical for this) has met both essential theoretical and numerical difficulties (see, for example, [1; 2]). As far as the theoretical ones are concerned, then the main problem is generated by the fact that the state constraints lead to the discontinuity (jump) of the trajectories of the adjoint system. This yields, in part, that the formulation of the optimality conditions uses the heavy mathematical tools such as the Borel measures, constrained variation functions as Lagrange multiplayers, etc. In addition, the necessary optimality conditions in this case can be degenerated for any admissible trajectory. The bibliography on the classic optimisation with a state constraint can be found, for example, in survey [3]. The nonsmooth character of the state constraints gives additional difficulties for the optimisation of these control models. Naturally, the effective numerical methods can not be constructed without a proper theoretical background based on the update nonsmooth optimisation theory (see [4; 5]). Note that there exists a good theory to construct the discrete approximation for optimisation problems under state constraints (see [6; 7] and the bibliography therein). It is well known that the approximation in this case demands a careful construction of the discrete model to guarantee the necessary adequacy of the model (the counterexample that demonstrates the noncorrect results of standard discretisation can be found in [7], for example). The nonsmooth character of the state constraint leads, in general, to an increasing number of the corresponding discrete finite-dimensional models of the constrained mathematical programming problem, and hence their solving by readily available software is exorbitantly expensive. An essential and key question for both theoretical and numerical aspects of approximations is the following: can one and in which sense approximate an optimal solution of the original control system using the approximate model? In order to achieve a well-posed discrete approximation ensuring the proper convergence of optimal solutions we need to admit, in general, the state perturbations in the discrete model. Another way to construct the proper approximation for the optimisation problem under the state constraint is to introduce a new (continuous in time) model without any state constraints. Surely, such simplification demands proper modification of the model: we make worse (in some sense) the right-hand side of the control system. Some aspects of this approach were considered in [8; 9].

Notation and model definition

In this section we use the idea of approximation by a continuous in time model for the following optimal control problem:
minimize

$$J(u) = \varphi(x(T)) \to \min_{u(\cdot)} \tag{1}$$

over absolutely continuous trajectories $x : [0, T] \to R^n$ for the differential equation

$$\frac{dx(t)}{dt} = f(x, u, t),\ x(0) = x_0,\ u(t) \in U, \text{ almost everywhere } t \in [0, T] \tag{2}$$

under the nonsmooth state constraint of the form

$$x(t) \in G,\ t \in [0, T], \text{ where } G = \{x \in R^n : g(x) \le 0\}, \tag{3}$$

where $\varphi : R^n \to R$ is a continuous function; $g : R^n \to R$ is a continuous and continuous directional differentiable function; x_0 is the given n-vector. The derivatives along the direction $l \in R^n$ we denote as $g'_x(l)$. Let us assume that the function $l \to g'_x(l)$ is continuous. We suppose that the function $f : R^n \times R^r \times R \to R^r$ satisfies the Caratheodory conditions (i. e. f is continuous on (x, u) and measurable on t) such that the initial Cauchy problem for differential equation (2) has an unique absolutely continuous solution.

Definition 1. We say that the function $u : T \to R^r$ is admissible for (2) if it is measurable and satisfies the constraint $u(t) \in U$ almost everywhere $t \in T$, where U is a given compact set from R^r. We say that the function $x : T \to R^n$ is an admissible solution of (2) corresponding to the given admissible control $u(t)$ if it is absolutely continuous with respect to $t \in T$ and satisfies (2) for almost all $t \in T$.

Next we suppose that the set of admissible controls $U(\cdot)$ is nonempty and assume also that the following assumption is satisfied.

Assumption 1. There exist the constants $\varepsilon_0 > 0$, $\alpha < 0$ such that for all $u \in U$ and almost all $t \in [0, T]$ the following inequality is fulfilled:

$$g'_x(f(x, u, t)) \le \alpha \text{ for } \forall x : 0 \le g(x) \le \varepsilon_0. \tag{4}$$

This condition can be treated as some normality or regularity conditions for the optimisation problem with state constraints. In fact, the assumption 1 coordinates the dynamic behaviour of system (2) with state restriction (3) in order to avoid the long time presence in the prohibited zone where the state constraints are disturbed. Also note that since $g(x)$, $g'_x(l)$, $f(x, u, t)$ are continuous then the assumption 1 is sufficiently to satisfy for $\forall x : g(x) = 0$ with some $\alpha < 0$. We consider this condition in the form (4) since the given number ε_0 will be used in the estimates below.

