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ВЕРОЯТНОСТНАЯ МОДЕЛЬ МАРШАЛЛА – ОЛКИНА, ПОРОЖДЕННАЯ РАСПРЕДЕЛЕНИЕМ ТЕЙССЬЕ, ДЛЯ ОПИСАНИЯ ДАННЫХ ВРЕМЕНИ ЖИЗНИ

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Точность математических выводов зависит от плана эксперимента и принятой модели. В настоящем исследовании для описания данных времени жизни использовано обобщенное распределение вероятностей Маршалла – Олкина, порожденное распределением Тейссье. Характеристики предложенной модели изучены и представлены в завершенной форме. Показано, что в частных случаях интенсивность отказов может иметь формы *J* и *U*, убывать и возрастать. Представлены результаты имитационного моделирования с использованием метода Монте-Карло для различных конфигураций параметров при меняющихся размерах выборки. На основании полученных результатов имитационного моделирования и проведенного анализа качества соответствия модели реальным данным времени жизни установлено, что исследованная в работе модель является гибкой, хорошо интерпретируемой и применимой в сравнении с другими двухпараметрическими распределениями вероятностей.

Ключевые слова: распределение Гомперца; распределение Маршалла – Олкина; распределения, порожденные распределением Тейссье; распределение Тейссье.

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THE MARSHALL – OLKIN TEISSIER GENERATED MODEL FOR LIFETIME DATA

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An accurate mathematical inference depends on the experimental design and the model adopted in the process. Thus, in this study Marshall – Olkin Teissier generated distribution was used to present the distribution of the true nature of lifetime data. The characteristics of the proposed model were examined in a closed form. The behaviour of the new model indicated that the hazard rate of the submodels could be *J*- and *U*-shaped, decreasing and increasing. Monte Carlo simulations were presented for different configurations of parameters with varying sizes. The results of the simulation and goodness-of-fit of the real lifetime data show that the Marshall – Olkin Teissier generated model is flexible, tractable and applicable when compared to some classical two parameters distributions.

Keywords: Gompertz distribution; Marshall – Olkin distribution; Teissier generated distributions; Teissier distribution.

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Introduction

An accurate mathematical conclusion is drawn when an accurate, flexible and applicable statistical distribution is used to express the true nature of the data set obtained from the scenario. However, many statistical distributions have been used in the fields of survival analysis, quality control, reliability theory, ecology, economics, medical sciences, actuarial science and others in modelling the behaviour of lifetime processes. Of utmost importance is the use of probability in ascertaining the quality, quantity and control of random processes. This is the result of the inherent stochastic nature of random processes. Thus a classical probability generator is a probability distribution that can generate several other recent probability densities.

The Teissier distribution is one of the statistical distributions with a parsimonious parameter like the exponential distribution in modelling lifespan scenarios and mean residual life functions with Gompertz, exponential and Weibull characteristics in modelling stochastic processes. Thus the Teissier model has been used in modelling failure and mortality from wear and ageing [1]. The Teissier distribution has an inherent monotone increasing function. However, lifetime processes are increasing non-monotone in nature (see [2–4]). Hence, there is a need to extend the classical Teissier distribution to account for the pitfall of the constant failure rate embedded in the Gompertz, Weibull and exponential distributions in modelling stochastic processes. The Teissier distribution does not account for a non-monotone increasing hazard rate such as the unimodal failure rates and the *U*-shaped rates that are commonly encountered in medical sciences, reliability theory, stochastic processes, thanatology and genealogy. Thus there is an urgent need to improve the Teissier distribution so that it can reflect the non-monotone increasing feature.

However, adding additional parameters to the existing classical statistical distribution has helped to improve the flexibility of many statistics models. Thus providing a very rich satisfactory statistical inference is germane to making good and reliable decisions. One of such interesting methods in stochastic modelling is the Marshall – Olkin model [5].

Nevertheless, for a random variable W and parameter μ the probability density function (PDF) of the Marshall – Olkin is specified as

$$f(w, \mu) = \frac{\mu g(w)}{\left[1 - \bar{\mu}\bar{G}(w)\right]^2}, \ \mu > 1, \ w > 0.$$

The cumulative distribution function (CDF) of the Marshall – Olkin generated model is designated as

$$F(w, \mu) = \frac{G(w)}{\left[1 - \bar{\mu}\bar{G}(w)\right]}, \ \mu > 1, \ w > 0,$$

where $g(w, \mu) = \dfrac{dG(w)}{dw}$ is the baseline and parent PDF, $\bar{\mu} = (1 - \mu)$ and $\bar{G}(w) = (1 - G(w))$ with the CDF $G(w)$. Thus, the transformed transformer ($T-X$) method [6] of generating a flexible family of classical distribution with a link function $-\log\left[1 - G(w)\right]$ can be adopted to obtain the Teissier generated family as the PDF and the CDF as

$$g(w) = \beta m(w)\left(1 - M(w)\right)^{-(\beta+1)}\left(\left[1 - M(w)\right]^{-\beta} - 1\right)e^{1-(1-M(w))^{-\beta}} \text{ for } \beta > 0$$

and

$$G(w) = 1 - \left(1 - M(w)\right)^{-\beta} e^{1-(1-M(w))^{-\beta}} \text{ for } \beta > 0,$$

where $m(w)$ and $M(w) = \int m(w)dw$ are the parent classical distribution PDF and CDF.

The development of a generalised family of distribution models that provides a parameterised mathematical function, a simple and efficient algorithm for the parameter estimation of data sets of various characteristics, has become an interest to researchers. Thus several Marshall – Olkin methods of adding parameters in the researched studies such as [7; 8–15] and alpha power Teissier in [16] were examined.

In this article the Marshall – Olkin Teissier (MT) generator is introduced with the prefix MT to improve the performance and flexibility of the parsimonious Teissier distribution. The MT generated model hazard rate function could be *J*- and *U*-shaped, decreasing and increasing in nature. Thus the proposed model extends and pushes forward the frontier of knowledge in many applied areas in statistics. However, few methods of improving the Teissier distribution has been in the existing literature. These methods include the Teissier distribution proposed in [17], the exponentiated Teissier distribution proposed by [18] and the bivariate Teissier distribution proposed by [19].

The Marshall – Olkin Teissier generated distribution

Let W be a random variable for $w \in \Re$. Then the PDF and CDF of the MT generated distribution can be defined as

$$f(w, \mu, \beta) = \frac{\mu\beta m(w)(1 - M(w))^{-(\beta+1)}\left((1 - M(w))^{-\beta} - 1\right)e^{1-(1-M(w))^{-\beta}}}{\left[1 - \bar{\mu}(1 - M(w))^{-\beta} e^{1-(1-M(w))^{-\beta}}\right]^2} \text{ for } \mu, \beta > 0,\ w > 0$$

and

$$F(w, \mu, \beta) = \frac{1 - (1 - M(w))^{-\beta} e^{1-(1-M(w))^{-\beta}}}{\left[1 - \bar{\mu}(1 - M(w))^{-\beta} e^{-(1-M(w))^{-\beta}}\right]} \text{ for } \mu, \beta > 0,\ w > 0,$$

where $m(w) = \frac{dM(w)}{dw}$ and $M(w)$ are the parent classical distribution and μ and β are the scale and threshold parameters respectively. The MT generated hazard rate function can take the form

$$h(w, \mu, \beta) = \frac{\left[\dfrac{\mu\beta m(w)(1 - M(w))^{-(\beta+1)}\left((1 - M(w))^{-\beta} - 1\right)e^{1-(1-M(w))^{-\beta}}}{\left[1 - \bar{\mu}[1 - M(w)]^{-\beta} e^{1-(1-M(w))^{-\beta}}\right]^2}\right]}{\left[1 - \dfrac{1 - (1 - M(w))^{-\beta} e^{1-(1-M(w))^{-\beta}}}{\left[1 - \bar{\mu}[1 - M(w)]^{-\beta} e^{1-(1-M(w))^{-\beta}}\right]}\right]}.$$

The MT generated hazard rate is increasing function if $\beta \geq 1$ and $\mu \geq 1$, while it is decreasing if $\beta \leq 1$ and $\mu \leq 1$. It can be observed that when $\mu = 1$, we obtain the Teissier generated family of distributions.

The quantile function of the MT generated model for a uniform interval $p \in (0, 1)$ can be obtained using the Lambert function W that satisfies the equation $W(t) = \exp(W(t)) = t \in [-1, \infty)$. Then for $(1 - M(w))^{-\beta} > 0$ we have $W_{-1}\left(\frac{p-1}{(p\bar{\mu} - 1)e}\right) = (1 - M(w))^{-\beta}$. Thus the quantile function of the MT generated model can be defined as

$$w_p = M^{-1}\left[1 - \left\{W_{-1}\left[\left(\frac{p-1}{p\bar{\mu} - 1}\right)e^{-1}\right]\right\}^{-\frac{1}{\beta}}\right],$$

where W_{-1} is the negative branch of the Lambert function W.

