ALMOST POSITIVITY AND UNIFORM BOUNDS FOR BIHARMONIC RESOLVENT KERNELS

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1. INTRODUCTION

It is known that in general, the positivity preserving property does not hold for higher (> 2) order elliptic equation problem, i.e. that the positivity of the datum does not necessarily lead to the positivity of the solution. Under certain conditions on the datum, however, at least some kind of positivity behaviour of the solution can be observed.

Let us consider a clamped plate equation problem:

$$\begin{cases} \Delta^2 u = f \ \text{ in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 \ \ on\ \partial\Omega \end{cases}, \tag{1}$$

where $\frac{\partial u}{\partial \nu}$ stands for the outward normal derivative.

The question to be investigated is whether the positivity of the datum f implies the positivity of the solution u, or, in other words, does the clamped plate bend upwards everywhere when it is pushed upwards?

This problem has been under investigation for over a century. In 1905, Boggio [2] could show that the positivity preserving property holds in $\Omega = B$ – unit ball, while in 1909 Hadamard [8] knew that the property fails to be true in annuli with small inner radius. Boggio and Hadamard suggested, however, that $f \geq 0$ implies $u \geq 0$ in convex 2-dimensional domains. After1949 this conjecture was disproved by several authors, as it turned out that one has in general change of sign even in mildly eccentric ellipses. [3, 5, 9] We can obtain certain results, however, by considering the Green function for this equation.

The Green function *G(x, y)* enables us to formally write the solution as:

$$u(x) = \int_\Omega G(x,y) f(y)\, dy \qquad (2)$$

Now it is evident that we are to investigate the positivity of the Green function, i.e. if *G(x, y)* ≥ 0 in Ω, then the positivity preserving property for the clamped plate equation in (1) holds.

In some cases, the Green function can be explicitly calculated and investigated for sign preserving property. For exampled, if Ω is a unit ball in $\mathbb{R}^n$, then the Green function is positive [6, Lemma 2.27].

In most cases, though, we can only try to obtain the so-called "almost positivity" property, i.e. there is change of sign, but the negative part can be estimated by "smaller functions" and is in absolute value significantly smaller than the positive part. This means that when a clamped plate is being pushed upwards, it can only bend downwards a very "small extent".

2. THE PROBLEMS CONSIDERED

2.1. The biharmonic heat problem

First we consider the biharmonic heat problem:

$$\begin{cases} \Delta^2 u + u_t = 0 \ \ in \ \ \Omega \times \mathbb{R}_+ \\ u(0,x) = \phi(x) \ for \ \ x \in \Omega \end{cases}, \qquad (3)$$

where Ω is a bounded smooth domain.

No positivity preserving property of the solution with respect to the initial datum holds true, nor it does in case of an unbounded domain (Ω = $\mathbb{R}^n$) [4]. Nevertheless, we can obtain certain positivity results for the fundamental solution of this problem. The fundamental solution is called the biharmonic heat kernel and is given by:

$$b(x,t) = \alpha_n \frac{f_n(\eta)}{t^{n/4}}, \qquad (4)$$

where $\eta = \frac{|x|}{t^{n/4}}$ and $\alpha_n^{-1} = \varpi_n \int_0^\infty r^{n-1} f_n(r) dr$, α_n being the normalization constant, and ϖ_n being the surface measure of the n-dimensional unit ball. Meanwhile

$$f_n(\eta) = \eta^{n-1} \int_0^\infty \exp(-s^4)\, (\eta s)^{n/2} J_{(n-2)/2}(\eta s) ds, \qquad (5)$$

where J_n is the n-th Bessel function.

