REGULARIZATION OF A COPOSITIVE OPTIMIZATION PROBLEM

O.I. Kostyukova[1], T.V. Tchemisova[2]

[1] Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus
kostyukova@im.bas-net.by

[2] Department of Mathematics, University of Aveiro, Portugal
tatiana@ua.pt

Introduction. Copositive Programming (CoP) deals with optimization over the convex cone of *copositive matrices*. Copositive models arise in many important applications, including $\mathcal{NP}$ -hard problems [1]. In convex and conic optimization, optimality conditions and duality results are usually formulated under certain regularity conditions which are often not satisfied. Thus, the idea of a *regularization* appears quite naturally and is aimed at obtaining an equivalent and more convenient reformulation of the problem with some required properties, one of which is the generalized Slater condition.

In this paper, we present and justify a regularization algorithm for linear CoP. It is based on the concept of immobile indices which play an important role in the feasible sets' characterization [2].

1. Problem's statement Consider a linear CoP problem

$$\min_{x} \ c'x \quad \text{s.t. } x \in X := \{x \in \mathbb{R}^n : \mathcal{A}(x) \in \mathcal{COP}^p\}, \qquad (1)$$

where $\mathcal{COP}^p := \{D \in \mathbb{R}^{p\times p} : t'Dt \geq 0 \ \forall t \in T\}$, $T := \{t \in \mathbb{R}^p_+ : \sum_{k=1}^{p} t_k = 1\}$, is the cone of copositive matrices, $\mathcal{A}(x) = \sum_{j=1}^{n} A_j x_j + A_0$, $A_0 \in \mathcal{COP}^p$, matrices $A_j, j = 0, ..., n$, and vector c are given.

The constraints of problem (1) satisfy **the Slater condition** if
$\exists \bar{x} \in \mathbb{R}^n : \ \mathcal{A}(\bar{x}) \in \operatorname{int} \mathcal{COP}^p = \{D \in \mathbb{R}^{p\times p} : t'\mathcal{A}(\bar{x})t > 0 \ \forall t \in T\}$.

Lemma 1. *Given the linear copositive problem (1),*
(i) *the Slater condition is equivalent to the emptiness of set of normalized immobile indices* $T_{im} := \{t \in T : t'\mathcal{A}(x)t = 0 \ \ \forall x \in X\}$*, and*
(ii) *the normalized immobile index set* T_{im} *is empty or can be represented as a union of a finite number of convex closed bounded polyhedra.*

Suppose that $T_{im} \neq \emptyset$ and $V = \{\tau(i), i \in I\} \subset T_{im}, \ 0 < |I| < \infty$. Set $\sigma(V) := \min\{\tau_k(i), k \in P_+(\tau(i)), i \in I\} > 0$,

$\Omega(V) := \{t \in T : \rho(t, \operatorname{conv} V) \geq \sigma(V)\}$, where $\rho(t, \mathcal{B}) = \min_{\tau\in\mathcal{B}} ||t - \tau||_1$,

$P_+(t) := \{k \in \{1, 2, ..., p\} : \ t_k > 0\}$ for $t = (t_k, \ k = 1, ..., p)' \in \mathbb{R}^p_+$.

2. Regularization Algorithm for CoP

Initialization: $m := 0,\ I_0 := \emptyset,\ W_0 := \emptyset,\ \Omega(W_0) := T.$

Iteration $\#\, m,\ m \geq 0.$ Solve a **regular** semi-infinite problem

$$\min_{(x,\mu)\in\mathbb{R}^{n+1}} \mu, \text{ s.t. } t'\mathcal{A}(x)t+\mu \geq 0,\ t\in\Omega(W_m),\ \mathcal{A}(x)\tau(i)\geq 0,\ i\in I_m. \quad (2)$$

If problem (2) admits a feasible solution $(\bar{x},\bar{\mu})$ with $\bar{\mu}<0$, then go to the *Final step* with $m_* := m$. Otherwise vector $(x=\mathbf{0},\mu=0)$ is an optimal solution of (2) and there exist indices, numbers, and vectors $\tau(i)\in\Omega(W_m),\ \gamma(i)>0, i\in\Delta I_m,\ 1\leq|\Delta I_m|\leq n+1;\ \lambda^m(i)\in\mathbb{R}^p_+,\ i\in I_m,$ such that

$$\sum_{i\in\Delta I_m}\gamma(i)(\tau(i))'A_j\tau(i)+\sum_{i\in I_m}(\lambda^m(i))'A_j\tau(i)=0,\ j=0,1,...,n.$$

Set $I_{m+1} := I_m\cup\Delta I_m,\ W_{m+1} := \{\tau(i), i\in I_{m+1}\}$ and go to iteration $\#(m+1)$.

Final step. For some $m=m_*\geq 0$, the problem (2) has a feasible solution $(\bar{x},\bar{\mu})$ with $\bar{\mu}<0$. We prove that the CoP problem

$$\min_{x\in\mathbb{R}^n} c'x, \quad \text{s.t. } t'\mathcal{A}(x)t\geq 0,\ t\in\Omega(W_{m_*}),\ \mathcal{A}(x)\tau(i)\geq 0, i\in I_{m_*},$$

is equivalent to the original CoP problem (1) being its **regularization** since (a) $\mathcal{X}(W_{m_*}) := \{x\in\mathbb{R}^n : \mathcal{A}(x)\tau(i)\geq 0, i\in I_{m_*};\ t'\mathcal{A}(x)t\geq 0,$ $t\in\Omega(W_{m_*})\}=X$, (b) it has a finite number of linear constraints, and (c) by construction, $t'\mathcal{A}(\bar{x})t>0, t\in\Omega(W_{m_*})$ for some $\bar{x}\in X$.

3. Conclusions. The proposed algorithm is based on the concept of immobile indices which permits to reduce a linear CoP problem to an equivalent regular SIP problem which can be solved numerically. The suggested approach permits to develop strong duality theory based on an explicit representation of the "regularized" feasible cone and the corresponding dual such as, e.g. the *Extended Lagrange Dual Problem* suggested for semidefinite problems in [3].

References

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