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ОБ АССОЦИИРОВАННЫХ РЕШЕНИЯХ СИСТЕМ НЕАВТОНОМНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ПРОСТРАНСТВАХ ЛЕБЕГА

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Исследуются системы неавтономных дифференциальных уравнений в алгебре новых обобщенных функций. Система неавтономных дифференциальных уравнений с обобщенными коэффициентами рассматривается как система уравнений в дифференциалах в алгебре новых обобщенных функций. Решением таких систем является новая обобщенная функция. Показывается, что различные интерпретации решений данных систем могут быть получены при помощи единственного подхода, использующего новые обобщенные функции. В настоящей статье, в отличие от предшествующих работ, описаны ассоциированные решения систем неавтономных дифференциальных уравнений с обобщенными коэффициентами в пространствах Лебега $L_p(T)$.

Ключевые слова: алгебра новых обобщенных функций; дифференциальные уравнения с обобщенными коэффициентами; функции ограниченной вариации.

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ON ASSOCIATED SOLUTIONS OF THE SYSTEM OF NON-AUTONOMOUS DIFFERENTIAL EQUATIONS IN THE LEBESGUE SPACES

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Herein, we investigate systems of non-autonomous differential equations with generalised coefficients using the algebra of new generalised functions. We consider a system of non-autonomous differential equations with generalised coefficients as a system of equations in differentials in the algebra of new generalised functions. The solution of such a system is a new generalised function. It is shown that the different interpretations of the solutions of the given systems can be described by a unique approach of the algebra of new generalised functions. In this paper, for the first time in the literature, we describe associated solutions of the system of non-autonomous differential equations with generalised coefficients in the Lebesgue spaces $L_p(T)$.

Keywords: algebra of new generalised functions; differential equations with generalised coefficients; functions of finite variation.

Introduction

The theory of generalised functions is one of the most powerful tools for investigating the systems of linear differential equations. However, from the outset the distribution theory has an essential disadvantage: it is inapplicable to the solution of non-linear problems. Therefore, various interpretations of the solutions of the systems of non-linear differential equations have been proposed by mathematicians. Unfortunately, different interpretation of the same equation lead, in general, to different solutions (see [1–3]). Usually, systems of differential equations are used to describe the dynamics of real systems or phenomena. In order to choose an adequate interpretation of such systems of equations one has to consider reasons for modelling the dynamics of real systems.

In this paper we will consider the following system of non-linear equations with generalised coefficients on $T \in [0; a] \subset R$:

$$\dot{x}^i(t) = \sum_{j=1}^{q} f^{ij}(t, x(t))\dot{L}^j(t), \; i = \overline{1, z}, \tag{1}$$

with the initial value $x(0) = x_0$, where $t \in T$ and $\dot{L}^j(t)$ are the derivative in the distributional sense, or we can say that $\dot{L}^j(t)$ are the derivative in the Schwartz space $D'(T)$, $x(t) = \left[x^1(t), x^2(t), \dots, x^z(t)\right]$. In general, since $\dot{L}^j(t)$ is the distribution and $f^{ij}(x(t))$ is not a smooth function, the products $f^{ij}(x(t))\dot{L}^j(t)$ are not well defined and the solution of system (1) essentially depends on the interpretation. System (1) describes the model of the rocket flight process. The generalised coefficients of equations correspond to the fact that mass is irregularly changed when the rocket stages are thrown. Also the control problems with impulse actions lead to such systems. Let us recall some approaches to the interpretation of system (1).

The first approach is concerned with considering the system of equations in the framework of the distribution theory. According to this approach, once the product of distributions from some classes is defined, then one tries to find the solution of the system of equations (1) in these classes of distributions. For example, in papers [1; 3] the product of some distributions and discontinuous functions was defined. See also monograph [4] for another definition. Notice that the solutions of system (1) obtained using the products from [1–4] are different.