Remark 1. The constraint of form (3) includes a wide class of the state restriction. In particular, this constraint is often given in the following form:

$$x(t) \in G,\ t \in [0, T], \text{ where } G = \left\{ x \in R^n : g(x) = \max_{1 \le i \le m} \varphi_i(x) \le 0 \right\}.$$

Usually it is assumed that the functions $\varphi_i(x)$ are continuously differentiable. It is known that in this case the function $g(x)$ is continuous, directional differentiable and the derivative along the direction $l \in R^n$ is given by the formula

$$g'_x(l) = \max_{i \in Q(x)} \left(\frac{\partial \varphi_i(x)}{\partial x},\ l \right), \text{ where } Q(x) = \{ i : g(x) = \varphi_i(x) \}.$$

Approximation by continuous in time optimisation problems

In order to construct the proper approximation for the optimisation problem with the state constraint we introduce a new model without any state constraints. Such simplification demands the corresponding modification of the right-hand side of the control system. This approach for optimisation problems of differential inclusions with the smooth state constraint can be proposed by [8]. In this paper, we use this idea for the control systems described by the ordinary differential equations in the presence of a nonsmooth state constraint.

For the optimisation problem, (1)–(3) introduce the following continuous time approximation: for each $n = 1, 2, \ldots$ one consider the sequence of the optimal control problems of the form

$$J(u) = \varphi(x(T)) \to \min_{u(\cdot)} \tag{5}$$

over the solutions of the equations

$$\frac{dx(t)}{dt} = \left(1 - nh^2(x)\right) f(x, u, t),\ u(t) \in U, \text{ almost everywhere } t \in [0, T],\ x(0) = x_0, \tag{6}$$

where $h(x) = \max\{0, g(x)\}$.

Thus, the original optimisation problem (1)–(3) is approximated by a sequence of continuous time optimisation problems without state constraints $x(t) \in G$. This relaxation is compensated by the modification of the right-hand side of the differential equation. Surely, it is interesting how the trajectories of the relaxed problem approximate the trajectories of the original system and the state constraint G. The following results are true.

Theorem 1. *Let* $n \ge \frac{1}{\varepsilon_0^2}$ *and the assumption 1 be held. Then each trajectory* $x(t)$ *of system (6) with the initial condition* $x(0) = x_1 : g(x_1) \le 0$ *and the fixed admissible control function* $u(t)$ *satisfies the inequality* $nh^2(x(t)) < 1$ *for* $\forall t \in [0, T]$.

P r o o f. On the contrary, let there exist a moment $\hat{t} \in [0, T]$ such that $nh^2(x(\hat{t})) \ge 1$. Since the parameter $nh^2(x)$ is continuous and $nh^2(x(0)) = nh^2(x_1) = 0$, then there is a minimal $t_* \in (0, T]$ such that $nh^2(x(t_*)) = 1$, and hence for any $\varepsilon > 0$ there is such $\delta > 0$ that $1 - \varepsilon \le nh^2(x(t)) < 1$ for $\forall t \in [t_* - \delta, t_*)$. Since $n \ge \frac{1}{\varepsilon_0^2}$ then

$0 \le g\big(x(t)\big) \le \varepsilon_0$ for $\forall t \in [t_* - \delta,\, t_*)$. Using the properties (see details in [7; 10], for example) of the function $h\big(x(t)\big)$ we can calculate the one-sided derivative $\frac{d^+h}{dt}$ for all $t \in [t_* - \delta,\, t_*)$:

$$\frac{d^+}{dt}\Big[nh^2\big(x(t)\big)\Big] = 2ng\big(x(t)\big)g'_{x(t)}\big(\dot{x}(t)\big). \tag{7}$$