Estimation

Parameter estimation. Several approaches for obtaining the parameter estimate of models have been proposed in the literature. However, the maximum likelihood method is one of the most commonly employed methods. Hence the maximum likelihood method was employed to obtain the parameter estimate of the proposed MT generated model in this section.

Let the likelihood of the MT generated model be denoted by Φ. Then the log-likelihood of Φ can be specified as

$$\log\Phi = n\log\mu + n\log\beta + \sum_{d=1}^{k}\log m(w_d, \varepsilon) - (\beta + 1)\sum_{d=1}^{k}\log(1 - M(w_d, \varepsilon)) +$$
$$+\sum_{d=1}^{k}\log R_d(w_d) - \sum_{d=1}^{k} R_d(w_d) - \sum_{d=1}^{k}\log T_d,$$

where $R_d = \left(1 - M\left(w_d\right)\right)^{-\beta} - 1$; ε is the parameter vector; $T_d = \left[\mu - (\mu - 1)\left[1 - \left(1 - \left(1 - M\left(w_d\right)\right)^{-\beta}\right) \times \times e^{1 - \left(1 - M\left(w_d\right)\right)^{-\beta}}\right]\right]^2$.

The parameters estimate of the MTG model can be obtained by taking the partial derivative of the $\log\Phi$ and equating it to zero. We have

$$\frac{\partial \log\Phi}{\partial\mu} = \frac{n}{\mu} - \sum_{d=1}^{k} \frac{T'_{d,\mu}\left(w_d, \varepsilon\right)}{T_d\left(w_d, \varepsilon\right)} = 0, \quad (1)$$

$$\frac{\partial \log\Phi}{\partial\beta} = \frac{n}{\beta} - \sum_{d=1}^{k} \log\left(1 - M\left(w_d, \varepsilon\right)\right) + \sum_{d=1}^{k} \frac{R'_{d,\beta}\left(w_d, \varepsilon\right)}{R_d\left(w_d, \varepsilon\right)} - \sum_{d=1}^{k} R'_{d,\beta}\left(w_d, \varepsilon\right) - \sum_{d=1}^{k} \frac{T'_{d,\beta}\left(w_d, \varepsilon\right)}{T_d\left(w_d, \varepsilon\right)} = 0 \quad (2)$$

and

$$\frac{\partial \log\Phi}{\partial\varepsilon} = \sum_{d=1}^{k} \frac{m'_{\varepsilon}\left(w_d, \varepsilon\right)}{m\left(w_d, \varepsilon\right)} + (\beta + 1)\sum_{d=1}^{k} \frac{M'_{\varepsilon}\left(w_d, \varepsilon\right)}{1 - M\left(w_d, \varepsilon\right)} + \sum_{d=1}^{k} \frac{R'_{d,\varepsilon}\left(w_d, \varepsilon\right)}{R_d\left(w_d, \varepsilon\right)} - $$
$$- \sum_{d=1}^{k} R'_{d,\varepsilon}\left(w_d, \varepsilon\right) - \sum_{d=1}^{k} \frac{T'_{d,\varepsilon}\left(w_d, \varepsilon\right)}{T_d\left(w_d, \varepsilon\right)} = 0. \quad (3)$$

However, in all cases symbol $'$ represents partial derivative of the corresponding parameter estimate. Equations (1)–(3) are non-linear. Thus the model parameters in the equations can be obtained using the Newton – Raphson method in *R* [20] and MatLab.

Special model

The performance of the MT generated model is assessed using the Weibull, Gompertz and Lomax distributions. Plots of equations (4)–(9) for some selected parameter values are given in fig. 1–3. The plots in fig. 1 show the density, hazard rate function and CDF of the MTG generated model. The plots in fig. 2 show the density, hazard rate function and CDF of the MTW generated model. More so, the plots in fig. 3 show the density, hazard rate function and CDF of the MTL generated model. The plots in fig. 1, *a*; 2, *a*, and 3, *a*, indicate that the MT generated model is very adaptable and flexible with the value of α having a weighty effect on the model kurtosis and skewness. On the plots in fig. 1, *b*; 2, *b*, and 3, *b*, we observe that MTG model can be used in solving a variety of statistical problems in modelling reliability data, because its hazard rate function can be expressed as *U*-shaped, increasing, decreasing, or initially increasing, then decreasing and eventually increasing.

The MTG distribution. Suppose for a random variable *X* the PDF and CDF of the Gompertz distribution is given as $m(x) = \rho e^{\theta x - \frac{\rho}{\theta}\left(e^{\theta x} - 1\right)}$ and $M(x) = 1 - e^{-\frac{\rho}{\theta}\left(e^{\theta x} - 1\right)}$ respectively for positive parameters ρ and θ. Then the PDF and hazard rate function of the MT generated model is specified as

$$f(w, \mu, \beta) = \frac{\mu\beta\rho\exp\left(\theta x - \frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\right)^{-(\beta+1)}}{\left[\mu - (\mu - 1)\left(1 - \left[\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\right]^{-\beta}\right) e^{1 - \left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\right)^{-\beta}}\right]^2} \times$$
$$\times \left(\left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\right)^{-\beta} - 1\right) e^{1 - \left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\right)^{-\beta}} \quad (4)$$

and

$$h(w, \mu, \beta) = \frac{\mu\beta\rho\exp\left(\theta x - \frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\right)^{-(\beta+1)}}{\left[\mu - (\mu - 1)\left(1 - \left[\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\right]^{-\beta}\right) e^{1 - \left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x) - 1\right)\right)\right)^{-\beta}}\right]^2} \times$$

$$\times\left(\left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x)-1\right)\right)\right)^{-\beta}-1\right)e^{1-\left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x)-1\right)\right)\right)^{-\beta}}\times$$

$$\times\left(\left[1-\frac{1-\left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x)-1\right)\right)\right)^{-\beta}e^{1-\left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x)-1\right)\right)\right)^{-\beta}}}{\left[\mu-(\mu-1)\left(1-\left[\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x)-1\right)\right)\right]^{-\beta}\right)e^{1-\left(\exp\left(-\frac{\rho}{\theta}\left(\exp(\theta x)-1\right)\right)\right)^{-\beta}}\right]}\right]\right)^{-1}. \quad (5)$$

The MTW distribution. Suppose for a random variable X the PDF and CDF (for $x \geq 0$), say $m(x) = \rho\theta^{\rho}x^{\rho-1}e^{-(\theta x)^{\rho}}$ and $M(x) = 1 - e^{-(\theta x)^{\rho}}$ respectively (for $\theta > 0$, $\rho > 0$) of the Weibull distribution. Then the PDF and hazard rate function of the MT generated model is specified as

$$f(w, \mu, \beta) = \mu\beta\rho\theta^{\rho}x^{\rho-1}\exp\left(-(\theta x)^{\rho}\right)\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-(\beta+1)}\left(\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-\beta}-1\right)\times$$

$$\times e^{1-\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-\beta}}\left(\left[\mu-(\mu-1)\left(1-\left[\exp\left(-(\theta x)^{\rho}\right)\right]^{-\beta}\right)e^{1-\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-\beta}}\right]\right)^{-2} \quad (6)$$

and

$$h(w, \mu, \beta) = \left[\frac{\mu\beta\rho\theta^{\rho}x^{\rho-1}\exp\left(-(\theta x)^{\rho}\right)\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-(\beta+1)}\left(\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-\beta}-1\right)}{\left[\mu-(\mu-1)\left(1-\left[\exp\left(-(\theta x)^{\rho}\right)\right]^{-\beta}\right)e^{1-\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-\beta}}\right]^{2}}\right]\times$$

$$\times e^{1-\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-\beta}}\left(\left[1-\frac{1-\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-\beta}e^{1-\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-\beta}}}{\left[\mu-(\mu-1)\left(1-\left[\exp\left(-(\theta x)^{\rho}\right)\right]^{-\beta}\right)e^{1-\left(\exp\left(-(\theta x)^{\rho}\right)\right)^{-\beta}}\right]}\right]\right)^{-1}. \quad (7)$$