The second approach is to interpret system (1) as the following system of integral equations:

$$x^i(t) = x_0^i + \sum_{j=1}^{q} \int_0^t f^{ij}(s, x(s))\dot{L}^j(s), \; i = \overline{1, z},$$

where the integrals are understood in the Lebesgue – Stieltjes sense, Perron – Stieltjes sense, etc. [2; 5]. But in this approach the solution of the system of integral equations depends on the interpretation of the integral and the definition of the functions $x^i(t)$ in the discontinuity points of $L^j(t)$.

The third approach is based on the idea of the approximation of the solution of system (1) by the solutions of the system of ordinary differential equations, which are constructed using the smooth approximation of the functions $L^j(t)$. In the monograph [4] it is shown that in this case the limit of the solutions of the smoothed equations exists.

In this paper we will consider system of equations (1) using the algebra of new generalised functions from [6]. Thus we will interpret system of equations (1) as a system of equations in the differentials in the algebra of new generalised functions. Such interpretation says that the solution of system (1) is a new generalised function.

The main purpose of this article is to show that under some conditions this new generalised function associates with some ordinary function, which is natural to call the solution of system (1). Also it will be shown that the solutions of system (1) in the sense of the previous approaches can be obtained from the solution of the system of equations in the differentials in the algebra of new generalised functions. We will describe associated solutions of the approximated systems used in previous similar articles, but we will obtain the main results in the Lebesgue spaces $L_p(T)$.

The algebra of new generalised functions

In this section we recall the definition of the algebra of new generalised functions from [6]. At first, we define an extended real line $\tilde{\mathfrak{R}}$ using a construction typical for non-standard analysis.

Let $\bar{\mathfrak{R}} = \left\{ (x_n)_{n=1}^{\infty} : x_n \in R \text{ for all } n \in N \right\}$ be a set of real sequences. We call two sequences $\{x_n\} \in \bar{\mathfrak{R}}$ and $\{y_n\} \in \bar{\mathfrak{R}}$ equivalent if there is a natural number N such that $x_n = y_n$ for all $n > N$.

The set $\tilde{\mathfrak{R}}$ of equivalence classes is called the extended real line, and any of the classes – a generalised real number.

It is easy to see that $R \subset \tilde{\mathfrak{R}}$ because one may associate with any ordinary number $x \in R$ a class containing a stationary sequence with $x_n = x$. It is evident that $\tilde{\mathfrak{R}}$ is an algebra. The product $\tilde{x}\tilde{y}$ of two generalised real numbers is defined as the class of sequences equivalent to the sequence $\{x_n y_n\}$, where $\{x_n\}$ and $\{y_n\}$ are the arbitrary representatives of the classes $\tilde{x}$ and $\tilde{y}$ respectively.

For any segment $T = [0; a] \subset R$ one can construct an extended segment $\tilde{T}$ in a similar way. Let H denote the subset of $\tilde{\mathfrak{R}}$ of non-negative «infinitely small numbers»:

$$H = \left\{ \tilde{h} \in \tilde{\mathfrak{R}} : \tilde{h} = [\{h_n\}],\ h_n > 0 \text{ for all } n \in N,\ \lim_{n \to \infty} h_n = 0 \right\}. \quad (2)$$

Consider the set of sequences of infinity differentiable functions $\{f_n(x)\}$ on R. We will call two sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ equivalent if for each compact set $K \subset R$ there is a natural number N such that $f_n(x) = g_n(x)$ for all $n > N$ and $x \in K$. The set of classes of equivalent functions is denoted by $\Im(R)$ and its elements are called new generalised functions. Similarly, one can define the space $\Im(T)$ for any interval $T = [0; a]$.

For each distribution f we can construct a sequence $\{f_n\}$ of smooth functions such that f_n converges to f (i. e. one can consider the convolution of f with some δ-sequence). This sequence defines the new generalised function that corresponds to the distribution f. Thus the space of distribution is a subset of the algebra of new generalised functions. However, in this case infinitely many new generalised functions correspond to one distribution (e. g. by taking a different δ-sequence). We will say that the new generalised function $\tilde{f} = [\{f_n\}]$ associates with the ordinary function or distribution f if f_n converges to f in some sense.