Using (4), we have

$$\frac{d^+}{dt}\Big[nh^2\big(x(t)\big)\Big] = 2ng\big(x(t)\big)\big(1 - nh^2\big(x(t)\big)\big)g'_{x(t)}\big(f\big(x(t),\, u(t),\, t\big)\big) \le 2ng\big(x(t)\big)\alpha\varepsilon \le 0.$$

The obtained inequality contradicts to the the given condition

$$1 = nh^2\big(x(t_*)\big) = \max\Big\{nh^2\big(x(t)\big) : t \in [t_* - \delta,\, t_*]\Big\}.$$

Since we have the function $nh^2\big(x(t)\big)$ that does not increase on the interval $[t_* - \delta,\, t_*]$, and hence $nh^2\big(x(t)\big) \ge 1$ for $\forall t \in [t_* - \delta,\, t_*]$ that is false since t_* is the minimal time where $nh^2\big(x(t_*)\big) = 1$. The theorem is proved.

Remark 2. Differential equation (6) can be presented in a different way in the general form

$$\frac{dx(t)}{dt} = H(x,\, u,\, t),\ u(t) \in U, \text{ almost everywhere } t \in [0,\, T],\ x(0) = x_0,$$

where the choice of H can be used to improve in the proper sense the necessary properties of the produced approximation. In general, we may variate the right-hand side of the differential equation (6) in the wide margins. In particular, differential equality (6) can be replaced by a differential inclusion that gives the wide margin to use this choice to guarantee the necessary approximation properties. As an example, we present the following modification. Let us assume that there are constants $\alpha < 0$, $\varepsilon_0 > 0$, and the continuous function $r(x)$ such that the following inequality

$$g'_x\big(r(x)\big) \le \alpha \text{ for } \forall x : 0 \le g(x) \le \varepsilon_0$$

is held. Now we consider the sequence of the optimal control problem f (5), (6) where differential equation (6) is replaced by the following

$$\frac{dx(t)}{dt} = \big(1 - nh^2(x)\big)f(x,\, u,\, t) + nh^2(x)r(x).$$

We can use the choice of the function $r(x)$ to improve the properties of the produced differential equation. It is shown that the statement of theorem 1 is true in this case. The corresponding changes of the proof after (7) are given as follows:

$$\frac{d^+}{dt}\Big[h^2\big(x(t)\big)\Big] = 2ng\big(x(t)\big)\Big[g'_{x(t)}\big(f\big(x(t),\, u(t),\, t\big)\big)\big(1 - h^2\big(x(t)\big)\big) + h^2(x)g'_{x(t)}\big(rx(t)\big)\Big].$$

Since the functions $f(x, u, t)$, $x(t)$, $g'_x(l)$ are continuous then there is a constant $M > 0$ such that $g'_{x(t)}\big(f\big(x(t),\, u(t),\, t\big)\big) \le M$ for $\forall t \in [0,\, T]$. Hence, choosing $\varepsilon > 0$ and $\delta > 0$, we have again

$$\frac{d^+}{dt}\Big[h^2\big(x(t)\big)\Big] = 2ng\big(x(t)\big)\big[\varepsilon M + \alpha(1 - \varepsilon)\big] \le 0,\ t \in [t_* - \delta,\, t_*].$$

So, the required statement is obtained.

Theorem 2. *Let the given assumptions in theorem 1 and the following condition* $|f(x, u, t)| \le M$ *for* $\forall x \in \mathbb{R}^n$, $\forall u \in \mathbb{R}^r$, $\forall t \in [0,\, T]$ *are held with* $M > 0$. *Then for any fixed control* $u(t)$ *the trajectory of system (6) with the initial condition* $x(0) = x_1 : g(x_1) \le 0$ *satisfies the following estimation:*

$$\rho\big(x(t),\, G\big) \le -\frac{M}{\alpha}\sqrt{\frac{1}{n}} \text{ for } \forall t \in [0,\, T],$$

starting from some $n > n_0$, *where* n_0 *is some integer,* $\rho(x,\, G) = \min\limits_{y \in G}\|x - y\|$ *is the distance between the point* x *and the set* G.

