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THE SMOOTHNESS CRITERION FOR THE CLASSICAL SOLUTION TO INHOMOGENEOUS MODEL TELEGRAPH EQUATION WITH THE RATE $a(x, t)$ ON THE HALF-LINE

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The smoothness criterion is derived on the right-hand side f for an explicit solution F to

$$u_{tt}(x,t) - a^2(x,t)u_{xx}(x,t) - a^{-1}(x,t)a_t(x,t)u_t(x,t) - a(x,t)a_x(x,t)u_x(x,t) = f(x,t) \quad (1)$$

with the variable rate $a(x, t)$ in the first quarter of the plane $\dot{G}_\infty =]0, +\infty[\times]0, +\infty[$. The smoothness criterion consists of the necessary and sufficient smoothness requirements for the right-hand side f to this model telegraph equation. The necessary smoothness requirements on f are found as derivatives of F along two families of implicit characteristics of the given equation. Hence, by differentiation, we derive their sufficiency for twice continuous differentiability of F. The function F satisfies equation (1) pointwise, since it satisfies its canonical form pointwise. When f depends only on x or on t, then this smoothness criterion is equivalent to continuity f respectively, with respect to x or to t. For the equation, a general integral is constructed under a smoothness criterion of its right-hand side f.

Keywords: *implicit characteristics; smoothness criterion; general integral.*

Mathematics Subject Classification (2020): Primary 35L10, 35B65.

КРИТЕРИЙ ГЛАДКОСТИ КЛАССИЧЕСКОГО РЕШЕНИЯ НЕОДНОРОДНОГО МОДЕЛЬНОГО ТЕЛЕГРАФНОГО УРАВНЕНИЯ СО СКОРОСТЬЮ $a(x, t)$ НА ПОЛУПРЯМОЙ

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Выведен критерий гладкости на правую часть f для явного решения F уравнения

$$u_{tt}(x,t) - a^2(x,t)u_{xx}(x,t) - a^{-1}(x,t)a_t(x,t)u_t(x,t) - a(x,t)a_x(x,t)u_x(x,t) = f(x,t) \quad (1)$$

с переменной скоростью $a(x, t)$ в первой четверти плоскости $\dot{G}_\infty =]0, +\infty[\times]0, +\infty[$. Критерий гладкости состоит из необходимых и достаточных требований гладкости на правую часть f этого модельного телеграфного уравнения. Необходимые требования гладкости на f найдены, как производные от F вдоль двух семейств неявных характеристик данного уравнения. Отсюда дифференцированием выводится их достаточность для дважды непрерывной дифференцируемости F. Функция F поточечно удовлетворяет уравнению (1), так как она поточечно удовлетворяет его каноническому виду. Когда f зависит только от x или t, тогда критерий гладкости равносилен непрерывности f соответственно по x или t. Для уравнения (1) построен общий интеграл с критерием гладкости его правой части f.

Ключевые слова: *неявные характеристики; критерий гладкости; общий интеграл.*

Introduction

In this paper, we derive a smoothness criterion (necessary and sufficient conditions) on the right-hand side to an inhomogeneous model telegraph equation with variable rate $a(x, t)$ in the first quarter of the plane for its explicit classical solution. Derivatives along two families of implicit characteristics of this equation from the investigated function of the form of a double integral over the characteristic triangle represent the necessary integral smoothness requirements on the continuous right-hand side of the equation. The sufficiency of all established necessary smoothness requirements for twice continuous differentiability of the function under study follows from the properties of solutions of a linear system of two equations with respect to its first partial derivatives with continuously differentiable right-hand sides. The sufficiency of all established necessary smoothness requirements on the right-hand side for the pointwise validity of the equation is obtained by substituting this function under study into the canonical form of an inhomogeneous model telegraph equation due to the non-degeneracy of the variables change. With the help of the obtained classical solution to the model telegraph equation with variable rate, its general integral in the first quarter of the plane is constructed. All proofs are essentially based on the inversion identities of implicit function characteristics of an equation and their implicit inverse functions from the article [1]. In this article, the classical solution was used to solve explicitly the first mixed problem to the model telegraph equation in the first quarter of the plane without continuing the original data. In it, the model telegraph equation is borrowed from Ph.D. thesis [2], where the first mixed problem was solved by the continuation method of the input data. In Ph.D. thesis [2], due to the special properties of the coefficients of the model telegraph equation on the half-strip of the plane, the first mixed problem is reduced to the upper half-plane by periodic continuations of the coefficients and the input data to the Cauchy problem and the d'Alembert formula on the set G_+, which are absent in [1].

1. Model telegraph equation with rate $a(x,t)$

In the first quarter of the plane $\dot{G}_\infty =]0,+\infty[\times]0,+\infty[$ it is searched for a classical solution $F = F(x,t)$ with minimal smoothness of the right-hand side $f = f(x,t)$ of the equation

$$u_{tt}(x,t) - a^2(x,t)u_{xx}(x,t) - a^{-1}(x,t)a_t(x,t)u_t(x,t) - a(x,t)a_x(x,t)u_x(x,t) = f(x,t) \qquad (1.1)$$

where f is a given real function of variables x and t, coefficient $a(x,t) \geq a_0 > 0$, $(x,t) \in G_\infty =$ $= [0,+\infty[\times[0,+\infty[$, and $a \in C^2(G_\infty)$. We denote by the number of subscripts of functions the corresponding orders of their partial derivatives. Here $C^k(\Omega)$ is the set of k times continuously differentiable functions on the subset $\Omega \subset R^2$, $R =]-\infty,+\infty[$, and $C^0(\Omega) = C(\Omega)$.