Let $\tilde{f} = [\{f_n\}]$ and $\tilde{g} = [\{g_n\}]$ be generalised functions. Then the composition $\tilde{f} \circ \tilde{g}$ is defined by $\tilde{f} \circ \tilde{g} = [\{f_n \circ g_n\}] \in \Im(R)$. Similarly, one can define the value of the new generalised function $\tilde{f}$ at the generalised real point $\tilde{x} = [\{x_n\}] \in \tilde{\mathfrak{R}}$ as $\tilde{f}(\tilde{x}) = [\{f_n(x_n)\}]$.

For each $\tilde{h} = [\{h_n\}] \in H$ and $\tilde{f} = [\{f_n\}] \in \Im(R)$ we define a differential $d_{\tilde{h}}\tilde{f} \in \Im(R)$ by $d_{\tilde{h}}\tilde{f} = [\{f_n(x + h_n) - f_n(x)\}]$. The construction of the differential was proposed by N. Lazakovich (see [6]).

Now we can give an interpretation of system of equations (1) using the introduced algebras. Let $L(t)$, $t \in [0; a] = T$, be a right-continuous function of finite variation. We replace ordinary functions in system (1) by the corresponding new generalised functions and then write differentials in the algebra. So we have

$$d_{\tilde{h}}\tilde{x}^i(\tilde{t}) = \sum_{j=1}^{q} \tilde{f}^{ij}(\tilde{t},\ \tilde{x}(\tilde{t}))\, d_{\tilde{h}}\tilde{L}^j(\tilde{t}),\ i = \overline{1,\ p}, \quad (3)$$

with the initial value $\tilde{x}\big|_{[\tilde{0};\,\tilde{h})} = \tilde{x}_0$, where $\tilde{h} = [\{h_n\}] \in H$, $\tilde{t} = [\{t_n\}] \in T$, $\tilde{x} = [\{x_n\}]$, $\tilde{f} = [\{f_n\}]$, $\tilde{x}_0 = [\{x_{0n}\}]$ and $\tilde{L} = [\{L_n\}]$ are elements of $\Im(R)$. Moreover $\tilde{f}$ and $\tilde{L}$ are associated with f and L respectively. If $\tilde{x}$ is associated with some function x then we say that x is a solution of system (3).

The following theorem from [7] gives necessary and sufficient conditions for the existence and uniqueness of the solutions of system (3).

Theorem 1. *If the following equality holds for some representatives* $\{f_n^{ij}\} \in \tilde{f}^{ij}$, $\{L_n^j\} \in \tilde{L}^j$, $\{x_n^i\} \in \tilde{x}^i$, $\{x_{0n}^i\} \in \tilde{x}_0^i$, *for all sufficiently large* $n \in N$ *and for all* $l = 0, 1, \ldots$:

$$\lim_{t \to 0^+}\left(\frac{d^l}{dt^l}\left[x_{0n}^i(h_n - t) - x_{0n}^i(t)\right] - \sum_{j=1}^{q}\frac{d^l}{dt^l}\left[f_n^{ij}(t,\, x_{0n}(t))\left[L_n^j(h_n + t) - L_n^j(t)\right]\right]\right) = 0,$$

then a solution of system (3) exists and is unique.

The purpose of the present paper is to investigate the case when the solution $\tilde{x}$ of system (3) is associated with some function and to describe all possible associated solutions.