P r o o f. From theorem 1 it follows that for any $t \in [0, T]$ the inequality $nh^2(x(t)) < 1$ is held, starting from $n \geq \frac{1}{\varepsilon_0^2}$, and hence $g(x(t)) \leq \sqrt{\frac{1}{n}}$. Thus for any ε_*, $0 \leq \varepsilon_* \leq \varepsilon_0$, the inequality $g(x(t)) \leq \sqrt{\frac{1}{n}} < \varepsilon_*$ is held for all $t \in [0, T]$, starting from some $n > n_0$. If $g(x(t)) \leq 0$ for $\forall t \in [0, T]$ then $\rho(x, G) = 0$ since $x(t) \in G$ for $\forall t \in [0, T]$. Let the inequality $g(x(t)) \leq 0$ is not fulfilled on the interval $[0, T]$. We pick an arbitrary value $\tau \in [0, T]$, where $0 < g(x(\tau)) < \varepsilon_*$. Let us consider now the following Cauchy problem:

$$\dot{y} = f(y, u(t), t),\ y(0) = x(\tau),\ t \geq 0, \tag{8}$$

where $u(t)$ is the control function corresponding to the given trajectory $x(t)$. This problem has a unique solution defined on the interval $[0, T]$. Since $0 < g(y(0)) = g(x(\tau)) < \varepsilon_*$ and then the function $g(y)$ is continuous for small values $0 < \varepsilon \leq \varepsilon_*$ and for almost all $t \in [0, \varepsilon]$, we have $0 \leq g(y(t)) \leq \varepsilon_*$. Then due to the assumption 1 we have

$$g'_{y(t)}(f(y(t), u(t), t)) \leq \alpha < 0$$

for all $y(t)$ such that $0 \leq g(y(t)) \leq \varepsilon_*$ for $\forall t \in [0, \varepsilon]$. Then calculating the directional derivative of the function $g(y(t))$ yields

$$\frac{d^+}{dt}\left[g(y(t))\right] = g'_{y(t)}(\dot{y}(t)) = g'_{y(t)}(f(y(t), u(t), t)) \leq \alpha < 0,\ t \in [0, \varepsilon].$$

Therefore the following decomposition

$$g(y(t)) = g(y(0)) + t\frac{d^+}{dt}\left[g(y(0))\right] + o(t),\ \frac{o(t)}{t} \to 0 \text{ at } t \downarrow 0,$$

is fulfilled. Since $\frac{d^+}{dt}\left[g(y(0))\right] \leq \alpha$, where $\alpha < 0$ is a constant, then for a small ε_* the inequality

$$0 \leq g(y(t)) = g(x(\tau)) + t\alpha + o(t) \leq \varepsilon_* + \alpha\varepsilon_* \leq 0$$

is held for all $t \in [0, \varepsilon]$, $\varepsilon \leq \varepsilon_*$. This yields that $\exists \hat{\tau},\ 0 < \hat{\tau} \leq -\frac{1}{\alpha} g(x(\tau))$ such that $g(y(\hat{\tau})) = 0$. This says that $y(\hat{\tau}) \in G$. Hence, integrating system (8) leads to the following estimation:

$$\rho(x(\tau), G) = \min_{y : g(y) \leq 0} \left\| x(\tau) - y \right\| \leq \left\| x(\tau) - y(\hat{\tau}) \right\| \leq$$

$$\leq \left\| x(\tau) - x(\tau) - \int_0^{\hat{\tau}} f(y, u, t)\,dt \right\| \leq M\hat{\tau} \leq -\frac{M}{\alpha} g(x(\tau)) \leq -\frac{M}{\alpha}\sqrt{\frac{1}{n}}$$

for all $t \in [0, T]$ and $n \geq n_0$ for some integer n_0. The proof is completed.

Discrete approximation

In the introduction of this paper it is noted that the well posed discrete approximation based on finite differences can be achieved if some perturbations of the state constraints are admitted in the produced discrete models. In this paragraph on the basis of the approach proposed in [6] we construct the well-posed discrete model for the control model (1), (2) with nonsmooth state constraint (3). Here we adopt this result for using in the numerical solution of the robot pass planning in the presence of state constraints and comparing the latter with the solution based on the proposed relaxation approach approximation.

