ACTIVE CONTROL OF AN IMPROVED BOUSSINESQ SYSTEM

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Introduction. In this study, optimal control of excessive water waves in a canal system, modeled by a nonlinear improved Boussinesq equation, is considered. Suppressing of the waves in the canal system is successfully obtained by means of optimally determining of canal depth control function via Pontryagin's maximum principle, which transforms to optimal control problem to solving an initial-boundary-terminal value problem. In order to show effectiveness and robustness of the control actuation, a numerical example is given in the table form.

1. Mathematical Formulation of the Control Problem. Consider two lakes/two separate seas in a region. Due to several reasons, designers need to open a canal between two lakes/two separate seas. As estimated, this canal will have the some effects, simply, such as economically due to digging cost and physically due to excessive waves. Canal system, in Fig. 1, is fully filled with water and subject to wind as an external excitation. In order to prevent excessive water waves and unnecessary cost, we need to optimally determine the depth of the canal. The main aim of the present control problem is to damp out the excessive water waves via optimal control of the canal depth.

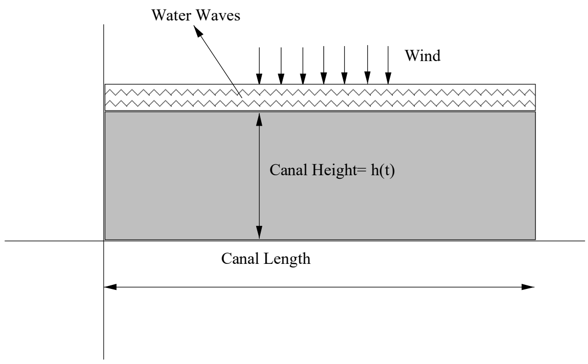

Figure 1 — A Canal System

Consider the system of equation in general form as follows [1, 2]

$$u_{tt} + \alpha(u_{ttxx} - u_{ttxxxx}) + \beta u_{txx} + \gamma u_{xxxx} + u_{xx} + [\mathcal{N}(u)]_{xx} = f(t,x) + \mathcal{H}(t,x), \qquad (1)$$

where state variable u is the elevation of the free surface of water at $(t,x) \in \Omega = \{t \in (0,tf), x \in (0,\ell)\}$, t is time variable, t_f is predetermined terminal time, x is space variable, ℓ is the length of the canal, α is a constant in $\mathcal{R}^+$, $\beta > 0$ is an internal damping constant, $\gamma > 0$ is a constant depending on the depth of water, $\mathcal{N}$ is a nonlinear function of u, f is an external excitation function, $\mathcal{H}(t,x) = \hbar(t)\theta(x)$ in which $\hbar(t)$ is the optimal canal depth control function and $\theta(x)$ is a function, affecting on canal depth function. Equation (1) is subject to the following boundary conditions $u(t,x) = 0$, $u_{xx}(t,x) = 0$ at $x = 0,\ell$, and initial conditions $u(t,x) = u_0(x)$, $u_t(t,x) = u_1(x)$ at $t = 0$. Then, the performance index functional of the system to be minimized on the control duration is given by as follows;

$$\mathcal{J}(\hbar(t)) = \int_0^\ell [\vartheta_1 u^2(t_f,x) + \vartheta_2 u_t^2(t_f,x)]dx + \vartheta_3 \int_0^{t_f} \hbar^2(t)dt \qquad (2)$$

in which $\vartheta_1, \vartheta_2 \geq 0$, $\vartheta_1 + \vartheta_2 \neq 0$ and $\vartheta_3 > 0$ are weighting constants. First integral on the left-hand side in equation (2) represents the modified dynamics response of the water waves system. First and second terms in this integral are quadratic functional of the displacement and velocity of the water wave, respectively. Second integral on the left-hand side in equation (2) is the measure of the total canal depth on the $(0,t_f)$.

2. Numerical Results. By using the Pontryagin's maximum principle, optimal canal depth function is obtained as follows;

$$\mathcal{H}(t,x) = \hbar(t)\theta(x), \quad \hbar(t) = \frac{-\Phi(t)}{2\vartheta_3}, \quad \Phi(t) = \int_0^\ell w(t,x)\theta(x)dx, \qquad (3)$$

in which w is the solution of the following system;

$$w_{tt} + \alpha(w_{ttxx} - w_{ttxxxx}) - \beta w_{txx} + \gamma w_{xxxx} + w_{xx} = 0,$$

$$w(t,x) = 0, \quad w_{xx}(t,x) = 0 \text{ at } x = 0,\ell,$$

$$-2\vartheta_1 u(t_f,x) = w_t(t_f,x) + \alpha[w_{txx}(t_f,x) - w_{txxxx}(t_f,x)] - \beta w_{xx}(t_f,x),$$

$$2\vartheta_2 u_t(t_f,x) = w(t_f,x) + \alpha[w_{xx}(t_f,x) - w_{xxxx}(t_f,x)].$$

Before giving the numerical example, consider the optimal canal depth control function given by equation (3), in which, it is clear that as the value of ϑ_3 is decreasing, the value of the canal depth is increasing. As a conclusion of this situation, dynamic response of the excessive water waves given by first integral on the left side of equation (2) is minimized by using minimum canal depth. Also, in numerical computations, $\mathcal{N}(u)$ is considered as 0 due to difficulties on solving system of equations (1) with the respective initial and boundary conditions. Weighted coefficients are taken into account as $\vartheta_{1,2} = 1$ and $\vartheta_3 = 10^4$ and $\vartheta_3 = 10^{-4}$ for uncontrolled and controlled case, respectively. Numerical values are computed on the middle of the canal, $x = 0.5$. The introduced control algorithm is valid for all coefficients in the system but due the stability of the solutions of equations (1), following coefficients are imposed. In the numerical example, followings are taken into account: $\alpha = 0.01$, $\beta = 0.001$, $\gamma = 0.0001$, $\ell = 1$, $t_f = 3$, $\theta(x) = 1$, $f(t,x) = te^x$, $u_0(x) = \cos(\pi x)$, $u_1(x) = \sqrt{2}\sin(\pi x)$.

Let us give the dynamic response of the wave in the canal system and used canal depth accumulates over $(0, t_f)$, respectively, as follows;

$$\mathcal{J}(u) = \int_0^1 [u^2(t_f, x) + u_t^2(t_f, x)]dx, \quad \mathcal{J}(\hbar) = \int_0^{t_f} \hbar^2(t)dt. \tag{4}$$

The dynamic response of the wave in the canal system is given by table form and it seemed from Table 1 that as weighted coefficient ϑ_3 in canal depth control function decreases, dynamic response of the wave decreases due to an increasing in the value of canal depth control function. These results indicate that introduced control actuation is very effective and applicable to other waves control system including nonlinear terms.

Table 1 — The values of $\mathcal{J}(u)$ and $\mathcal{J}(\hbar)$ for different values of ϑ_3

ϑ_3	$\mathcal{J}(u)$	$\mathcal{J}(\hbar)$
10^4	1.8 e-3	2.7 e-9
10^0	5.0 e-4	2.2 e-3
10^{-4}	3.9 e-10	6.0 e-2

References

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