Main results

In this paper we consider specific types of representatives of the new generalised functions (mnemofunctions). We take the following convolutions as representatives of $\tilde{L}$ from system (3):

$$L_n^j(t) = \left(L^j * \rho_n^j\right)(t) = \int_0^{1/(\gamma^j(n))} L^j(t+s)\rho_n^j(s)\,ds, \tag{4}$$

where $\rho_n^j(t) = \gamma^j(n)\rho^j\left(\gamma^j(n)t\right)$; $\rho^j \geq 0$; $\operatorname{supp}\rho^j \subseteq [0;\,1]$; $\int_0^1 \rho^j(s)\,ds = 1$ and $f_n = f * \tilde{\rho}_n$; $\tilde{\rho} \in C^\infty\left(R^{z+1}\right)$; $\int_{[0;\,1]^{z+1}} \tilde{\rho}(x_0,\, x_1,\, \ldots,\, x_z)\,dx_0\,dx_1 \ldots dx_z = 1$; $\tilde{\rho} \geq 0$; $\operatorname{supp}\tilde{\rho} \subseteq [0;\,1]^{z+1}$.

If the function $\gamma^j(n)$ is some monotonic function such as $\lim\limits_{\substack{n\to\infty\\ h_n \to 0}} \gamma^j(n) = \infty$, we will assume that $\lim\limits_{\substack{n\to\infty\\ h_n \to 0}} \gamma^j(n)h_n = \infty$ for $j = \overline{1,\, w}$ and $\lim\limits_{\substack{n\to\infty\\ h_n \to 0}} \gamma^j(n)h_n = 0$ for $j = \overline{w+1,\, q}$.

Using representatives, we can rewrite system (3) in the following form:

$$\begin{cases} x_n^i(t + h_n) - x_n^i(t) = \sum\limits_{j=1}^{q} f_n^{ij}(t,\, x_n(t))\left[L_n^j(t + h_n) - L_n^j(t)\right],\ i = \overline{1,\, z}, \\ x_n(t)\big|_{[0;\,h_n)} = x_{0n}(t). \end{cases} \tag{5}$$

The solution $\tilde{x}$ of system (3) is associated with some function if and only if the sequence $\{x_n\}$ of the solutions of system (5) converges. Therefore, we have to investigate the limiting behaviour of the sequence $\{x_n\}$.

Let t be an arbitrary point of T. There exist $m_t \in N$ and $\tau_t \in [0;\, h_n)$ such that $t = \tau_t + m_t h_n$. Set $t_k = \tau_t + kh_n$ for $k = 0, 1, \ldots, m_t$. Then the solution of system (5) can be written as

$$x_n^i(t) = x_{0n}^i(\tau_t) + \sum_{j=1}^{q}\sum_{k=0}^{m_t - 1} f_n^{ij}(t_k,\, x_n(t_k))\left[L_n^j(t_{k+1}) - L_n^j(t_k)\right],\ i = \overline{1,\, z}. \tag{6}$$

Let $L^j(t)$, $j = \overline{1,\, q}$, $t \in T = [0;\, a]$, be a right-continuous function of finite variation. We will assume that $L^j(t) = L(a)$ if $t > a$ and $L^j(t) = L(0)$ if $t < 0$. Let us denote by $\operatorname*{var}\limits_{u \in T} L(u) = \sum\limits_{j=1}^{q}\operatorname*{var}\limits_{u \in T} L^j(u)$ the total variation of the function $L = \left[L^1,\, L^2,\, \ldots,\, L^q\right]$ on the interval T. Suppose that f is a Lipschitz continuous function with a constant M and for all $x_1, x_2 \in R$ and $t \in T$:

$$\left|f(x_1)-f(x_2)\right|\le M\left|x_1-x_2\right|. \tag{7}$$

In order to describe the limits of the sequence x_n of (6), we consider the following system of integral equations:

$$x^i(t)=x_0^i+\sum_{j=1}^{q}\int_0^t f^{ij}(s, x(s))dL^{jc}(s)+\sum_{\mu_r\le t}S^i(\mu_r, x(\mu_r-), \Delta L(\mu_r)), i=\overline{1, z}, \tag{8}$$