Let us replace the derivatives in (2) by the Euler finite difference

$$\dot{x}(t) \approx \frac{1}{h}\left[x(t+h) - x(t)\right] \text{ as } h \to 0.$$

Put $N = 1, 2, 3, \ldots$. Let $T_N \doteq \{0, h_N, 2h_N, \ldots, T - h_N\}$ be a uniform grid on $[0, T]$ with the stepsize $h_N \doteq \frac{T}{N}$, and let

$$x_N(t+h_N) = x_N(t) + h_N f\left(x_N(t),\ u_N(t),\ t\right) \text{ for } t \in T_N,\ N = 1,\ 2,\ \ldots, \tag{9}$$

be an associated sequence of discrete equations. State constraints (3) are replaced by the following disturbed discrete analogous

$$g\left(x(t)\right) \le \varepsilon_N. \tag{10}$$

We say that the sequence of the problems of (1), (9), (10) is a discrete approximation of problem (1)–(3) if $h_N \to 0$ and $\varepsilon_N \to 0$ as $N \to \infty$.

Our purpose is to state the condition that guarantees the convergence of the optimal trajectories and optimal criteria values for the given discrete approximation at $N \to \infty$.

First of all, we establish that any admissible trajectory of (2) can be uniformly approximated by a sequence of discrete trajectories (9). This can be done on the basis of the known results of the optimal control theory. Next, using the results obtained in [6] we show that the so-called relaxation stability property is sufficient for the value convergence of discrete approximation under the proper perturbation of the state constraints. It should be noted that the requirement of the proper state constraints perturbation in the discrete scheme is essential for the value convergence (some details and corresponding counterexamples can be found in [6], for example).

Let $x(t),\ t \in T_N$, be a trajectory of discrete equation (9), and for any $t \in [0,\ T]$ we denoted by t^N and t_N the points of the grid T_N nearest to t from left and right, respectively. Now we consider the following piecewise-linear extension of the discrete trajectories (the so-called Euler's broken line):

$$x_N(t) = x_N\left(t^N\right) + \frac{1}{h_N}\left[x_N(t_N) - x_N\left(t^N\right)\right]\left(t - t^N\right) \text{ for } t \in [0,\ T]. \tag{11}$$

The following result for the pointwise convergence of the extended trajectories is true.

Lemma 1. *Let $x(t),\ t \in [0,\ T]$, be admissible absolutely continuous trajectory of (2). Then for any partition T_N of the interval $[0,\ T]$ with $h_N \to 0$ as $N \to \infty$ there exists a subsequence $\{x_N(t)\},\ t \in T_N$, of admissible solutions of discrete equation (9), piecewise-linear extensions (11) of which converge uniformly to $x(t)$ on the interval $[0,\ T]$.*

A well-posed approximation ensuring a proper convergence of the optimal discrete trajectories of (1), (9), (10) to the optimal solution of the original problem (1)–(3) exploits its following relaxation stability property. Along with the optimisation problem (1)–(3) we consider the following relaxation (in the Gamkrelidze form): to minimise cost functional (1) over the set of couples of measurable functions $\{\alpha_i(t),\ u_i(t),\ i = 1,\ 2,\ \ldots,\ n+1\}$ and the set of absolutely continuous trajectories $x(t),\ t \in [0,\ T]$, which satisfy constraints (3) and the following convexified differential equations:

$$\begin{gathered} \frac{dx(t)}{dt} = \sum_{i=1}^{n+1} \alpha_i(t) f\left(x,\ u_i,\ t\right), \text{ almost everywhere } t \in [0,\ T],\ x(0) = x_0, \\ \alpha_i(t) \ge 0,\ \sum_{i=1}^{n+1} \alpha_i(t) = 1,\ u_i(t) \in U,\ i = 1,\ 2,\ \ldots,\ n+1. \end{gathered} \tag{12}$$

Let $J_C^0,\ J_R^0,\ J_N^0,\ N = 1,\ 2,\ \ldots$, be the minimal values of cost functional (1) in problems (2), (3), (12) and (9), (10), respectively.