It is well known that to the equation (1.1) corresponds the two characteristic equations

$$dx = (-1)^i a(x,t)dt,\ i = 1,\ 2, \qquad (1.2)$$

which have common integrals $g_i(x,t) = C_i,\ C_i \in R,\ i = 1,\ 2$. If the coefficient a is strictly positive, i.e. $a(x,t) \geq a_0 > 0$, $(x,t) \in G_\infty)$, then the variable t on the characteristics $g_1(x,t) = C_1,\ C_1 \in R$, strictly decreases and on the characteristics $g_2(x,t) = C_2,\ C_2 \in R$, it strictly increases with the growth of x. Therefore, implicit characteristic functions

$$y_i = g_i(x,t) = C_i,\ x \geq 0,\ t \geq 0, \quad i = 1,\ 2, \qquad (1.3)$$

have strictly monotonic inverse functions $x = h_i\{y_i,t\}, t \geq 0$, $t = h^{(i)}[x,y_i], x \geq 0$, $i = 1,\ 2$. By the definition of inverse mappings, the following inversion identities from [1] are true:

$$g_i(h_i\{y_i,\ t\},\ t) = y_i,\ \forall\ y_i,\ h_i\{g_i(x,\ t),\ t\} = x,\ x \geq 0,\ i = 1,\ 2, \qquad (1.4)$$

$$g_i(x,\ h^{(i)}[x,y_i]) = y_i,\ \forall\ y_i,\ h^{(i)}[x,\ g_i(x,t)] = t,\ t \geq 0,\ i = 1,\ 2, \qquad (1.5)$$

$$h_i\{y_i,\ h^{(i)}[x,y_i]\} = x,\ x \geq 0,\ dh^{(i)}[h_i\{y_i,\ t\},y_i] = t,\ t \geq 0,\ i = 1,\ 2. \qquad (1.6)$$

On the right-hand sides of identities (1.4)–(1.6), mutually inverse functions and variables that repeat twice on the left-hand sides are excluded, even if only one of the possible value of these variables repeats twice on the left-hand sides of these identities. If the coefficient satisfies the inequality $a(x,t) \geq a_0 > 0$, $(x,t) \in G_\infty$), $a \in C^2(G_\infty)$, then the functions g_i, h_i, $h^{(i)}$ belong to C^2 with respect to x, t, y_i, $i = 1,\ 2$ [1].

In case $a(x,t) = a = const > 0$ they have the forms $g_1(x,t) = x + at$, $g_2(x,t) = x - at$, $h_1\{y_1, t\} = y_1 - at$, $h_2\{y_2, t\} = y_2 + at$, $h^{(1)}[x, y_1] = (y_1 - x)/a$, $h^{(2)}[x, y_2] = (x - y_2)/a$ in the article [3].

Definition 1.1. The function $u = u(x,t)$ is called a classical solution to an equation (1.1) on the set G_∞, if $u \in C^2(G_\infty)$ and satisfies this equation at each point $(x,t) \in \dot{G}_\infty$.

First, if there exists at least one classical solution $u \in C^2(G_\infty)$ to the inhomogeneous equation (1.1) in G_∞, then its right-hand side must obviously be continuous $f \in C(G_\infty)$. Second, according to Definition 1.1, a particular solution F must be twice continuously differentiable $F \in C^2(G_\infty)$ and satisfies equation (1.1) pointwise on the set $\dot{G}_\infty$.

Let us find a particular classical solution to the inhomogeneous equation (1.1) in $\dot{G}_\infty$, a smoothness criterion (necessary and sufficient requirements) on f in G_∞ and its general integral in G_∞.

The characteristic $g_2(x,t) = g_2(0,0)$ divides the first quarter of the plane G_∞ into sets [1]:

$$G_- = \{(x,t) \in G_\infty : \quad g_2(x,t) > g_2(0,0)\},\ G_+ = \{(x,t) \in G_\infty : \quad g_2(x,t) \leq g_2(0,0)\}.$$

Just for the sake of simplicity, let the functions $\breve{a}$, $\breve{f}$ be the even extensions respectively of the functions a, f in x from G_∞ to all $x < 0$. As a result, we will not need these extensions due to the modulus of $|x|$ in these functions a, f.

Remark 1.1. An explicit formula for the classical solution of the first mixed problem for the model telegraph equation (1.1) in the first quarter of the plane $\dot{G}_\infty$ was derived in [1].

2. Smoothness criterion for a particular classical solution to an inhomogeneous model telegraph equation

Let us indicate the classical solution to equation (1.1) and its smoothness criterion on f in G_∞.

Theorem 2.1. [4] *Let the coefficient of (1.1) is strictly positive, i.e.* $a(x,t) \geq a_0 > 0$, $(x,t) \in G_\infty$), $a \in C^2(G_\infty)$. *The function*

$$F(x,t) = \frac{1}{2} \int_0^t \int_{h_2\{g_2(x,t),\tau\}}^{h_1\{g_1(x,t),\tau\}} \frac{f(|s|,\tau)}{a(|s|,\tau)}\, ds\, d\tau \tag{2.1}$$

is a classical solution to an inhomogeneous equation (1.1) in G_∞ *if and only if its right-hand side is continuous* $f \in C(G_\infty)$, *and*

$$H_i(x,t) \equiv \int_0^t \frac{f(|h_i\{g_i(x,t),\tau\}|,\tau)}{a(|h_i\{g_i(x,t),\tau\}|,\tau)} \frac{\partial h_i\{g_i(x,t),\tau\}}{\partial g_i}\, d\tau \in C^1(G_\infty),\ i = 1,\ 2. \tag{2.2}$$

Proof. Necessity. If the function f is a classical solution to equation (1.1) on the set G_∞, then equation (1.1) implies the continuity of f on the set G_∞, i.e. $f \in C(G_\infty)$.

In equation (1.1) on G_∞ we make a non-degenerate change of independent variables [5]

$$\xi = g_1(x,t), \quad \eta = g_2(x,t), \tag{2.3}$$

where $g_i(x,t)$ are described in (1.3). Its Jacobian $J(x,t) = \xi_x \eta_t - \xi_t \eta_x \neq 0$ non-degenerate in G_∞, because $a(x,t) \geq a_0 > 0$ in G_∞.