The MTL distribution. Suppose for a random variable X the PDF and CDF (for $x \geq 0$), say $m(x) = \frac{\theta}{\rho}\left[1+\frac{x}{\rho}\right]^{-(\theta+1)}$ and $M(x) = 1 - \left[1+\frac{x}{\rho}\right]^{-\theta}$ respectively (for a scale parameter $\rho > 0$ and a shape parameter $\theta > 0$) of the Lomax distribution. Then the PDF and hazard rate function of the MT generated model is specified as

$$f(w, \mu, \beta) = \frac{\mu\beta\frac{\theta}{\rho}\left[1+\frac{x}{\rho}\right]^{-(\theta+1)}\left(\left[1+\frac{x}{\rho}\right]^{-\theta}\right)^{-(\beta+1)}\left(\left(\left[1+\frac{x}{\rho}\right]^{-\theta}\right)^{-\beta}-1\right)e^{1-\left(\left[1+\frac{x}{\rho}\right]^{-\theta}\right)^{-\beta}}}{\left[\mu-(\mu-1)\left(1-\left[\left[1+\frac{x}{\rho}\right]^{-\theta}\right]^{-\beta}\right)e^{1-\left(\left[1+\frac{x}{\rho}\right]^{-\theta}\right)^{-\beta}}\right]^{2}} \quad (8)$$

and

$$h(w, \mu, \beta) = \frac{\left[\dfrac{\mu\beta\dfrac{\theta}{\rho}\left[1+\dfrac{x}{\rho}\right]^{-(\theta+1)}\left(\left[1+\dfrac{x}{\rho}\right]^{-\theta}\right)^{-(\beta+1)}\left(\left(\left[1+\dfrac{x}{\rho}\right]^{-\theta}\right)^{-\beta}-1\right)e^{1-\left(\left[1+\frac{x}{\rho}\right]^{-\theta}\right)^{-\beta}}}{\left[\mu-(\mu-1)\left(1-\left[\left[1+\dfrac{x}{\rho}\right]^{-\theta}\right]^{-\beta}\right)e^{1-\left(\left[1+\frac{x}{\rho}\right]^{-\theta}\right)^{-\beta}}\right]^2}\right]}{\left[1-\dfrac{1-\left(\left[1+\dfrac{x}{\rho}\right]^{-\theta}\right)^{-\beta}e^{1-\left(\left[1+\frac{x}{\rho}\right]^{-\theta}\right)^{-\beta}}}{\left[\mu-(\mu-1)\left(1-\left[\left[1+\dfrac{x}{\rho}\right]^{-\theta}\right]^{-\beta}\right)e^{1-\left(\left[1+\frac{x}{\rho}\right]^{-\theta}\right)^{-\beta}}\right]}\right]}. \quad (9)$$

General statistical properties

In this section the statistical properties of the MT generated model are discussed. However, simplification is carried out on the proposed model to enable the proposed model to be presented in a simple manner.

Let $|s| < 1$ and $c > 0$. Then the expression $(1-s)^{-c}$ can be simplified as

$$(1-s)^{-c} = \sum_{d=0}^{\infty} \frac{\Gamma(c+d)}{\Gamma(c)d!} s^d,$$

where $\Gamma(\cdot)$ is the gamma function. Thus the PDF of the MT generated family can be specified as a linear combination of the Teissier generated distribution as

$$f(w) = \sum_{d=0}^{\infty} \frac{\Gamma(d+2)}{d!} \mu(1-\mu)^d g(w, \beta(d+1)), \ \mu > 0, \ w > 0.$$

Also the CDF can be expressed as

$$F(w) = \sum_{t=0}^{\infty} \frac{\Gamma(t+1)}{t!} (1-\mu)^t G(w) \bar{G}^t(w).$$

Moments. The MT generated r moment can be obtained as

$$\mu_r' = \sum_{d=0}^{\infty} \frac{\Gamma(d+2)}{d!} \mu(1-\mu)^d K(r),$$

where $K(r) = \int_0^{\infty} w^r g(w, \beta(d+1)) dw$. The mean of the MT generated model is obtained when $r = 1$. Various moments can be obtained by varying the values of r.

Probability generating function. Let $w_1, w_2, \ldots, w_n$ be the random variable sampled from the MT generated model. Then the MT generated probability generating function is given as

$$p(w) = \sum_{d, a=0}^{\infty} \frac{(\log t)^a \Gamma(d+2)}{a!d!} \mu(1-\mu)^d B(a),$$

where

$$B(a) = \int_1^{\infty} w^a g(w, \beta(d+1)) dw.$$

Probability weighted moments. The parameters and quantile of the MT generated model can be obtained using the probability weighted moments (PWM). Thus the MT generated PWM for $r \geq 1$ and $s \geq 0$ can be obtained as

$$P_{sr}(w) = \sum_{d,\,t=0}^{\infty} \frac{\Gamma(d+2)}{d!}\left[\frac{\Gamma(t+1)}{t!}\right]^{s} \mu(1-\mu)^{ts+d} R(t,\, d),$$

where

$$R(t,\, d) = \int_{0}^{\infty} w^{r} g(w,\, \beta(d+1)) G^{s}(w) \bar{G}^{ts}(w) dw.$$

Moment of the residual. The r moment of the residual life of the MT generated model, say $d_r(t) = E\left[(W-t)^{r} | W > t\right]$ for $r = 1, 2, 3, 4, \ldots$ can be obtained uniquely as

$$d_r(t) = \frac{1}{1-F(w)} \sum_{d=0}^{\infty} \sum_{c=0}^{r} \binom{r}{c} t^{n-c} \frac{\Gamma(d+2)}{d!} \mu(1-\mu)^{d} R(c,\, d),$$

where

$$R(c,\, d) = \int_{t}^{\infty} w^{c} g(w,\, \beta(d+1)) dw.$$

However, the r moment of the reversed residual life, say $D_r(t) = E\left[(t-W)^{r} | W \leq t\right]$ for $t > 0$ can be defined as

$$D_r(t) = \frac{1}{F(w)} \sum_{d=0}^{\infty} \sum_{a=0}^{r} \binom{r}{a} t^{n-a} \frac{\Gamma(d+2)}{d!} \mu(1-\mu)^{d} R(a,\, d),$$

where

$$R(a,\, d) = \int_{0}^{t} w^{a} g(w,\, \beta(d+1)) dw.$$

Entropy. Entropy is the measure of uncertainty. Thus the Renyi entropy is expressed as

$$R_e(\upsilon) = \frac{1}{(1-\upsilon)} \log\left(\sum_{d=0}^{\infty} \frac{\Gamma(d+2)}{d!} \mu(1-\mu)^{d}\right)^{\upsilon} D(\upsilon),$$

where $D(\upsilon) = \int_{1}^{\infty} g^{\upsilon}(w,\, \beta(d+1)) dw.$

Order statistics. Let $W_1, W_2, \ldots, W_n$ be MT generated random samples of size n and $W_{(1)}, W_{(2)}, \ldots, W_{(n)}$ the order statistics of the processes. Then the PDF of the j order statistic $W_{(j)}$, say $f_j(w)$ is defined as

$$f_j(w) = \frac{n!\mu\beta m(w)}{(j-1)!(n-j)!} \frac{(1-M(w))^{-(\beta+1)}\left((1-M(w))^{-\beta}-1\right) e^{1-(1-M(w))^{-\beta}}}{\left[1-\bar{\mu}(1-M(w))^{-\beta} e^{1-(1-M(w))^{-\beta}}\right]^{2}} \times$$

$$\times \left[\frac{1-(1-M(w))^{-\beta} e^{1-(1-M(w))^{-\beta}}}{\left[1-\bar{\mu}(1-M(w))^{-\beta} e^{1-(1-M(w))^{-\beta}}\right]}\right]^{j-1} \left[1-\frac{1-(1-M(w))^{-\beta} e^{1-(1-M(w))^{-\beta}}}{\left[1-\bar{\mu}(1-M(w))^{-\beta} e^{1-(1-M(w))^{-\beta}}\right]}\right]^{n-j}, \quad w > 0,\, \beta > 0.$$

The minimum and maximum order statistics are obtained when $j = 1$ and $j = n$ respectively.

Application

An application to demonstrate the tractability and flexibility is given in this section. The MT generated submodels are compared with some existing generating models of Weibull, Gompertz and Kumaraswamy. A simulation and real-life applications are considered in this section.