where $L^{jc}(t)$ is the continuous part and $L^{jd}(t)$ is discontinuous part of the function $L^j(t)$; μ_r, $r=1, 2, \ldots$, is discontinuity points of the function $L^j(t)$, $j=\overline{1, q}$; $\Delta L^j(\mu_r)=L^{jd}(\mu_r+)-L^{jd}(\mu_r-)$, $j=\overline{1, q}$, is the size of the jump

$$S^i(\mu, x, u)=\varphi^i(1, \mu, x, u)-\varphi^i(0, \mu, x, u),$$

where $\varphi^i(t, \mu, x, u)$ is the solution of the integral equation

$$\varphi^i(t, \mu, x, u)=x^i+\sum_{j=1}^{w}u^j\int_0^t f^{ij}(\mu, \varphi(s-, \mu, x, u))dH(s-1)+$$

$$+\sum_{j=w+1}^{q}u^j\int_0^t f^{ij}(\mu, \varphi(s, \mu, x, u))ds, i=\overline{1, z}.$$

Here and in what follows all integrals are understood in the Lebesgue – Stieltjes sense.

Theorem 2. *Let f^{ij}, $i=\overline{1, z}$, $j=\overline{1, q}$, be Lipschitz continuous functions satisfying (7) and L^j be right-continuous functions of finite variation. Suppose that $\int_T\left|x_{n0}(\tau_t)-x_0\right|dt\to 0$ in the space $L_p(T)$ as $n\to\infty$, $h_n\to 0$, $\gamma^j(n)\to\infty$ and $\gamma^j(n)h_n\to\infty$ for $j=\overline{1, w}$ and $\gamma^j(n)h_n\to 0$ for $j=\overline{w+1, q}$, then the solution $x_n(t)$ of (5) converges to the solution $x(t)$ of (8) in $L_p(T)$.*

Theorem 3. *Under the condition of theorem 1 let f^{ij}, $i=\overline{1, z}$, $j=\overline{1, q}$, be Lipschitz continuous functions satisfying (7) and L^j be right-continuous functions of finite variation. Suppose that $\int_T\left|x_{n0}(\tau_t)-x_0\right|dt\to 0$ in the space $L_p(T)$ as $n\to\infty$, $h_n\to 0$, $\gamma^j(n)\to\infty$ and $\gamma^j(n)h_n\to\infty$ for $j=\overline{1, w}$ and $\gamma^j(n)h_n\to 0$ for $j=\overline{w+1, q}$, then the associated solution of (3) is the solution of (8) in the space $L_p(T)$.*

Similar results for the system of non-autonomous differential equations in the space $L_1(t)$ have been obtained in [8].

Definition 1. We say that the function $x(t)$ is an *I*-associated (*S*-associated) solution of the system of equations in differentials (3) if it is associated solution of (3) under conditions that $\lim_{\substack{n\to\infty\\ h_n\to 0}}\gamma^j(n)h_n=\infty$ ($\lim_{\substack{n\to\infty\\ h_n\to 0}}\gamma^j(n)h_n=0$) and the representatives of the functions $\tilde{f}$ and $\tilde{L}$ are set by formula (4). In this case we name $d_{\tilde{h}}\tilde{L}^j$ as an *I*-associated (*S*-associated) coefficient.

Let $f: R^z\to R$. We set

$$f_n(t)=(f*\tilde{\rho}_n)(t)=\int_{[0, 1/n]^z}f(t+s)\tilde{\rho}_n(s)ds,$$

where $\tilde{\rho}_n(t)\in C^\infty(R^z)$; $\tilde{\rho}_n(t)\ge 0$; $\operatorname{supp}\tilde{\rho}_n(t)\subset\left[0, \frac{1}{n}\right]^z$; $\int_{[0, 1/n]^z}\tilde{\rho}_n(s)ds=1$, $n\in N$.

Consider the case when $\gamma^j(n)=n$, then $\tilde{\rho}_n(t)\in n^z\tilde{\rho}(nt)$, $\tilde{\rho}_n(t)\in C^\infty(R^z)$, $\operatorname{supp}\tilde{\rho}\subset[0, 1]^z$, $\int_{[0, 1]^z}\tilde{\rho}_n(s)ds=1$, $n\in N$.