Now, in order to identify additional necessary smoothness requirements (2.2) on the right-hand side of f to the continuity of $f \in C(G_\infty)$, we calculate the derivative of F from (2.1) along the characteristics $g_i(x,t) = C_i$ from (1.3), i.e. along the vectors $\vec{\sigma}_i = \{(g_i)_t,\ -(g_i)_x\}$, $i = 1,\ 2$. Gradients

$\overrightarrow{grad}\ \ g_i(x,t) = \{(g_i)_x,\ (g_i)_t\}$, $i = 1,\ 2$, are orthogonal to them, since $\left(\overrightarrow{grad}\ \ g_i(x,t),\ \vec{\sigma}_i\right) =$ $(g_i)_x(g_i)_t - (g_i)_t(g_i)_x = 0$, $(x,t) \in G_\infty$, and are directed along the normals to these characteristics. By virtue of the second inversion identities (1.4), the first partial derivatives of the function F are

$$F_t = \frac{1}{2}\int\limits_0^t \left[\frac{f(|h_1\{g_1(x,t),\tau\}|,\tau)}{a(|h_1\{g_1(x,t),\tau\}|,\tau)}\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial t} - \frac{f(|h_2\{g_2(x,t),\tau\}|,\tau)}{a(|h_2\{g_2(x,t),\tau\}|,\tau)}\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial t}\right] d\tau,$$

$$F_x = \frac{1}{2}\int\limits_0^t \left[\frac{f(|h_1\{g_1(x,t),\tau\}|,\tau)}{a(|h_1\{g_1(x,t),\tau\}|,\tau)}\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial x} - \frac{f(|h_2\{g_2(x,t),\tau\}|,\tau)}{a(|h_2\{g_2(x,t),\tau\}|,\tau)}\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial x}\right] d\tau.$$

The derivatives along the characteristics (1.3) of the twicely continuously differentiable function $F \in C^2(G_\infty)$ are the continuously differentiable functions:

$$(g_1)_t F_x - (g_1)_x F_t =$$

$$= \frac{1}{2}\int\limits_0^t \frac{f(|h_2\{g_2(x,t),\tau\}|,\tau)}{a(|h_2\{g_2(x,t),\tau\}|,\tau)}\left[(g_1)_x\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial t} - (g_1)_t\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial x}\right] d\tau =$$

$$= \frac{1}{2}J(x,t)\int\limits_0^t \frac{f(|h_2\{g_2(x,t),\tau\}|\,,\tau)}{a(|h_2\{g_2(x,t),\tau\}|\,,\tau)}\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial g_2}\ \ d\tau \in C^1(G_\infty), \tag{2.4}$$

$$(g_2)_t F_x - (g_2)_x F_t =$$

$$= \frac{1}{2}\int\limits_0^t \frac{f(|h_1\{g_1(x,t),\tau\}|\,,\tau)}{a(|h_1\{g_1(x,t),\tau\}|\,,\tau)}\left[(g_2)_t\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial x} - (g_2)_x\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial t}\right] d\tau =$$

$$= \frac{1}{2}J(x,t)\ \int\limits_0^t \frac{f(|h_1\{g_1(x,t),\ \tau\}|\,,\ \tau)}{a(|h_1\{g_1(x,t),\ \tau\}|\,,\ \tau)}\ \frac{\partial h_1\{g_1(x,t),\ \tau\}}{\partial g_1}\ \ d\tau \in C^1(G_\infty), \tag{2.5}$$

since for partial derivatives of functions $h_i = h_i\{g_i(x,t),\ \tau\}$ the folowing relations

$$(g_i)_x\frac{\partial h_i\{g_i(x,t),\ \tau\}}{\partial t} - (g_i)_t\frac{\partial h_i\{g_i(x,t),\ \tau\}}{\partial x}\ \ =$$

$$= (g_i)_x\frac{\partial h_i\{g_i(x,t),\ \tau\}}{\partial g_i}(g_i)_t - (g_i)_t\frac{\partial h_i\{g_i(x,t),\ \tau\}}{\partial g_i}(g_i)_x \equiv 0,\quad i = 1,\ 2,$$

$$(g_1)_x\frac{\partial h_2\{g_2(x,t),\ \tau\}}{\partial t} - (g_1)_t\frac{\partial h_2\{g_2(x,t),\ \tau\}}{\partial x}\ \ =$$

$$= [(g_1)_x(g_2)_t - (g_1)_t(g_2)_x\ ]\frac{\partial h_2\{g_2(x,t),\ \tau\}}{\partial g_2} = J(x,t)\frac{\partial h_2\{g_2(x,t),\ \tau\}}{\partial g_2},$$

$$(g_2)_t\frac{\partial h_1\{g_1(x,t),\ \tau\}}{\partial x} - (g_2)_x\frac{\partial h_1\{g_1(x,t),\ \tau\}}{\partial t}\ \ =$$

$$= [(g_1)_x(g_2)_t - (g_1)_t(g_2)_x\ ]\frac{\partial h_1\{g_1(x,t),\ \tau\}}{\partial g_1} = J(x,t)\frac{\partial h_1\{g_1(x,t),\ \tau\}}{\partial g_1}$$

are true. This implies the validity of the inclusions (2.2), since the Jacobian $J(x,t) \neq 0$ in G_∞ and $J \in C^1(G_\infty)$. The necessity of the continuity $f \in C(G_\infty)$ and the smoothness requirements (2.2) is proved.

Sufficiency of continuity $f \in C(G_\infty)$ and integral smoothness requirements (2.2) for twicely continuous differentiability of $F \in C^2(G_\infty)$ follows from the continuous differentiability in G_∞ of the first partial derivatives F_t and F_x of the function F. Since they are solutions of the linear equation system (2.4), (2.5) with continuously differentiable right-hand sides of this system due to the smoothness of (2.2) and the Jacobian $J \in C^1(G_\infty)$.

It remains to verify that function (2.1) satisfies equation (1.1) pointwise on G_∞. By virtue of the established twicely continuous differentiability of the function $F \in C^2(G_\infty)$, the latter is equivalent to the fact that after the change (2.3) the function F satisfies the corresponding canonical form of the equation (1.1) (see equation (2.13) below).