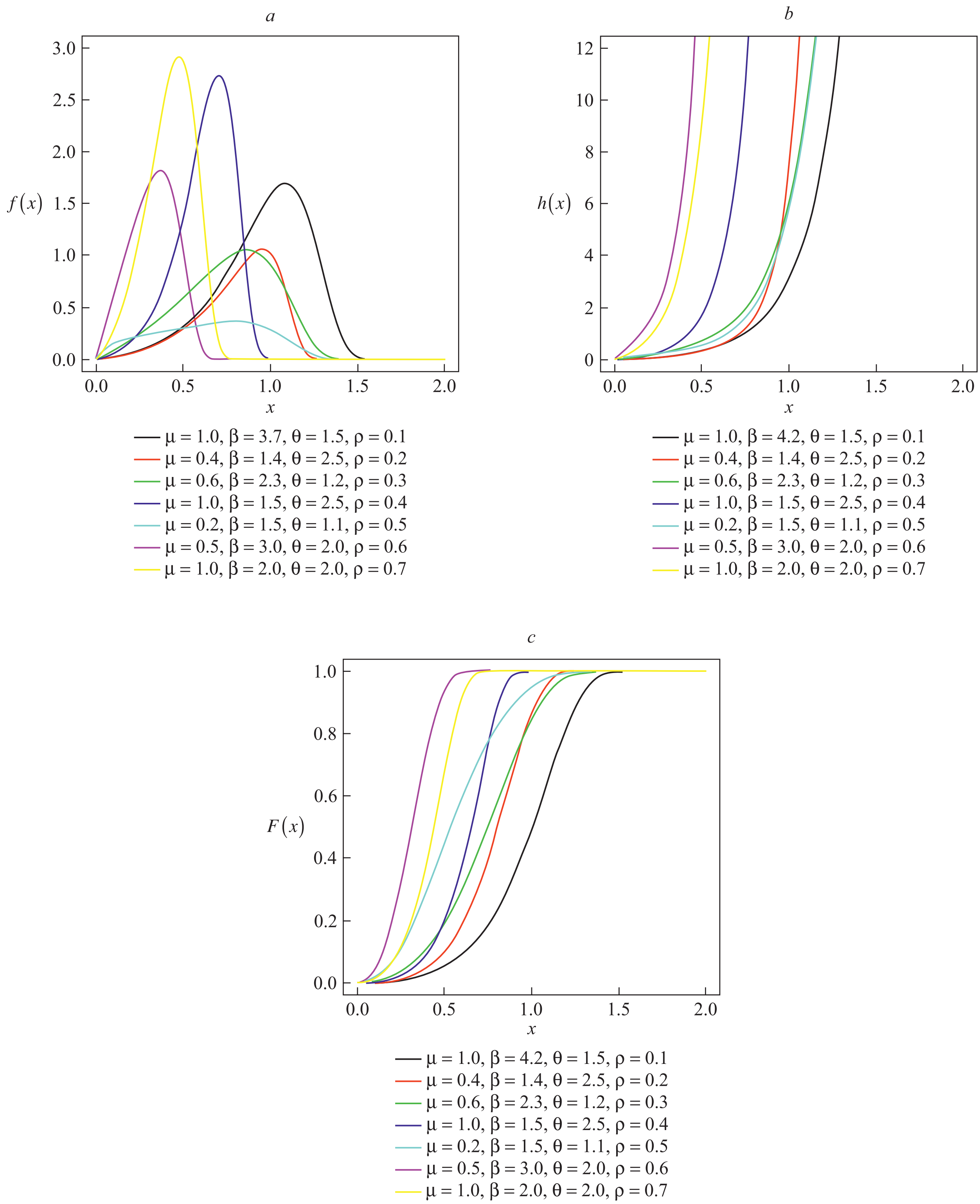

Fig. 1. MT generated plots
for the Gompertz model with various parameter values:
a – MTG density; *b* – MTG hazard rate function; *c* – MTG CDF

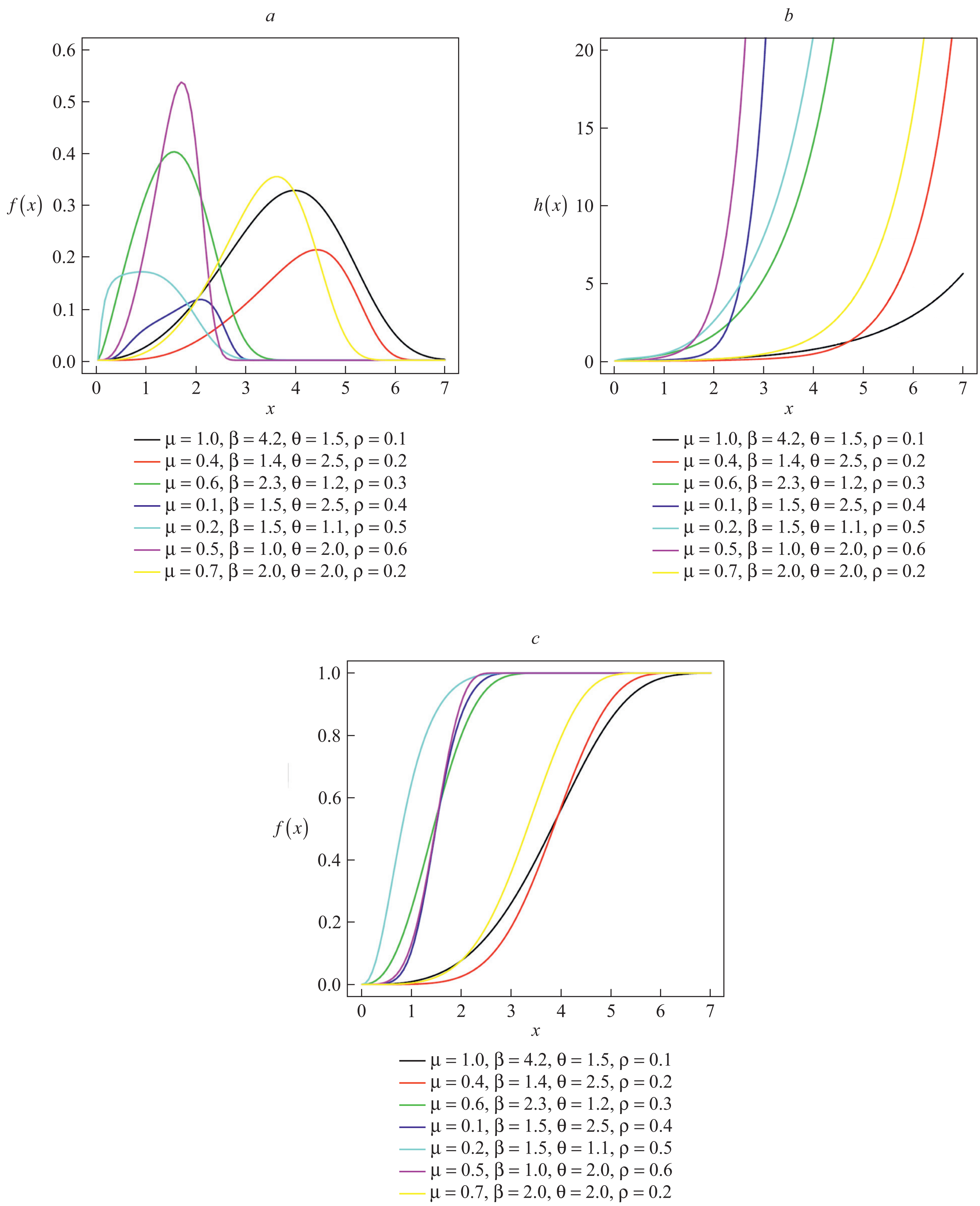

Fig. 2. MT generated plots
for the Weibull model with various parameter values:
a – MTW density; *b* – MTW hazard rate function; *c* – MTW CDF

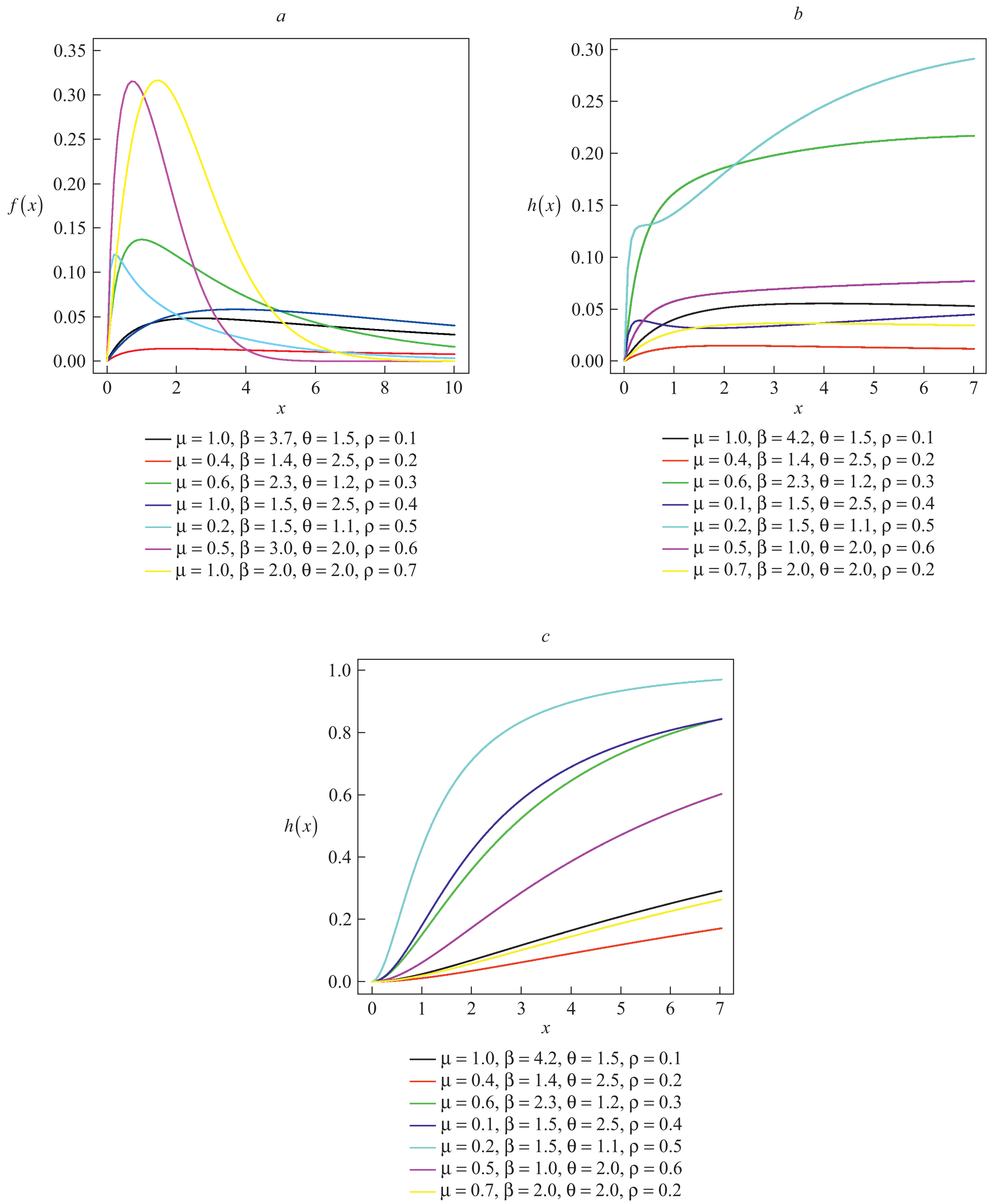

Fig. 3. MT generated plots
for the Lomax model with various parameter values:
a – MTL density; *b* – MTL hazard rate function; *c* – MTL CDF