It is said that the original optimisation problem (1)–(3) is stable with respect to relaxation if $J_C^0 = J_R^0$.

This property is connected with the so-called hidden convexity [6] of the nonconvex differential systems and it holds for a wide class of the control systems such as linear systems, nonlinear systems in the absence of state constraints and some others. Thus, the necessary value convergence is given by the following lemma.

Lemma 2. *Let us assume problem (1)–(3) is stable with respect to relaxation. Then there is a sequence of perturbations $\varepsilon_N \to 0$ as $N \to \infty$ in (9), (10) such that $\lim\limits_{N \to \infty} J_N^0 = J_C^0$.*

The outlined in this section discrete approximation will be used for numerical tests and the results will be reported in due course.

Conclusion

This paper has used the continuous in time approximation to design numerical methods for solutions of the optimisation problems with min-max state constraints. The main advantage of using the proposed approximation is that it eliminates the need for solving a potentially very large collection of the constrained nonlinear programming problems which usually arise under standard approximation schemes. We present the theoretical background to construct the scheme with proper trajectory convergence. It is conjectured that our approach accompanied by the modern methods of nonsmooth optimisation (see [4; 7; 10]), computational theory for optimal control (see [5]) and some results for the optimisation of special repetitive processes (see [11–13]) will be effective for the solution of optimal control problems with state constraints. The proposed approximation will be tested for the robot trajectory planning and the results of numerical tests will be reported in due course.

Библиографические ссылки

1. Arutyunov AV, Aseev SM. Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints. *SIAM Journal on Control and Optimization.* 1997;35(3):930–952. DOI: 10.1137/S036301299426996X.

2. Mordukhovich BS. Discrete approximations and refined Euler – Lagrange conditions for nonconvex differential inclusions. *SIAM Journal on Control and Optimization.* 1995;33(3):882–915. DOI: 10.1137/S0363012993245665.

3. Hartl RF, Sethi SP, Vickson RG. A survey of the maximum principles for optimal control problems with state constraints. *SIAM Review.* 1995;37(2):181–218. DOI: 10.1137/1037043.

4. Clarke FH. *Optimization and nonsmooth analysis.* New York: Willey; 1983. 308 p.

5. Teo KL, Goh CJ, Wong KH. *A unified computational approach to optimal control problems.* Harlow: Longman Scientific & Technical; 1991. 329 p. (Pitman monographs and surveys in pure and applied mathematics; volume 55).

6. Мордухович БШ. *Методы аппроксимаций в задачах оптимизации и управления.* Москва: Наука; 1988. 360 с.

7. Гороховик ВВ. *Конечномерные задачи оптимизации.* Минск: Издательский центр БГУ; 2007. 240 с.

8. Асеев СМ. Задача оптимального управления для дифференциального включения с фазовым ограничением. Гладкие аппроксимации и необходимые условия оптимальности. В: *Труды международной конференции, посвященной 90-летию со дня рождения Л. С. Понтрягина; 31 августа – 6 сентября 1998 г.; Москва, Россия. Том 3. Геометрическая теория управления.* Москва: ВИНИТИ; 1999. с. 57–81 (Итоги науки и техники. Серия: Современная математика и ее приложения. Тематические обзоры; том 64).

9. Dymkou S. Approximation of the constrained nonsmooth optimal control problems and their applications for the robot planning motion. In: Moore RR, editor. *Book of abstracts of 5th International congress on industrial and applied mathematics; 7–11 July 2003; Sydney, Australia.* Sydney: ICIAM 2003 Management Committee; c2003. p. 184.

10. Demyanov V, Rubinov A. *Constructive nonsmooth analysis.* Frankfurt am Main: Published Verlag Peter Lang; 1995. 416 p.

11. Dymkou S, Dymkov M, Rogers E, Galkowski K. Optimal control of non-stationary differential linear repetitive processes. *Integral Equations and Operator Theory.* 2008;60(2):201–216. DOI: 10.1007/s00020-008-1554-0.