Equation (1.1) is reduced by non-degenerate change (2.3) to the form

$$[(\xi_t)^2 - a^2(\xi_x)^2]\tilde{u}_{\xi\xi} + 2aJ(x,t)\tilde{u}_{\xi\eta} + [(\eta_t)^2 - a^2(\eta_x)^2]\tilde{u}_{\eta\eta}+$$
$$+[\xi_{tt} - a^2\xi_{xx} - a^{-1}a_t\xi_t - aa_x\xi_x]\tilde{u}_\xi + [\eta_{tt} - a^2\eta_{xx} - a^{-1}a_t\eta_t - aa_x\eta_x]\tilde{u}_\eta =$$
$$= \tilde{\breve{f}}(\xi,\eta) = f(x(\xi,\eta), t(\xi,\eta)) \tag{2.6}$$

with respect to the function $\tilde{u}(\xi,\eta) = u(x(\xi,\eta),\ t(\xi,\eta)) \in C^2(\tilde{G}_\infty)$. Here the set $\tilde{G}_\infty$ is the image of the first quarter of the plane G_∞ under changing (2.3).

The total differentials of the functions g_i, $i = 1,\ 2$, of characteristics (1.3) are obviously identically equal to zero and, according to the characteristic equations (1.2), we have the representations

$$dg_i = (g_i)_x dx + (g_i)_t dt = [(g_i)_t + (-1)^i a(x,t)(g_i)_x]dt \equiv 0,\ (x,t) \in G_\infty,\ i = 1,\ 2,$$

and, therefore, in view of (2.3) we have the relations

$$(g_i)_t \equiv (-1)^{i+1}a(x,t)(g_i)_x,\ (x,t) \in G_\infty,\ i = 1,\ 2, \tag{2.7}$$

$$\xi_t - a(x,t)\xi_x = 0,\ \eta_t + a(x,t)\eta_x = 0,\ (x,t) \in G_\infty. \tag{2.8}$$

We differentiate once the equations (2.8) with respect to t

$$\xi_{tt} - a_t(x,t)\xi_x - a(x,t)\xi_{xt} = 0, \quad \eta_{tt} + a_t(x,t)\eta_x + a(x,t)\eta_{xt} = 0 \tag{2.9}$$

and then differentiate the from (2.8) once with respect to x

$$\xi_{tx} - a_x(x,t)\xi_x - a(x,t)\xi_{xx} = 0,\ \eta_{tx} + a_x(x,t)\eta_x + a(x,t)\eta_{xx} = 0. \tag{2.10}$$

We sum and subtract the equations (2.9), respectively, with the equations (2.10) multiplied by the coefficient $a(x,t)$, and arrive at the equations

$$\xi_{tt} - a_t(x,t)\xi_x - a(x,t)a_x(x,t)\xi_x - a^2(x,t)\xi_{xx} = 0,$$
$$\eta_{tt} - a_t(x,t)\eta_x - a(x,t)a_x(x,t)\eta_x - a^2(x,t)\eta_{xx} = 0. \tag{2.11}$$

By virtue of equalities (2.8) we derive from equations (2.11), respectively, for $(x,t) \in G_\infty$

$$\xi_{tt} - a^2(x,t)\xi_{xx} = a_t(x,t)\xi_x + a(x,t)a_x(x,t)\xi_x = a^{-1}(x,t)a_t(x,t)\xi_t + a(x,t)a_x(x,t)\xi_x,$$
$$\eta_{tt} - a^2(x,t)\eta_{xx} = a_t(x,t)\eta_x + a(x,t)a_x(x,t)\eta_x = a^{-1}(x,t)a_t(x,t)\eta_t + a(x,t)a_x(x,t)\eta_x. \tag{2.12}$$

Based on the identities (2.12), the equation (2.6) becomes the equation

$$\tilde{u}_{\xi\eta}(\xi,\eta) = \tilde{f}(\xi,\eta)/[2\tilde{a}(\xi,\eta)J(x,t)], \quad (\xi,\mu) \in \tilde{G}_\infty, \tag{2.13}$$

where the coefficient $\tilde{a}(\xi,\eta) = \breve{a}(x(\xi,\eta), t(\xi,\eta))$ and the replacement Jacobian $J(x,t) = \xi_x\eta_t - \xi_t\eta_x \neq$ $= 0$ on the set $\tilde{G}_\infty = \{(\xi,\eta) : h_2\{\eta,0\} \leq h_1\{\xi,0\},\ \eta \geq 0; h^{(2)}[\eta,0] < h^{(1)}[\xi,0],\ \eta < 0;\ \xi \geq 0\}$.

For any point $M(x,t) \in G_\infty$, the double iterated integral (2.1) is equal to the double integral over the curvilinear characteristic triangle ΔMPQ with vertex $M(x,t) \in G_\infty$ and the vertices of its base $P\,(h_2\{g_2(x,t),0\},0)$, $Q\,(h_1\{g_1(x,t),0\},0)$ (Fig. 1, a; Fig. 2, a):

$$F(x,t) = \frac{1}{2}\int\limits_{\Delta MPQ} \frac{f(|x|,t)}{a(|x|,t)}dxdt = \frac{1}{2}\int\limits_{\Delta MPQ} \frac{\tilde{\breve{f}}(x,t)}{\breve{a}(x,t)}dxdt, \tag{2.14}$$

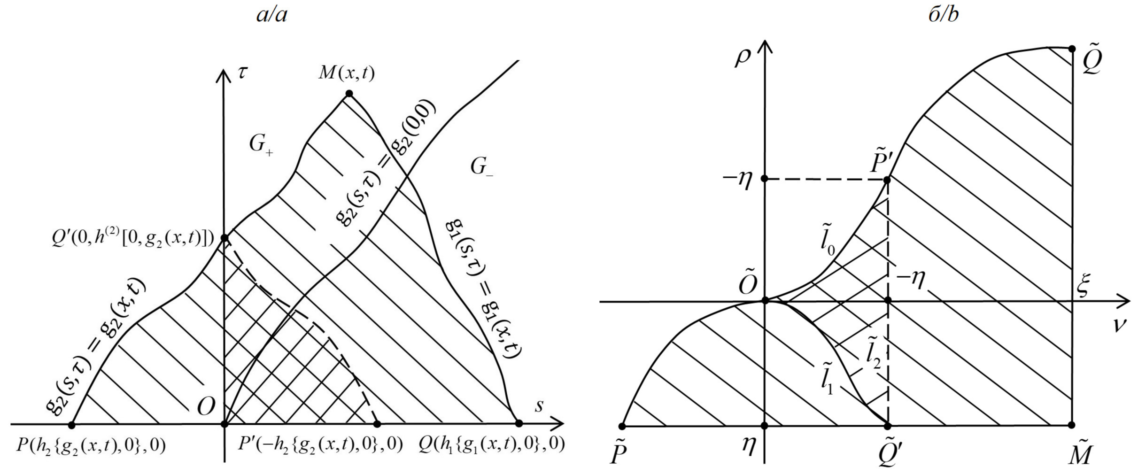

Fig. 1: The region of integration on the set G_+: a — for the function F; b — for the function $\tilde{F}$