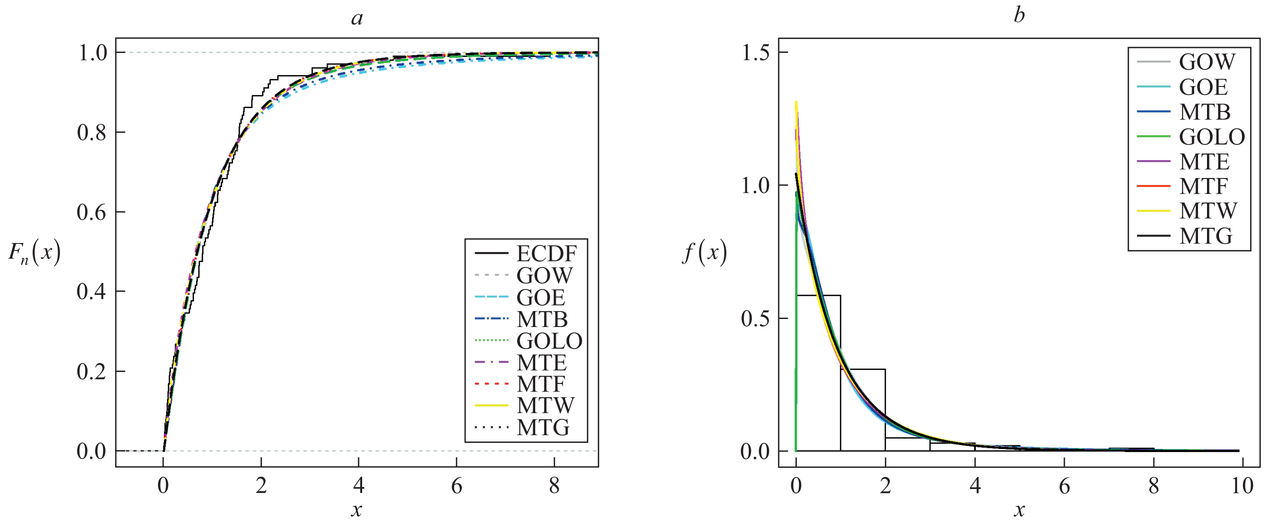

Fig. 4. Empirical estimate for the first dataset:
a – empirical densities; *b* – empirical CDFs

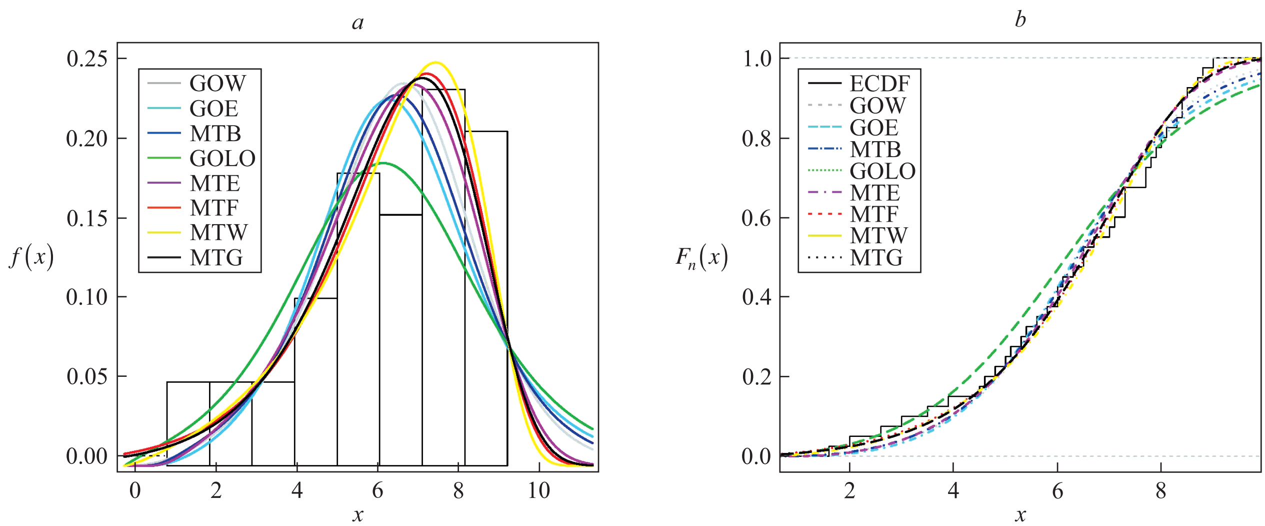

Fig. 5. Empirical estimate for the second dataset:
a – empirical densities; *b* – empirical CDFs

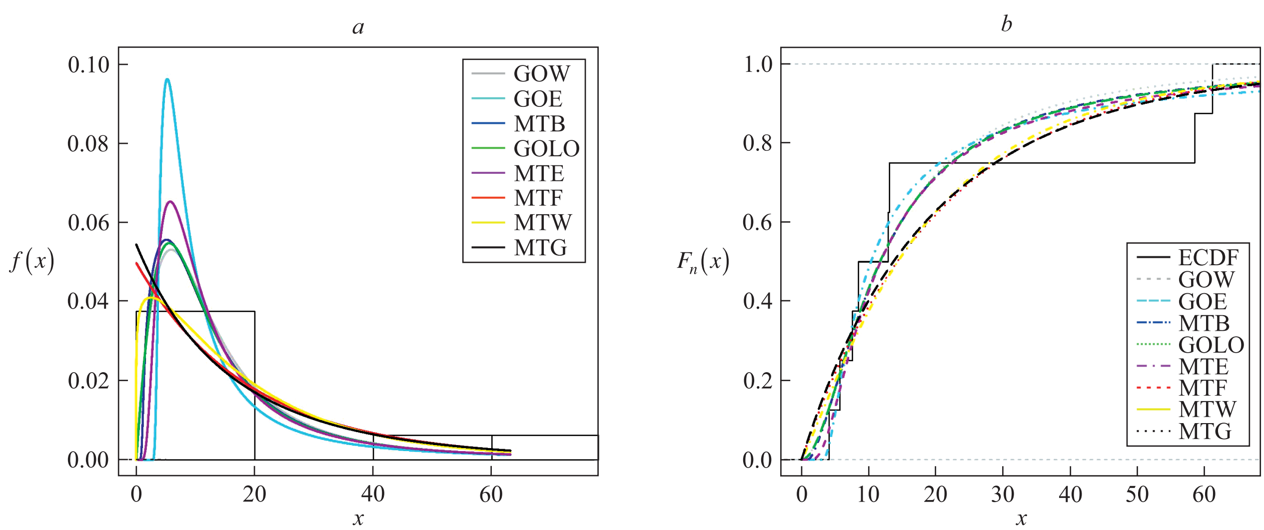

Fig. 6. Empirical estimate for the third dataset:
a – empirical densities; *b* – empirical CDFs

Simulation study. A simulation is used to examine the tractability, flexibility and performance of the proposed model with the submodels, Gompertz, Weibull and Lomax distributions. In the simulation study the estimated mean estimate is denoted by ME, the bias is denoted by Bias and mean squared errors by MSE [20], software is used to obtain the simulation results. Random sample sizes of 5, 10, 20, 50, 100, 200, 250, 300, 450, 500 and 600 are used. The sample sizes are replicated 6000 times for the value of the estimated parameter $\mu = 0.25$, $\beta = 1.50$, $\rho = 1.25$ and $\theta = 2.00$ for the Gompertz model, $\mu = 2.50$, $\beta = 1.60$, $\rho = 0.60$ and $\theta = 0.70$ for the Weibull model and $\mu = 0.50$, $\beta = 0.40$, $\rho = 0.70$ and $\theta = 0.70$ for the Lomax model using the quantile function. Table 1 shows the results of the simulation.