Remark 1. Let $\gamma^j(n)=n$, then we can define the set H from (2) using the following subsets:

$$I=\left\{\tilde{h}\in H: \frac{1}{n}=o(h_n), n\to\infty, h_n\to 0 \text{ for all } h_n\in\tilde{h}\right\},$$

$$S=\left\{\tilde{h}\in H: h_n=o\left(\frac{1}{n}\right), h_n\to 0, n\to\infty \text{ for all } h_n\in\tilde{h}\right\}.$$

We name the generalised differential $d_{\tilde{h}}$ as *I*-generalised (*S*-generalised) differential and denote $d_{\tilde{h}}^{I}\ \left(d_{\tilde{h}}^{S}\right)$, if $\tilde{h} \in I\ \left(\tilde{h} \in S\right)$. Note, that the *I*-generalised (*S*-generalised) differential makes sense only for the new generalised function $\tilde{L}^j$ with representatives (4), where $\gamma^j(n) = n$.

According to equation (3), we will consider the systems of equations with *I*-generalised and *S*-generalised differentials:

$$\begin{cases} d_{\tilde{h}}^{I}\tilde{x}^i(\tilde{t}) = \sum\limits_{j=1}^{q} \tilde{f}^{ij}\left(\tilde{t}, \tilde{x}(\tilde{t})\right) d_{\tilde{h}}^{I}\tilde{L}^j(\tilde{t}), \\ \tilde{x}\big|_{\left[0, \tilde{h}\right)} = \tilde{x}^0, \end{cases} \tag{9}$$

$$\begin{cases} d_{\tilde{h}}^{S}\tilde{x}^i(\tilde{t}) = \sum\limits_{j=1}^{q} \tilde{f}^{ij}\left(\tilde{t}, \tilde{x}(\tilde{t})\right) d_{\tilde{h}}^{S}\tilde{L}^j(\tilde{t}), \\ \tilde{x}\big|_{\left[0, \tilde{h}\right)} = \tilde{x}^0. \end{cases} \tag{10}$$

Remark 2. In case $\gamma^j(n) = n$ definition 1 will take the following form: we will say that the function $x(t)$ is the *I*-associated or *S*-associated solution of a system of equations in differentials (3) if it is associated solution of (9) or (10) respectively.

Let $\gamma^j(n) = n$. In order to describe the limits of the sequence x_n we consider the following system of integral equations:

$$x^i(t) = x_0^i + \sum_{j=1}^{q} \int_0^t f^{ij}\left(s, x(s)\right) dL^j(s),\ i = \overline{1, z}. \tag{11}$$

Theorem 4. *Let $f^{ij},\ i = \overline{1, z},\ j = \overline{1, q}$, be Lipschitz continuous functions satisfying (7) and L^j be continuous functions of finite variation. Suppose that $\int_T \left|x_{n0}(\tau_t) - x_0\right| dt \to 0$ in the space $L_p(T)$, then the solution $x_n(t)$ of (5) converges to the solution $x(t)$ of (11) in the space $L_p(T)$ as $n \to \infty$, $h_n \to 0$.*

Theorem 5. *Under the condition of theorem 1 let $f^{ij},\ i = \overline{1, z},\ j = \overline{1, q}$, be Lipschitz continuous functions satisfying (7) and L^j be continuous functions of finite variation. Suppose that $\int_T \left|x_{n0}(\tau_t) - x_0\right| dt \to 0$ in the space $L_p(T)$, then the associated solution of (3) is the solution of (11) in the space $L_p(T)$ as $n \to \infty$, $h_n \to 0$.*

The proof of a similar theorem in another space and in an autonomous case can be seen in [9].