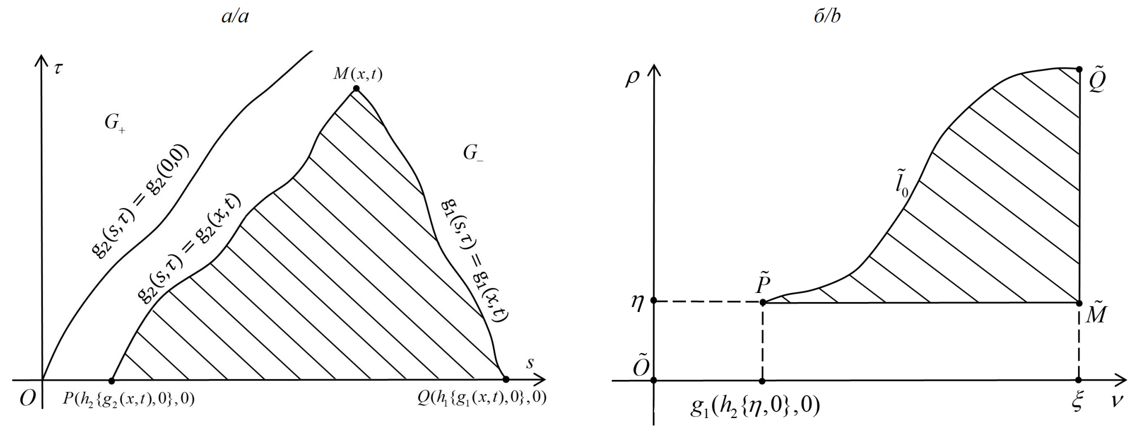

Fig. 2: The region of integration on the set G_-a — for the function F; b — for the function $\tilde{F}$

By passing to new variables of type (2.3) in function (2.14):

$$\nu = g_1(s,\tau),\ \rho = g_2(s,\tau), \tag{2.15}$$

in the plane $\tilde{O}\nu\rho$, we find the image $\Delta\ \tilde{M}\tilde{P}\tilde{Q}$ of triangle $\Delta\ MPQ$. Two characteristics $g_2(s,\tau) = g_2(x,t)$ and $g_1(s,\tau) = g_1(x,t)$, which intersect axis Os at $\tau = 0$, respectively, at the base points $P\,(h_2\{g_2(x,t),0\},0)$ and $Q\,(h_1\{g_1(x,t),0\},0)$.

The mapping (2.3) maps a point $M(x,t)$ of the plane $Os\tau$ to a point $\tilde{M}(\xi,\eta)$ of the plane $\tilde{O}\nu\rho$. After replacement (2.15) of the characteristic equations $g_i(s,\tau) = g_i(x,t)$, $i = 1,\ 2$, of the sides MP and MQ of the triangle ΔMPQ become respectively the equations $\nu = g_1(s,\tau) = g_1(x,t) = \xi$, $\rho = g_2(s,\tau) = g_2(x,t) = \eta$ of the sides $\tilde{M}\tilde{P}$ and $\tilde{M}\tilde{Q}$ for the triangle $\Delta\tilde{M}\tilde{P}\tilde{Q}$ in the plane $\tilde{O}\nu\rho$. Replace (2.15) with the coordinates of the base vertices $P(h_2\{g_2(x,t),0\},0)$, $Q(h_1\{g_1(x,t),0\},0)$ and by virtue of the first inversion identities from (1.4) we find the coordinates

$$\nu = g_1\left(h_2\{\eta,0\},0\right),\ \rho = g_2\left(h_2\{\eta,0\},0\right) = \eta;\ \nu = g_1\left(h_1\{\xi,0\},0\right) = \xi,\ \rho = g_2\left(h_1\{\xi,0\},0\right)$$

of the vertex $\tilde{P}\left(g_1\left(h_2\{\eta,0\},0\right),\eta\right)$, $\tilde{Q}\left(\xi, g_2\left(h_1\{\xi,0\},0\right)\right)$ of the triangle $\Delta\tilde{M}\tilde{P}\tilde{Q}$ in the plane $\tilde{O}\nu\rho$, since $g_1(x,t) = \xi$, $g_2(x,t) = \eta$ in (2.3) (Fig. 1, b; Fig. 2, b).

Putting $\tau = 0$ in change (2.15), we arrive at the equations $\nu = g_1(s,0)$, $\rho = g_2(s,0)$, of which, due to the uniqueness of the solutions $s = h_1\{\nu,0\}$, $s = h_2\{\rho,0\}$ of the equation system (2.15) with respect to (s,τ) and the second inversion identities from (1.4), we derive the equation $\nu = g_1(h_2\{\rho,0\},0)$ of the curvilinear base $\tilde{P}\tilde{Q}$ of the triangle $\Delta\tilde{M}\tilde{P}\tilde{Q}$ in $\tilde{O}\nu\rho$ (Fig. 1, b; Fig. 2, b).