Table 1

The mean estimates, biases and mean squared errors Monte Carlo simulation results

Model	*n*	ME	Bias	MSE
Gompertz	5	1.0157, 1.1885, 2.0465, 1.3275	0.0157, 0.1885, 0.5465, –0.1725	2.2387, 0.5569, 1.1076, 1.5291
	10	0.5503, 1.2813, 1.6780, 1.7002	–0.4497, 0.2813, 0.1780, 0.2002	1.1246, 0.4875, 0.4573, 0.6351
	20	0.3554, 1.3426, 1.4933, 1.8498	–0.6446, 0.3426, –0.0067, 0.3498	0.7829, 0.4013, 0.2439, 0.3848
	50	0.2592, 1.4001, 1.3639, 1.8996	–0.7408, 0.4001, –0.1361, 0.3996	0.6683, 0.3312, 0.1260, 0.2721
	100	0.2456, 1.4397, 1.3071, 1.8912	–0.7544, 0.4397, –0.1929, 0.3912	0.0220, 0.1862, 0.0319, 0.1096
	200	0.2433, 1.4656, 1.2763, 1.8766	–0.7567, 0.4656, –0.2237, 0.3766	0.0163, 0.1779, 0.0275, 0.0316
	250	0.2437, 1.4746, 1.2671, 1.8721	–0.7563, 0.4746, –0.2329, 0.3721	0.0108, 0.0162, 0.0154, 0.0240
	300	0.2422, 1.4744, 1.2657, 1.8710	–0.7578, 0.4744, –0.2343, 0.3710	0.0093, 0.0143, 0.0140, 0.0210
	450	0.2452, 1.4860, 1.2562, 1.8682	–0.7548, 0.4860, –0.2438, 0.3682	0.0077, 0.0134, 0.0127, 0.0171
	500	0.2474, 1.4938, 1.2521, 1.8678	–0.7526, 0.4938, –0.2479, 0.3678	0.0059, 0.0099, 0.0058, 0.0158
	600	0.2495, 1.5006, 1.2478, 1.8651	–0.7505, 0.5006, –0.2522, 0.3651	0.0019, 0.0037, 0.0041, 0.0136
Weibull	5	3.0984, 1.8692, 0.7350, 0.7248	2.0984, 0.8692, 0.2350, 0.2248	1.5445, 0.3159, 0.1937, 0.1211
	10	3.2731, 1.7685, 0.5964, 0.7243	2.2731, 0.7685, 0.0964, 0.2243	1.4101, 0.2169, 0.0537, 0.1063
	20	3.2434, 1.7141, 0.5627, 0.7238	2.2434, 0.7141, 0.0627, 0.2238	1.1715, 0.1583, 0.0275, 0.0917
	50	3.0367, 1.7106, 0.5590, 0.6734	2.0367, 0.7106, 0.0590, 0.1734	0.8990, 0.0981, 0.0161, 0.0532
	100	2.8336, 1.7070, 0.5812, 0.6448	1.8336, 0.7070, 0.0812, 0.1448	0.6782, 0.0722, 0.0122, 0.0292
	200	2.6601, 1.6648, 0.6073, 0.6492	1.6601, 0.6648, 0.1073, 0.1492	0.4692, 0.0497, 0.0092, 0.0145
	250	2.6203, 1.6505, 0.6157, 0.6412	1.6203, 0.6505, 0.1157, 0.1512	0.3899, 0.0432, 0.0085, 0.0120
	300	2.5085, 1.6336, 0.6211, 0.6401	1.5885, 0.6336, 0.1111, 0.1501	0.3364, 0.0399, 0.0075, 0.0105
	450	2.5038, 1.6023, 0.6156, 0.6694	1.5338, 0.6023, 0.1356, 0.1694	0.2147, 0.0298, 0.0058, 0.0074
	500	2.5024, 1.5966, 0.6040, 0.6717	1.5354, 0.5966, 0.1380, 0.1717	0.1949, 0.0287, 0.0054, 0.0069
	600	2.5005, 1.5859, 0.6005, 0.6745	1.5235, 0.5859, 0.1435, 0.1745	0.1649, 0.0255, 0.0048, 0.0061
Lomax	5	1.1175, 1.8662, 0.7310, 0.7188	2.1175, 0.8662, 0.2310, 0.2188	1.5439, 0.2928, 0.1876, 0.1158
	10	1.2638, 1.7546, 0.5955, 0.7364	2.2638, 0.7546, 0.0955, 0.2364	1.3417, 0.2172, 0.0556, 0.1102
	20	1.2585, 1.7148, 0.5596, 0.7236	2.2585, 0.7148, 0.0596, 0.2236	1.1901, 0.1568, 0.0272, 0.0908
	50	1.0350, 0.7174, 0.5579, 0.6710	2.0350, 0.7174, 0.0579, 0.1710	0.9278, 0.0361, 0.0163, 0.0531
	100	0.8403, 0.7092, 0.5806, 0.6420	1.8403, 0.7092, 0.0806, 0.1420	0.2769, 0.0292, 0.0126, 0.0277
	200	0.6719, 0.6662, 0.6079, 0.6477	1.6719, 0.6662, 0.1079, 0.1477	0.1852, 0.0194, 0.0093, 0.0144
	250	0.6215, 0.6490, 0.6159, 0.6526	1.6215, 0.6490, 0.1159, 0.1526	0.0958, 0.0132, 0.0082, 0.0113
	300	0.5030, 0.6357, 0.6218, 0.6581	1.5930, 0.6357, 0.1218, 0.1581	0.0428, 0.0103, 0.0077, 0.0104
	450	0.5052, 0.5052, 0.6342, 0.6680	1.5452, 0.6052, 0.1342, 0.1680	0.0139, 0.0096, 0.0060, 0.0077
	500	0.5005, 0.4009, 0.6385, 0.6716	1.5305, 0.5959, 0.1385, 0.1716	0.0086, 0.0084, 0.0056, 0.0070
	600	0.5002, 0.4002, 0.6424, 0.6757	1.5202, 0.5842, 0.1424, 0.1757	0.0020, 0.0059, 0.0050, 0.0062

The bias is calculated (for X) by

$$\widehat{\text{Bias}}_W = \frac{1}{6000}\sum_{i=1}^{6000}\left(\hat{W}_i - W\right).$$

Also the MSE is obtained as

$$\widehat{\text{MSE}}_W = \frac{1}{6000}\sum_{i=1}^{6000}\left(\hat{W}_i - W\right)^2.$$

In table 1 the performance of the proposed model is examined. The mean estimated value tends to the true values in all cases considered as the sample sizes increase. More so, the MSE decrease as the sample size increases.

Real-life. A real-life dataset is used to illustrate the performance of the MT generated model. In the illustrations given the goodness-of-fit of the submodels is classified using their p-values, Kolmogorov – Smirnov (KS) test statistic, negative log-likelihood (log-lik), Cramér-von Mises statistic (W) and Anderson – Darling statistic (A). Figures 4–6 show the empirical densities and CDFs of the first, second and third dataset. Several models are compared with the MT generated models of Gompertz, Weibull, Frechet, Burr XII and Lomax. These include Gompertz – Weibull (GW), Weibull – Gompertz (WG), Gompertz – Lomax (GL), alpha power Gompertz (APG), Weibull – Burr XII (WB), Kumaraswamy – Burr XII (KB), Kumaraswamy – Frechet (KF), Kumaraswamy – Gompertz (KG), Kumaraswamy – Lomax (KL), Weibull – Frechet (WF), Gompertz exponential (GE), gamma (Ga), Gompertz – Lomax and Gompertz – Burr XII (GB).

The first real data set refers to the stress-rupture life of kevlar 49 and epoxy strands subjected to constant sustained pressure at the 90 % stress level until all had failed. This data set was studied by [21–25]. The data are as follows

1.8, 1.8, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.2, 4.69, 7.89, 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.1, 0.1, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.2, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.4, 0.42, 0.43, 0.52, 0.54, 0.56, 0.6, 0.6, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.8, 0.8, 0.83, 0.85, 0.9, 0.92, 0.95, 0.99, 1, 1.01, 1.02, 1.03, 1.05, 1.1, 1.1, 1.11, 1.15, 1.18, 1.2, 1.29, 1.31, 1.33, 1.34, 1.4, 1.43, 1.45, 1.5, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64.

Table 2 shows the MLEs with standard errors in parentheses and the measures A, W, AIC, KS, negative log-likelihood and p-values with the stress-rupture data.