Let L^j be right-continuous functions of finite variation, $\gamma^j(n) = n$ and $\frac{1}{n} = o(h_n)$ as $n \to \infty$, $h_n \to 0$. In order to describe the limits of the sequence x_n, we consider the following system of integral equations:

$$x^i(t) = x_0^i + \sum_{j=1}^{q} \int_0^{t+} f^{ij}\left(s, x(s-)\right) dL^j(s),\ i = \overline{1, z}. \tag{12}$$

Theorem 6. *Let $f^{ij},\ i = \overline{1, z},\ j = \overline{1, q}$, be Lipschitz continuous functions satisfying (7) and L^j be right-continuous functions of finite variation. Suppose that $\int_T \left|x_{n0}(\tau_t) - x_0\right| dt \to 0$ in the space $L_p(T)$, then the solution $x_n(t)$ of (5) converges to the solution $x(t)$ in the space $L_p(T)$ of (12) as $n \to \infty$, $h_n \to 0$ and $\frac{1}{n} = o(h_n)$.*

Theorem 7. *Under the condition of theorem 1 let $f^{ij},\ i = \overline{1, z},\ j = \overline{1, q}$, be Lipschitz continuous functions satisfying (7) and L^j be right-continuous functions of finite variation. Suppose that $\int_T \left|x_{n0}(\tau_t) - x_0\right| dt \to 0$ in the space $L_p(T)$ as $n \to \infty$, $h_n \to 0$, then the I-associated solution of (3) is the solution of (12) in the space $L_p(T)$ as $n \to \infty$, $h_n \to 0$.*

Similar results for the system of autonomous differential equations in other spaces have been obtained in [10; 11].

Let L^j be right-continuous functions of finite variation, $\gamma^j(n) = n$ and $h_n = o\left(\frac{1}{n}\right)$ as $n \to \infty$, $h_n \to 0$. In order to describe the limits of the sequence x_n, we consider the following system of integral equations:

$$x^i(t) = x_0^i + \sum_{j=1}^{q} \int_0^t f^{ij}(s, x(s)) dL^{jc}(s) + \sum_{\mu_r \le t} S^i(\mu_r, x(\mu_r -), \Delta L(\mu_r)), \; i = \overline{1, z}, \qquad (13)$$

where $S^i(\mu, x, u) = \varphi^i(1, \mu, x, u) - \varphi^i(0, \mu, x, u)$, and $\varphi^i(t, \mu, x, u)$ is the solution of the integral equation

$$\varphi^i(t, \mu, x, u) = x^i + \sum_{j=1}^{q} u^j \int_0^t f^{ij}(\mu, \varphi(s, \mu, x, u)) ds, \; i = \overline{1, z}.$$

Theorem 8. *Let f^{ij}, $i = \overline{1, z}$, $j = \overline{1, q}$, be Lipschitz continuous functions satisfying (7) and L^j be right-continuous functions of finite variation. Suppose that $\int_T |x_{n0}(\tau_t) - x_0| dt \to 0$ in the space $L_p(T)$, then the solution $x_n(t)$ of (5) converges to the solution $x(t)$ of (13) in the space $L_p(T)$ as $n \to \infty$, $h_n \to 0$ and $h_n = o\left(\frac{1}{n}\right)$.*

Theorem 9. *Under the condition of theorem 1 let f^{ij}, $i = \overline{1, z}$, $j = \overline{1, q}$, be Lipschitz continuous functions satisfying (7) and L^j be right-continuous functions of finite variation. Suppose that $\int_T |x_{n0}(\tau_t) - x_0| dt \to 0$ in the space $L_p(T)$ as $n \to \infty$, $h_n \to 0$, then the S-associated solution of (3) is the solution of (13) in the space $L_p(T)$ as $n \to \infty$, $h_n \to 0$.*

Similar results for such system of autonomous differential equations in another space have been obtained in [12].

Notice that the solution $x_n(t)$ of system (5) converges either to the solution of system (1) in the sense of paper [2; 5] if $\frac{1}{n} = o(h_n)$ or to the approximative solution of (1) in the sense of monograph [4] if $h_n = o\left(\frac{1}{n}\right)$.

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