Due to the mutually inverse orientation of the lateral sides of the triangles ΔMPQ and $\Delta\tilde{M}\tilde{P}\tilde{Q}$, the double integral (2.14) becomes an iterated integral as a result of the substitution (2.3)

$$\tilde{F}(\xi,\eta)=\frac{1}{2}\int\limits_{\Delta\,\tilde{M}\tilde{P}\tilde{Q}}\frac{\tilde{\breve{f}}(\xi,\eta)}{\tilde{\breve{a}}(\xi,\eta)}J(\xi,\eta)d\xi d\eta=\frac{1}{2}\int\limits_{g_2(h_1\{\xi,0\},0)}^{\eta}\int\limits_{g_1(h_2\{\rho,0\},0)}^{\xi}\frac{\tilde{\breve{f}}(\nu,\rho)}{\tilde{\breve{a}}(\nu,\rho)}J(\nu,\rho)d\nu d\rho, \qquad (2.16)$$

where the points $\tilde{O}(0,0)$, $\tilde{M}(\xi,\eta)$, $\tilde{P}\left(g_1\left(h_2\{\eta,0\},0\right),\eta\right)$, $\tilde{Q}\left(\xi,g_2\left(h_1\{\xi,0\},0\right)\right)$ on the plane $\tilde{O}\nu\rho$ are respectively the images of the points $O(0,0)$, $M(x,t)$, $P\left(h_2\{g_2(x,t),0\},0\right)$, $Q\left(h_1\{g_1(x,t),0\},0\right)$ from the plane $Os\tau$ after variable transformation (2.15). For the existence of the integral (2.16), it is important that if the function $f\in C(G_\infty)$ is continuous, then, due to the continuity of the Jacobian $J(\nu,\rho)\neq 0$ of the transformation (2.15), the function $\tilde{\breve{f}}\in C(\tilde{G}_\infty)$ in (2.16) is also continuous.

We take partial derivatives with respect to η and ξ of $\tilde{F}$ from (2.16) and have a mixed derivative

$$\frac{\partial^2\tilde{F}(\xi,\eta)}{\partial\xi\partial\eta}=\frac{1}{2}\frac{\tilde{\breve{f}}(\xi,\eta)}{\tilde{\breve{a}}(\xi,\eta)}J(\xi,\eta)=\frac{1}{2}\frac{\tilde{f}(\xi,\eta)}{\tilde{a}(\xi,\eta)J(x,t)},\ (\xi,\eta)\in\tilde{G}_\infty, \qquad (2.17)$$

since the inverse Jacobian $[J(\xi,\eta)]^{-1}=x_\xi t_\eta-x_\eta t_\xi=1/J(x,t)\neq 0$ on $\tilde{G}_\infty$. Thus, the function F after the change (2.3) satisfies equation (2.13) on $\tilde{G}_\infty$. Theorem 2.1 is proved.

C o r o l l a r y 2.1. [4] *Let the coefficient is strictly positive $a(x,t)\geq a_0>0$, $(x,t)\in G_\infty)$, $a\in C^2(G_\infty)$. If the right-hand side f of the equation (1.1) does not depend on x or on t in G_∞, then the continuity of f in t or in x, respectively, is necessary and is sufficient for the function F from (2.1) to be a classical solution of the inhomogeneous equation (1.1) in G_∞.*

P r o o f. The necessity of the continuity of $f\in C[0,+\infty[$ in t or in x is strictly justified in the proof of Theorem 2.1. In fact, it remains to show the sufficiency of this continuity of $f\in C[0,+\infty[$ in t or in x by the fact that in this case the corresponding integral smoothness requirements (2.2) are automatically satisfied.

If the right-hand side $f=f(t)$ does not depend on x, then function F from (2.1) takes the form

$$F(x,t)=\frac{1}{2}\int\limits_0^t\int\limits_{h_2\{g_2(x,t),\tau\}}^{h_1\{g_1(x,t),\tau\}}\frac{f(\tau)}{a(|s|,\tau)}\,ds\,d\tau. \qquad (2.18)$$

According to the proof of Theorem 2.1, the inclusion $F\in C^2(G_\infty)$ is sufficient for the function (2.18) to be a solution of the equation (1.1) in G_∞. Using the second inversion identities from (1.4), in G_∞ for all $f(t)\in C[0,+\infty[$ its first partial derivatives are obviously continuously differentiable

$$F_t=\frac{1}{2}\int\limits_0^t\left[\frac{f(\tau)}{a(|h_1\{g_1(x,t),\tau\}|,\tau)}\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial t}-\frac{f(\tau)}{a(|h_2\{g_2(x,t),\tau\}|,\tau)}\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial t}\right]d\tau,$$

$$F_x=\frac{1}{2}\int\limits_0^t\left[\frac{f(\tau)}{a(|h_1\{g_1(x,t),\tau\}|,\tau)}\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial x}-\frac{f(\tau)}{a(|h_2\{g_2(x,t),\tau\}|,\tau)}\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial x}\right]d\tau.$$

If the right side $f=f(x)$ does not depend on t, then the function F from (2.1) takes the form

$$F(x,t)=\frac{1}{2}\int\limits_0^t\int\limits_{h_2\{g_2(x,t),\tau\}}^{h_1\{g_1(x,t),\tau\}}\frac{\breve{f}(s)}{\breve{a}(s,\tau)}dsd\tau, \qquad (2.19)$$

where the functions $\breve{f}$ and $\breve{a}$ are even extensions of the functions f and a from $x\geq 0$ to all $x<0$. Same as at the beginning of the proof of the corollary 2.1, from Theorem 2.1 we have that if a function

$F\in C^2(G_\infty)$, then it satisfies equation (1.1) in G_∞. Let us check its twice continuous differentiability for $f(x)\in C[0,+\infty[$. The first partial derivative with respect to t of (2.19) is equal to the function

$$F_t=\frac{1}{2}\int\limits_0^t\left[\frac{\check f(h_1\{g_1(x,t),\tau\})}{\check a(h_1\{g_1(x,t),\tau\},\tau)}\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial t}-\frac{\check f(h_2\{g_2(x,t),\tau\})}{\check a(h_2\{g_2(x,t),\tau\},\tau)}\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial t}\right]d\tau,$$

to which we also applied the second inversion identities from (1.4). When we move here to the new integration variables $y=h_1\{g_1(x,t),\tau\}$, $z=h_2\{g_2(x,t),\tau\}$, then on G_∞ we obtain the obvious continuously differentiable representation of this derivative

$$F_t(x,t)=\frac{1}{2}\int\limits_{h_1\{g_1(x,t),\ 0\}}^{x}\left[\frac{\check f(y)}{\check a(y,\tau)}\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial t}\left(\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial\tau}\right)^{-1}\right]\Bigg|_{\tau=h^{(1)}[y,g_1(x,t)]}dy-$$

$$-\frac{1}{2}\int\limits_{h_2\{g_2(x,t),\ 0\}}^{x}\left[\frac{\check f(y)}{\check a(y,\tau)}\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial t}\left(\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial\tau}\right)^{-1}\right]\Bigg|_{\tau=h^{(2)}[z,g_2(x,t)]}dz\in C^1(G_\infty),$$

where we have used the second inversion identities from (1.4) and the identities $\tau=h^{(1)}[y,g_1(x,t)]$, $\tau=h^{(2)}[z,g_2(x,t)]$ due to the second inversion identities from (1.6).