Table 2

MLEs and the test statistic for first data

Distribution	Estimates	Log-lik	KS	W	A	p-Value
MTW	$\hat{\mu} = 5.8857$ (0.2620) $\hat{\beta} = 3.2885$ (0.8062) $\hat{\theta} = 1.0513$ (0.7767) $\hat{\varphi} = 0.4481$ (0.0795)	82.56	0.0035	0.0003	0.0019	0.9814
MTG	$\hat{\mu} = 0.0745$ (0.0403) $\hat{\beta} = 0.2209$ (0.1639) $\hat{\theta} = 0.6518$ (0.2371) $\hat{\varphi} = 0.3355$ (0.0002)	89.41	0.0054	0.0008	0.0022	0.9548
MTE	$\hat{\mu} = 0.1024$ (0.0826) $\hat{\beta} = 0.5816$ (0.2770) $\hat{\theta} = 0.1058$ (0.0983)	92.81	0.0704	0.0452	0.0160	0.8741
MTL	$\hat{\mu} = 0.3990$ (0.1653) $\hat{\beta} = 0.7400$ (0.0000) $\hat{\theta} = 1.3804$ (0.6561) $\hat{\varphi} = 0.0745$ (0.0000)	93.28	0.0906	0.2510	0.3536	0.7775

Continuation of the table 2

Distribution	Estimates	Log-lik	KS	W	A	*p*-Value
MTF	$\hat{\mu} = 0.9338\ (0.0693)$ $\hat{\beta} = 0.4721\ (0.1891)$ $\hat{\theta} = 0.1653\ (0.0205)$ $\hat{\varphi} = 0.6363\ (0.1023)$	96.08	0.0932	0.3503	0.5016	0.7639
MTB	$\hat{\mu} = 1.0961\ (0.6845)$ $\hat{\beta} = 0.0660\ (0.0000)$ $\hat{\theta} = 1.9499\ (0.0001)$ $\hat{\varphi} = 0.5214\ (0.0911)$	100.00	0.0995	0.3894	0.6024	0.7522
KB	$\hat{\alpha} = 0.1146\ (0.0523)$ $\hat{\beta} = 0.3254\ (0.0972)$ $\hat{\theta} = 1.4251\ (0.9151)$ $\hat{\varphi} = 4.5665\ (1.6679)$	101.12	0.0999	0.4904	0.6198	0.6443
WG	$\hat{\mu} = 0.3696\ (0.2326)$ $\hat{\beta} = 0.6902\ (0.1474)$ $\hat{\theta} = 2.0139\ (1.2141)$ $\hat{\varphi} = -0.4921\ (0.1481)$	101.15	1.0028	0.0805	0.7378	0.6205
WB	$\hat{\alpha} = 0.9141\ (0.1039)$ $\hat{\beta} = 0.1369\ (0.0118)$ $\hat{\theta} = 1.2445\ (0.1057)$ $\hat{\varphi} = 6.1457\ (0.0730)$	102.42	1.0821	0.1412	0.8443	0.5030
KG	$\hat{\mu} = 0.7920\ (0.1829)$ $\hat{\beta} = 0.2521\ (0.0446)$ $\hat{\theta} = 3.7911\ (0.3021)$ $\hat{\varphi} = -0.0274\ (0.0852)$	102.58	1.0883	0.1680	0.9658	0.4093
GL	$\hat{\mu} = 0.2666\ (0.4370)$ $\hat{\beta} = 0.7889\ (0.1795)$ $\hat{\theta} = 2.8184\ (5.2237)$ $\hat{\varphi} = 1.3583\ (0.4577)$	102.59	1.1788	0.8639	1.0043	0.4063
GB	$\hat{\mu} = 0.8795\ (0.2014)$ $\hat{\beta} = 0.3509\ (0.3214)$ $\hat{\theta} = 1.2267\ (0.3202)$ $\hat{\varphi} = 2.4355\ (2.2372)$	102.68	1.1842	0.9214	1.1573	0.4014
GW	$\hat{\mu} = 1.7833\ (12.0124)$ $\hat{\beta} = 4.7085\ (9.8791)$ $\hat{\theta} = 0.0442\ (0.1297)$ $\hat{\varphi} = 0.1470\ (0.3557)$	102.87	1.1854	1.0038	1.1598	0.4003
KL	$\hat{\mu} = 0.9253\ (0.0556)$ $\hat{\beta} = 5.7165\ (0.0000)$ $\hat{\theta} = 0.0294\ (0.0496)$ $\hat{\varphi} = 5.2205\ (0.0000)$	102.94	1.1926	1.1030	1.1804	0.3509

Ending table 2

Distribution	Estimates	Log-lik	KS	W	A	*p*-Value
WF	$\hat{\mu} = 20.2781\ (4.6040)$ $\hat{\beta} = -0.2712\ (0.0306)$ $\hat{\theta} = -3.4104\ (0.2777)$ $\hat{\varphi} = 25.6038\ (0.2329)$	102.97	1.1937	1.1187	1.1917	0.3760
GE	$\hat{\alpha} = 24.9400\ (65.1117)$ $\hat{\beta} = 0.9086\ (0.0868)$ $\hat{\theta} = 0.0286\ (0.0747)$	103.00	1.2894	1.1201	1.2739	0.3539
APG	$\hat{\alpha} = 1.4727\ (1.1890)$ $\hat{\beta} = 1.1825\ (0.3364)$ $\hat{\theta} = -0.0955\ (0.1127)$	103.10	1.3886	1.1483	1.2828	0.3053
KF	$\hat{\mu} = 5.5352\ (3.1267)$ $\hat{\beta} = 324.1089\ (327.7648)$ $\hat{\theta} = 0.1450\ (0.0244)$ $\hat{\varphi} = 1.2144\ (4.4818)$	106.67	1.4297	1.4038	2.1526	0.0666

The second data set below is obtained from [26] as used in [27]. It represents the time to failure (103 h) of the turbocharger of one type of engine. The data are

2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0, 8.7, 8.8, 1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 9.0.

The results of the goodness-of-fit are given in table 3.

Table 3

MLEs and the test statistic for second data

Distribution	Estimates	Log-lik	KS	W	A	*p*-Value
MTW	$\hat{\mu} = 0.4864\ (0.0289)$ $\hat{\beta} = 0.7589\ (0.9293)$ $\hat{\theta} = 0.3904\ (0.2390)$ $\hat{\varphi} = 1.6277\ (0.4479)$	62.61	0.0108	0.0037	0.0291	0.9957
MTL	$\hat{\mu} = 0.9698\ (0.7766)$ $\hat{\beta} = 0.7918\ (0.4993)$ $\hat{\theta} = 0.0122\ (0.0055)$ $\hat{\varphi} = 0.2731\ (0.1423)$	64.94	0.0203	0.0138	0.1397	0.9679
MTG	$\hat{\mu} = 1.2063\ (1.3997)$ $\hat{\beta} = 0.3300\ (0.1252)$ $\hat{\theta} = 0.2543\ (0.0001)$ $\hat{\varphi} = 0.1766\ (0.0001)$	69.76	0.0879	0.0283	0.2168	0.9165
MTB	$\hat{\mu} = 5.8081\ (0.4091)$ $\hat{\beta} = 2.8726\ (0.1930)$ $\hat{\theta} = 0.0977\ (0.0093)$ $\hat{\varphi} = 3.5877\ (0.4220)$	73.20	0.0928	0.0877	0.6326	0.8807

Continuation of the table 3

Distribution	Estimates	Log-lik	KS	W	A	p-Value
MTF	$\hat{\mu} = 0.7649\ (0.3309)$ $\hat{\beta} = 0.4293\ (0.0304)$ $\hat{\theta} = 0.4942\ (0.2396)$ $\hat{\varphi} = 1.7826\ (0.2474)$	74.79	0.1025	0.1186	0.8320	0.7944
GW	$\hat{\alpha} = 0.3016\ (0.0000)$ $\hat{\beta} = 1.3109\ (0.0000)$ $\hat{\theta} = 0.1550\ (0.0000)$ $\hat{\varphi} = 2.2756\ (0.8877)$	79.15	0.1796	0.2234	0.9714	0.7613
KG	$\hat{\mu} = 0.7554\ (0.3957)$ $\hat{\beta} = 0.3872\ (0.4886)$ $\hat{\theta} = 0.0097\ (0.0124)$ $\hat{\varphi} = 0.7163\ (0.1619)$	79.81	0.1826	0.2327	1.1237	0.7474
APG	$\hat{\alpha} = 1.7842\ (2.3698)$ $\hat{\beta} = 0.0108\ (0.0102)$ $\hat{\theta} = 0.5841\ (0.1158)$	79.94	0.1871	0.6125	1.2327	0.7213
GE	$\hat{\mu} = 0.0111\ (0.0122)$ $\hat{\beta} = 0.9725\ (1.1608)$ $\hat{\theta} = 0.6476\ (0.7419)$	79.95	0.1961	0.6229	1.2625	0.7045
GL	$\hat{\mu} = 0.0445\ (0.0587)$ $\hat{\beta} = 6.5649\ (7.9237)$ $\hat{\theta} = 0.0394\ (0.0000)$ $\hat{\varphi} = 3.0837\ (3.6737)$	80.22	0.2929	0.6431	1.2667	0.5795
GF	$\hat{\mu} = 0.8038\ (5.0665)$ $\hat{\beta} = 19.0220\ (31.5247)$ $\hat{\theta} = 0.6571\ (0.1867)$ $\hat{\varphi} = 18.2216\ (40.7352)$	80.28	0.3180	0.7364	1.2841	0.5368
WL	$\hat{\mu} = 0.0222\ (0.0490)$ $\hat{\beta} = 1.9855\ (1.2425)$ $\hat{\theta} = 0.0288\ (0.0000)$ $\hat{\varphi} = 11.1181\ (8.6584)$	80.51	0.3297	0.8395	1.3061	0.5306
WF	$\hat{\mu} = 0.0069\ (0.0047)$ $\hat{\beta} = 0.9576\ (0.6158)$ $\hat{\theta} = 3.8599\ (2.5857)$ $\hat{\varphi} = 1.7927\ (0.3768)$	82.59	0.4105	0.8888	1.6505	0.5125
WB	$\hat{\mu} = 0.0040\ (0.0019)$ $\hat{\beta} = 2.8798\ (2.0136)$ $\hat{\theta} = 0.6563\ (0.2780)$ $\hat{\varphi} = 1.6107\ (1.2637)$	83.39	0.4222	0.9009	1.7298	0.5079