12. Dymkov M, Rogers E, Dymkou S, Galkowski K. Constrained optimal control theory for differential linear repetitive processes. *SIAM Journal on Control and Optimization.* 2008;47(1):396–420. DOI: 10.1137/060668298.

13. Дымков МП. О структуре решения в линейной задаче движения газа в трубопроводе. *Журнал Белорусского государственного университета. Математика. Информатика*. 2017;3:27–37.

References

1. Arutyunov AV, Aseev SM. Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints. *SIAM Journal on Control and Optimization.* 1997;35(3):930–952. DOI: 10.1137/S036301299426996X.

2. Mordukhovich BS. Discrete approximations and refined Euler – Lagrange conditions for nonconvex differential inclusions. *SIAM Journal on Control and Optimization.* 1995;33(3):882–915. DOI: 10.1137/S0363012993245665.

3. Hartl RF, Sethi SP, Vickson RG. A survey of the maximum principles for optimal control problems with state constraints. *SIAM Review.* 1995;37(2):181–218. DOI: 10.1137/1037043.

4. Clarke FH. *Optimization and nonsmooth analysis.* New York: Willey; 1983. 308 p.

5. Teo KL, Goh CJ, Wong KH. *A unified computational approach to optimal control problems.* Harlow: Longman Scientific & Technical; 1991. 329 p. (Pitman monographs and surveys in pure and applied mathematics; volume 55).

6. Mordukhovich BS. *Metody approksimatsii v zadachakh optimizatsii i upravleniya* [Approximation methods in problems of optimization and control]. Moscow: Nauka; 1988. 360 p. Russian.

7. Gorokhovik VV. *Konechnomernye zadachi optimizatsii* [Finite dimensional optimization problems]. Minsk: Izdatel'skii tsentr BGU; 2007. 240 p. Russian.

8. Aseev SM. Zadacha optimal'nogo upravleniya dlya differentsial'nogo vklyucheniya s fazovym ogranicheniem. Gladkie approksimatsii i neobkhodimye usloviya optimal'nosti. In: *Trudy mezhdunarodnoi konferentsii, posvyashchennoi 90-letiyu so dnya rozhdeniya L. S. Pontryagina; 31 avgusta – 6 sentyabrya 1998 g.; Moskva, Rossiya. Tom 3. Geometricheskaya teoriya upravleniya* [An optimal control problem for a differential inclusion with a phase constraint. Smooth approximations and necessary optimality conditions. In: Proceedings of the international conference dedicated to the 90th anniversary of the birth of L. S. Pontryagin; 31 August – 6 September 1998; Moscow, Russia. Volume 3. Geometric control theory]. Moscow: VINITI; 1999. p. 57–81 (Itogi nauki i tekhniki. Seriya: Sovremennaya matematika i ee prilozheniya. Tematicheskie obzory; volume 64). Russian.

9. Dymkou S. Approximation of the constrained nonsmooth optimal control problems and their applications for the robot planning motion. In: Moore RR, editor. *Book of abstracts of 5th International congress on industrial and applied mathematics; 7–11 July 2003; Sydney, Australia.* Sydney: ICIAM 2003 Management Committee; c2003. p. 184.
10. Demyanov V, Rubinov A. *Constructive nonsmooth analysis.* Frankfurt am Main: Published Verlag Peter Lang; 1995. 416 p.
11. Dymkou S, Dymkov M, Rogers E, Galkowski K. Optimal control of non-stationary differential linear repetitive processes. *Integral Equations and Operator Theory.* 2008;60(2):201–216. DOI: 10.1007/s00020-008-1554-0.
12. Dymkov M, Rogers E, Dymkou S, Galkowski K. Constrained optimal control theory for differential linear repetitive processes. *SIAM Journal on Control and Optimization.* 2008;47(1):396–420. DOI: 10.1137/060668298.
13. Dymkov MP. Solution representation for a linear gas flow model in pipeline. *Journal of the Belarusian State University. Mathematics and Informatics.* 2017;3:27–37. Russian.

Received 04.04.2022 / revised 10.07.2022 / accepted 10.07.2022.