For all $f(x)\in C[0,+\infty[$ the first partial derivative with respect to x of (2.19) is the function

$$F_x=\frac{1}{2}\int\limits_0^t\left[\frac{\check f(h_1\{g_1(x,t),\tau\})}{\check a(h_1\{g_1(x,t),\tau\},\tau)}\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial x}-\frac{\check f(h_2\{g_2(x,t),\tau\})}{\check a(h_2\{g_2(x,t),\tau\},\tau)}\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial x}\right]d\tau,$$

in which we used the second inversion identities from (1.4). After passing here to the new integration variables $y=h_1\{g_1(x,t),\tau\}$, $z=h_2\{g_2(x,t),\tau\}$ this partial derivative acquires the obvious for $f(x)\in$ $\in C[0,+\infty[$ continuously differentiable representation

$$F_x=\frac{1}{2}\int\limits_{h_1\{g_1(x,t),\ 0\}}^{x}\left[\frac{\check f(y)}{\check a(y,\tau)}\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial x}\left(\frac{\partial h_1\{g_1(x,t),\tau\}}{\partial\tau}\right)^{-1}\right]\Bigg|_{\tau=h^{(1)}[y,g_1(x,t)]}dy-$$

$$-\frac{1}{2}\int\limits_{h_2\{g_2(x,t),\ 0\}}^{x}\left[\frac{\check f(y)}{\check a(y,\tau)}\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial x}\left(\frac{\partial h_2\{g_2(x,t),\tau\}}{\partial\tau}\right)^{-1}\right]\Bigg|_{\tau=h^{(2)}[z,g_2(x,t)]}dz\in C^1(G_\infty),$$

where we have applied the same identities as for the partial derivative of F with respect to t.

Thus, the sufficiency of $f\in C[0,+\infty[$ for $F\in C^2(G_\infty)$ is verified. Corollary 2.1 is proved.

C o r o l l a r y 2.2.[4] *Let the coefficient be strictly positive,* $a(x,t)\geq a_0>0$, $(x,t)\in G_\infty$, $a\in$ $\in C^2(G_\infty)$. *If the function f depends on x and on t, then for $f\in C(G_\infty)$ the requirements that the integrals from (2.2) belong to the space $C^1(G_\infty)$ are equivalent to the requirements that they belong to the space $C^{(1,0)}(G_\infty)$ or $C^{(0,1)}(G_\infty)$. Here $C^{(1,0)}(G_\infty)$, $C^{(0,1)}(G_\infty)$ are, respectively, the spaces of continuously differentiable with respect to x and t and continuous with respect to t and x functions on G_∞.*

P r o o f. The continuously differentiable right-hand sides $f\in C^1(G_\infty)$ obviously satisfy the integral requirements from (2.2). By replacing $s_i=h_i\{g_i(x,t),\tau\}$ by the integration variable τ, for example, the integrals from (2.2) are reduced to the integrals

$$H_i(x,t)=\int\limits_0^t\frac{f(|h_i\{g_i(x,t),\tau\}|,\tau)}{a(|h_i\{g_i(x,t),\tau\}|,\tau)}\frac{\partial h_i\{g_i(x,t),\tau\}}{\partial g_i}\,d\tau=$$

$$= \int\limits_{h_i\{g_i(x,t),0\}}^{x} \left[\frac{f(|s_i|,\tau)}{a(|s_i|,\tau)} \frac{\partial h_i\{g_i(x,t),\tau\}}{\partial g_i} \left(\frac{\partial h_i\{g_i(x,t),\tau\}}{\partial \tau} \right)^{-1} \right] \Bigg|_{\tau=h^{(i)}[s_i,g_i(x,t)]} ds_i, \ i=1,\ 2, \quad (2.20)$$

which for $f \in C^1(G_\infty)$ are indeed continuously differentiable with respect to x and t in G_∞, because in the last integrals (2.20) under the modulus $|s_i|$ the variables x and t are missing. Otherwise, the module would give a discontinuity of derivatives. Here we have applied the second inversion identities from (1.4) and the equalities $\tau = h^{(i)}[s_i, g_i(x,t)]$, $i = 1,\ 2$, due to the second identities from (1.6).

First, for smoother $f \in C^1(G_\infty)$, we take the derivative of the integrals (2.2)

$$\frac{\partial H_i}{\partial t} = \frac{f(|h_i\{g_i(x,t),t\}|\,,t)}{a(|h_i\{g_i(x,t),t\}|\,,t)} \frac{\partial h_i\{g_i(x,t),t\}}{\partial g_i} + \int\limits_0^t \left[\frac{f(|h_i\{g_i(x,t),\tau\}|,\tau)}{a(|h_i\{g_i(x,t),\tau\}|,\tau)} \frac{\partial h_i\{g_i(x,t),\tau\}}{\partial g_i} \right]'_t d\tau =$$

$$= \frac{f(x,t)}{a(x,t)} \frac{1}{(g_i)_x} + (-1)^{i+1} a(x,t) \int\limits_0^t \left[\frac{f(|h_i\{g_i(x,t),\tau\}|,\tau)}{a(|h_i\{g_i(x,t),\tau\}|,\tau)} \frac{\partial h_i\{g_i(x,t),\tau\}}{\partial g_i} \right]'_x d\tau =$$