Ending table 3

Distribution	Estimates	Log-lik	KS	W	A	*p*-Value
KL	$\hat{\mu} = 4.9797$ (1.0742) $\hat{\beta} = 33.6129$ (25.9336) $\hat{\theta} = 0.0140$ (0.0066) $\hat{\varphi} = 7.3936$ (3.5097)	84.49	0.5089	0.9269	1.8130	0.5007
KF	$\hat{\mu} = 6.2625$ (0.0000) $\hat{\beta} = 581.3908$ (351.4526) $\hat{\theta} = 0.4868$ (0.0675) $\hat{\varphi} = 7.0997$ (0.0000)	86.18	0.6110	1.0731	1.9007	0.4078
KB	$\hat{\mu} = 43.9133$ (0.0000) $\hat{\beta} = 85.7011$ (54.5877) $\hat{\theta} = 1.8023$ (0.0000) $\hat{\varphi} = 0.5092$ (0.0000)	86.67	0.6114	1.1034	2.2040	0.4034
Ga	$\hat{\alpha} = 7.718$ (1.6891) $\hat{\beta} = 1.234$ (0.2790)	87.41	0.7201	0.2051	1.3607	0.3239

The third data consists of the lists of the number of deaths caused as reported by the Centers for Disease Control and Prevention on 6 February 2015 in ten of thousands (*www.cdc.gov*) (see [28, p. 6]). The data are as follows

61.1105, 58.4881, 13.0557, 12.8978, 8.4767, 7.5578, 5.6979, 4.1149.

Table 4 shows the MLEs with standard errors in parentheses and the measures A, W, AIC, KS, negative log-likelihood and *p*-values with the lists of number of deaths data for the MT generated models and some classical statistical distribution models.

Table 4

MLEs and the test statistic for third data

Distribution	Estimates	Log-lik	KS	W	A	*p*-Value
MTF	$\hat{\mu} = 0.6960$ (0.1720) $\hat{\beta} = 0.1200$ (0.0831) $\hat{\theta} = 1.0843$ (0.2830) $\hat{\varphi} = 3.3191$ (0.0428)	28.42	0.1679	0.0332	0.2590	0.9508
MTL	$\hat{\mu} = 0.1288$ (0.0115) $\hat{\beta} = 1.6198$ (0.1285) $\hat{\theta} = 0.0223$ (0.0041) $\hat{\varphi} = 1.4292$ (0.0838)	28.54	0.2013	0.0732	0.4999	0.8429
MTB	$\hat{\mu} = 2.5840$ (1.5241) $\hat{\beta} = 0.8732$ (0.2504) $\hat{\theta} = 0.3290$ (0.0779) $\hat{\varphi} = 5.8648$ (2.7369)	29.56	0.2027	0.0739	0.5033	0.8371
MTW	$\hat{\mu} = 0.0657$ (0.1995) $\hat{\beta} = 0.1176$ (3.0468) $\hat{\theta} = 0.2982$ (8.3294) $\hat{\varphi} = 0.9293$ (0.2828)	29.61	0.2050	0.0834	0.5567	0.8274

Continuation of the table 4

Distribution	Estimates	Log-lik	KS	W	A	p-Value
MTG	$\hat{\mu} = 0.0242\ (0.0953)$ $\hat{\beta} = 0.0431\ (0.4355)$ $\hat{\theta} = 0.4551\ (4.7291)$ $\hat{\varphi} = 0.0129\ (0.0256)$	29.71	0.207 1	0.095 7	0.575 9	0.817 9
KF	$\hat{\alpha} = 4.0904\ (0.0000)$ $\hat{\beta} = 0.0854\ (0.0302)$ $\hat{\theta} = 10.0374\ (0.1181)$ $\hat{\varphi} = 3.5943\ (0.1132)$	30.93	0.253 9	0.106 8	0.585 6	0.775 9
WF	$\hat{\mu} = 0.9088\ (0.9376)$ $\hat{\beta} = 0.2686\ (0.1664)$ $\hat{\theta} = 2.3470\ (1.7385)$ $\hat{\varphi} = 11.7384\ (8.9583)$	31.11	0.269 1	0.137 4	0.618 8	0.582 0
KG	$\hat{\mu} = 16.0638\ (0.5676)$ $\hat{\beta} = 0.1098\ (0.0633)$ $\hat{\theta} = 0.7655\ (0.1092)$ $\hat{\varphi} = -0.0170\ (0.0210)$	31.22	0.276 9	0.146 3	0.623 3	0.580 5
GB	$\hat{\mu} = 11.1513\ (75.0203)$ $\hat{\beta} = 2.5241\ (1.0054)$ $\hat{\theta} = 0.0134\ (0.0000)$ $\hat{\varphi} = 8.3168\ (0.0000)$	31.42	0.285 8	0.188 0	0.576 2	0.575 9
WB	$\hat{\mu} = 5.3909\ (22.7691)$ $\hat{\beta} = 2.3987\ (0.8263)$ $\hat{\theta} = 0.0289\ (0.1102)$ $\hat{\varphi} = 4.7381\ (16.6691)$	31.44	0.294 4	0.188 9	0.581 0	0.552 5
KB	$\hat{\mu} = 44.0261\ (104.6614)$ $\hat{\beta} = 0.5269\ (0.6554)$ $\hat{\theta} = 0.3317\ (3.5855)$ $\hat{\varphi} = 6.3291\ (68.2626)$	31.56	0.298 8	0.195 0	0.590 1	0.449 1
GL	$\hat{\mu} = 5.9856\ (24.8726)$ $\hat{\beta} = 2.3249\ (1.8700)$ $\hat{\theta} = 0.5917\ (1.7966)$ $\hat{\varphi} = 0.1505\ (0.2068)$	31.82	0.300 4	0.199 9	0.659 1	0.437 6
APG	$\hat{\alpha} = 0.1887\ (0.5945)$ $\hat{\beta} = 0.0258\ (0.0315)$ $\hat{\theta} = 0.0076\ (0.0223)$	32.38	0.310 8	0.229 7	0.801 8	0.434 2
WG	$\hat{\mu} = 7.5695\ (17.8991)$ $\hat{\beta} = 1.1883\ (0.6145)$ $\hat{\theta} = 0.0099\ (0.0166)$ $\hat{\varphi} = -0.0187\ (0.0352)$	32.48	0.315 9	0.230 5	0.846 5	0.418 8

Ending table 4

Distribution	Estimates	Log-lik	KS	W	A	*p*-Value
GE	$\hat{\alpha} = 6.2977\ (9.9683)$ $\hat{\beta} = 0.9516\ (0.2607)$ $\hat{\theta} = 0.0059\ (0.0077)$	32.59	0.3173	0.2438	0.8748	0.4162
GW	$\hat{\mu} = 0.1534\ (1.3781)$ $\hat{\beta} = 6.0629\ (14.0956)$ $\hat{\theta} = 0.0118\ (0.0379)$ $\hat{\varphi} = 0.1129\ (0.2816)$	32.64	0.3221	0.2459	0.8859	0.3695
KL	$\hat{\mu} = 37.8065\ (79.7357)$ $\hat{\beta} = 0.2458\ (0.3821)$ $\hat{\theta} = 0.1812\ (0.4026)$ $\hat{\varphi} = 5.7881\ (11.1232)$	32.65	0.3322	0.2510	0.8967	0.3124

In the three datasets illustrated the MT generated models have the highest *p*-values and with the smallest Akaiki information criteria. Thus it is chosen to be the best model for the data under consideration.

Conclusions

A family of distribution models that provides a parameterised mathematical function, simple and efficient, has been the trend. Thus a model with the algorithm for the parameter estimation of data sets of various characteristics and decision making has become an interest to researchers. However, Marshall – Olkin [5] proposed a major transformation for adding a parameter to a classical statistical distribution. Thus a two-parameter method is introduced for generating efficient, improved, and flexible classical models in distribution theory. The Lambert *W* function is implored to obtain the MT generated quantile function. The parameter of the proposed model is acquired using the maximum likelihood. The outcomes of the real-life and simulation study show the relevance and performance of the MT generated model. The results indicated that the MT generated is flexible and tractable in terms of their goodness-of-fit. Thus stochastic processes in quality control and reliability studies can be modelled using the MT generated distribution because of its *U*- and *J*-shaped hazard rate function.

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