There exist constants *K*, μ_n such that the f_n functions have also exponential decay at infinity according to the following estimation [4]:

$$|f_n(\eta)| \leq K \exp\left(\mu\, \eta^{4/3}\right) \tag{6}$$

Another important property is that for all $n \geq 1$

$$f_n'(\eta) = -\eta f_{n+2}(\eta) \tag{7}$$

This can be obtained by direct computation:

$$\frac{d}{d\eta} f_n(\eta) = \frac{d}{d\eta}\left(\int_0^\infty \exp(-s^4)\, \eta^{1-n/2} s^{n/2} J_{(n-2)/2}(\eta s) ds\right) =$$

$$= \frac{d}{d\eta}\left(\int_0^\infty \exp(-s^4)\, (\eta s)^{(2-n)/2}\, s^{n-1} J_{(n-2)/2}(\eta s) ds\right) =$$

$$= \int_0^\infty \exp(-s^4)\, s^{n-1} \frac{d}{d\eta}\left((\eta s)^{(2-n)/2} J_{(n-2)/2}(\eta s)\right) ds =$$

$$= [\text{substituting } \tau := \eta s] =$$

$$= \int_0^\infty \exp(-s^4)\, s^{n-1} s \frac{d}{d\tau}\left(\tau^{(2-n)/2} J_{(n-2)/2}(\tau)\right) ds =$$

$$= [\textit{remembering the properties of Bessel functions}] =$$

$$= \int_0^\infty \exp(-s^4)\, s^n \tau^{(2-n)/2} J_{n/2}(\tau) ds =$$

$$= \int_0^\infty \exp(-s^4)\, s^n (\eta s)^{(2-n)/2} J_{n/2}(\eta s) ds = -\eta f_{n+2}(\eta)$$

Now we put

$$C_{n,\beta} = \int_0^\infty \eta^{n-1-\beta} f_n(\eta) d\eta \tag{8}$$

and obtain the positivity result that we shall prove in the next section:

Theorem 1

For all integer $n \geq 1$ and all $\beta \in [0, n)$ we have $C_{n,\beta} > 0$.

2.2. The biharmonic resolvent problem

Let us consider the following resolvent problem:

$$\begin{cases} \lambda u + \Delta^2 u = 0 \ in\ \mathbb{R}^n \\ u(x) = \frac{\partial u}{\partial \nu} = 0\ for\ |x| \to \infty \end{cases}, \tag{9}$$

where $\lambda > 0$ is the resolvent parameter, and

$$\Gamma_\lambda(x) = \int_0^\infty e^{-\lambda t} b(x,t) dt \qquad (10)$$

is the Green function in $\mathbb{R}^n$ for this problem.

We can see that

$$\int_{\mathbb{R}^n} \Gamma_\lambda(x) dx = \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^n} b(x,t) dx\, dt = \frac{1}{\lambda} > 0 \qquad (11)$$

One must pay attention, however, to the justification of applying the Fubini-Tonelli theorem in this case, as the decay at infinity is quite slow ($J_{(n-2)/2}(\sigma)$ decays at infinity like $\sigma^{1/2}$).

This result means that, for any $\lambda > 0$ positivity dominates for $\Gamma_\lambda(x)$ in the L^1-sense. We can also obtain an important estimation of the Greenfunction:

Theorem 2

There exist constants $C_n > 0, \mu_n > 0$ such that we have uniformly in $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$ that

$$|\Gamma_\lambda(x)| \le \begin{cases} C_n |x|^{4-n} \exp(-\mu_n \sqrt[4]{\lambda}|x|) \; if\; n > 4 \\ C_n(1 + |\log(\sqrt[4]{\lambda}|x|)|) \exp(-\mu_n \sqrt[4]{\lambda}|x|) \; if\; n = 4 \\ C_n \lambda^{\frac{n}{4}-1} \exp(-\mu_n \sqrt[4]{\lambda}|x|) \; if\; n < 4 \end{cases} \qquad (12)$$

This theorem is proved in one of the sections that follow.

Then we shall also prove a more general result of the weighted positivity for Γ_λ:

Theorem 3

For $\beta \in [0, \min\{n, 4\})$

$$\int_{\mathbb{R}^n} |x|^{-\beta}\, \Gamma_\lambda(x) dx > 0. \qquad (13)$$

3. POSITIVITY IN DIFFERENT SENSES

The reason why we consider such problems is that we do not always get positivity results in L^∞ sense, i.e. we can not always estimate f_n or the Green-function in L^∞ sense like in Theorem 2. From this position, we can imagine that the positivity result we obtain in Theorem 1 is "weighted positivity"- the f_n functions are positive in a weighted sense. Meanwhile, Theorem 3 gives weighted positivity result for Γ_λ.

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