$$= \frac{f(x,t)}{a(x,\ t)} \frac{1}{(g_i(x,t))_x} + (-1)^{i+1} a(x,t) \frac{\partial H_i}{\partial x},\ i=1,\ 2, \quad (2.21)$$

due to the second inversion identities from (1.4), to the well known formula for the derivative of the inverse function, to the relations (2.7) and to the equalities

$$\frac{\partial f(|h_i\{g_i(x,t),\tau\}|,\tau)}{\partial t} = \frac{\partial f(|h_i\{g_i(x,t),\tau\}|,\tau)}{\partial h_i} \frac{\partial h_i\{g_i(x,t),\tau\}}{\partial g_i} (g_i)_t =$$

$$= (-1)^{i+1} a(x,t) \frac{\partial f(|h_i\{g_i(x,t),\tau\}|,\tau)}{\partial h_i} \frac{\partial h_i\{g_i(x,t),\tau\}}{\partial g_i} (g_i)_x = (-1)^{i+1} a(x,t) \frac{\partial f(|h_i\{g_i(x,t),\tau\}|,\tau)}{\partial x},$$

$$\frac{\partial^2 h_i\{g_i(x,t),\tau\}}{\partial t \partial g_i} = \frac{\partial^2 h_i\{g_i(x,t),\ \tau\}}{\partial g_i^2} (g_i(x,t))_t =$$

$$= (-1)^{i+1} a(x,t) \frac{\partial^2 h_i\{g_i(x,t),\tau\}}{\partial g_i^2} (g_i(x,t))_x = (-1)^{i+1} a(x,t) \frac{\partial^2 h_i\{g_i(x,t),\tau\}}{\partial x \partial g_i},\ i=1,\ 2.$$

Then two equalities (2.21) of the first and last parts, which do not contain explicit derivatives of function f with respect to x and t in G_∞, are extended by passing to the limit in f smoother $f \in C^1(G_\infty)$ into continuous functions $f \in C(G_\infty)$, satisfying (2.2) in G_∞ [6]. Two equalities (2.21) obtained after passing to limit confirm the assertion of Corollary 2.2 on G_∞. Corollary 2.2 is proved.

3. General integral of the model telegraph equation

When solving mixed (initial-boundary) problems for the model telegraph equation (1.1) on half-strip plane by "method of auxiliary mixed problems for wave equations on the half-line" from [7], it is important to know its general integral (the set of all twicely continuously differentiable solutions).

T h e o r e m 3.1. [4] *Let $a(x,t) \geq a_0 > 0$, $(x,t) \in G_\infty)$, $a \in C^2(G_\infty)$ and (2.2) for $f \in C(G_\infty)$. Then the general integral of equation (1.1) in G_∞ in the set of classical solutions are the functions*

$$u(x,t) = \tilde{f}_1(g_1(x,t)) + \tilde{f}_2(g_2(x,t)) + F(x,t),\ (x,t) \in G_\infty, \quad (3.1)$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are any twicely continuously differentiable functions of the variables ξ, η having the form

$$\tilde{f}_1(\xi) = f_1(\xi) + f_2(g_2(0,0)),\ \tilde{f}_2(\eta) = f_2(\eta) - f_2(g_2(0,0)). \quad (3.2)$$

P r o o f. For the continuous right-hand side $f \in C(G_\infty)$, the integral smoothness requirements (2.2) from Theorem 2.1 are obviously equivalent to the integral smoothness requirements (2.2) with the requirement that the first partial derivatives of the functions $H_i(x,t)$, $i = 1, 2$, be continuous on the sets G_+, G_- and in some neighborhood of characteristic $g_2(x,t) = g_2(0,0)$ (Fig. 1, a; Fig. 2, a). By Theorem 2.1, for the right-hand side $f \in C(G_\infty)$ and $H_i(x,t) \in C^1(G_\infty)$, $i = 1, 2$, the function F of the form (2.1) is twicely continuously differentiable and satisfies the equation (1.1) pointwise on $\dot{G}_\infty$ and the equation (2.13) on $\tilde{G}_\infty$ due to the identity (2.17) established above.

Therefore, formulas (3.1), (3.2) are indeed the set of all classical solutions to the model telegraph equation (1.1) on G_∞. The classical solutions (3.2) to the homogeneous equation (1.1) are obtained by "the method of immersion in solutions with fixed values", proposed in [8] in order to simplify the calculation of explicit solutions to systems of differential equations. It is clear that, in equalities (3.2) the functions f_1, f_2 and $\tilde{f}_1$, $\tilde{f}_2$, respectively, are twicely continuously differentiable at the same time. The general integral (3.1) of all classical solutions to the inhomogeneous equation (1.1) is the sum of an general integral $u_0(x,t) = \tilde{f}_1(g_1(x,t)) + \tilde{f}_2(g_2(x,t))$ to an homogeneous equation (1.1) and the particular classical solution F of the form (2.2) to the inhomogeneous equation (1.1). Theorem 3.1 is proved.

The smooth non-degenerate coefficient a simplifies requirements (2.2) on $f \in C(G_\infty)$.

Remark 3.1. For the coefficient $a(x,t) \geq a_0 > 0$, $(x,t) \in G_\infty)$, $a \in C^2(G_\infty)$ the integral smoothness requirements (2.2) on the continuous $f \in C(G_\infty)$ are equivalent to the requirements

$$\int_0^t f(|h_i\{g_i(x,t),\tau\}|,\tau)\, d\tau \in C^1(G_\infty),\ i = 1,\ 2.$$

4. Conclusion

The criterion for twice continuous differentiability of the solution F of the form (2.1) to the inhomogeneous model telegraph equation (1.1) with a variable rate $a(x,t)$ in the first quarter of the plane G_∞ is found. It consists of the continuity requirement right-hand side $f \in C(G_\infty)$ and two integral smoothness requirements (2.2) on the set G_∞. The general integral (the general solution) (3.1), (3.2) from twicely continuously differentiable functions determines the explicit resolution of various mixed (initial-boundary) problems for the inhomogeneous model telegraph equation (1.1) on the set G_∞.

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