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ВКЛАД ЙОНАСА КУБИЛЮСА В МЕТРИЧЕСКУЮ ТЕОРИЮ ДИОФАНТОВЫХ ПРИБЛИЖЕНИЙ ЗАВИСИМЫХ ПЕРЕМЕННЫХ

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Посвящается 100-летию со дня рождения академика Йонаса Кубилюса, который является основоположником метрической теории диофантовых приближений. Проводится обзор наиболее важных результатов, полученных в метрической теории диофантовых приближений. Отмечается, что за последние 70 лет в области диофантовых приближений сделано много выдающихся достижений. Упоминаются работы лауреатов Филдсовской премии Алана Бейкера и Григория Маргулиса, а также ученика Й. Кубилюса, академика АН БССР Владимира Спринджука, который в 1964 г. решил известную проблему Малера и стал основателем белорусской школы теории чисел.

Ключевые слова: Й. Кубилюс; диофантовы приближения; проблема Малера; метрическая теория чисел; трансцендентные и алгебраические числа.

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CONTRIBUTION OF JONAS KUBILIUS TO THE METRIC THEORY OF DIOPHANTINE APPROXIMATION OF DEPENDENT VARIABLES

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The article is devoted to the latest results in metric theory of Diophantine approximation. One of the first major result in area of number theory was a theorem by academician Jonas Kubilius. This paper is dedicated to centenary of his birth. Over the last 70 years, the area of Diophantine approximation yielded a number of significant results by great mathematicians, including Fields prize winners Alan Baker and Grigori Margulis. In 1964 academician of the Academy of Sciences of BSSR Vladimir Sprindžuk, who was a pupil of academician J. Kubilius, solved the well-known Mahler's conjecture on the measure of the set of *S*-numbers under Mahler's classification, thus becoming the founder of the Belarusian academic school of number theory in 1962.

Keywords: J. Kubilius; Diophantine approximation; Mahler's conjecture; metric number theory; transcendence and algebraic numbers.

Dedicated to the centenary of academician Jonas Kubilius's birth

Introduction

Academician Jonas Kubilius devoted his life to research in theory of probability and number theory. He was one of the founders of metric theory of Diophantine approximation, obtaining one of the earliest major results in this field [1; 2] and influencing the work of his pupil Vladimir Sprindžuk, who in 1964 proved the famous Mahler's conjecture [3–5].

First results in metric theory of Diophantine approximation were obtained by Émile Borel in the beginning of the 20th century [6], and they were later significantly improved in a seminal work of Alexander Khintchine [7].

Let $\psi(x)$ be a monotonic decreasing function of $x > 0$, and let μB denote the Lebesgue measure of a measurable set $B \subset \mathbb{R}$. Let $\mathcal{L}_1(\psi)$ denote the set of real numbers in the interval $I \subset \mathbb{R}$ such that the inequality

$$\left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q} \tag{1}$$

has infinitely many solutions in integers $p \in \mathbb{Z}$ and positive integers $q \in \mathbb{N}$.

Theorem 1 (Khintchine's theorem). *The Lebesque measure of the set* $\mathcal{L}_1(\psi)$ *satisfies*

$$\mu\mathcal{L}_1(\psi) = 0 \text{ if } \sum_{q=1}^{\infty} \psi(q) < \infty, \tag{2}$$

$$\mu\mathcal{L}_1(\psi) = \mu I \text{ if } \sum_{q=1}^{\infty} \psi(q) = \infty. \tag{3}$$

Note that in the case of convergence (2) the theorem also holds without the monotonicity requirement on the function $\psi(x)$. Khintchine's theorem was later generalised by A. V. Groshev for system of linear forms [8].

One important generalisation of the above setting considered by A. Khintchine concerns small values of integral polynomials, see articles [3; 5] and monographs [4; 9–11]. In particular, in [12] A. Khintchine proved the following theorem.

Theorem 2. *For all* $\varepsilon > 0$ *and an arbitrary interval I, the inequality*

$$|P(x)| < \varepsilon H^{-n} \tag{4}$$

has infinitely many solutions in polynomials

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,\ P(x) \in \mathbb{Z}[x],$$

of degree n and height $H = H(P) = \max_{0 \le j \le n} |a_j|$ *for almost all points* $x \in I$.

Clearly, we can find $x \in \mathbb{R}$ such that for $\varepsilon = \varepsilon(x)$ the inequality opposite to (4) is satisfied. Taking, for example, $x_1 = \sqrt[n+1]{2}$, we have

$$|P(x_1)| > c(n) H^{-n}$$

for an appropriate positive constant $c(n)$.

The inequality (4) can be viewed as the first result in metric Diophantine approximation of dependent variables since it relates to approximation of zero by values of an integer polynomial of an arbitrary degree.

In what follows, $c_1 = c_1(n), c_2, \ldots$ will denote quantities that depend on n and do not depend on H. Quantities A and B satisfy the inequality $A \ll B$, where $\ll$ is the Vinogradov symbol, if there exists a constant c such that $A < cB$.

In 1932 Kurt Mahler [13] proposed his classification of real and complex numbers based on the behaviour of best polynomial approximations of zero $P(\xi)$ and $P(z)$ as the height H tends to infinity at respectively real and complex points $\xi \in \mathbb{R}$, $z \in \mathbb{C}$. A detailed description of Mahler's classification can be found in Sprindžuk's [3; 4], Schmidt's [14] and Bugeaud's [15] monographs. K. Mahler divided the set of real numbers into four classes, namely the *A*-, *S*-, *T*- and *U*-numbers. He formulated the famous Mahler's conjecture, which became the main problem in metric theory of Diophantine approximations for several decades.

Let $M_n(w)$ be the set of real numbers $x \in I$ such that the inequality

$$|P(x)| < H^{-w},\ w > 0,$$

has infinitely many solutions in polynomials $P(x) \in \mathbb{Z}[x]$, $\deg P = n$.

Conjecture (Mahler's conjecture). For $w > n$ we have $\mu M_n(w) = 0$.

For complex numbers, conjecture 1 can be formulated as follows: define the set $\mathcal{K}_n(v)$ of $z \in \mathbb{C}$ such that the inequality $|P(z)| < H^{-v}$ has infinitely many solutions in $P(z) \in \mathbb{Z}[z]$, then for $v > \frac{n-1}{2}$ we have $\mu_2 \mathcal{K}_n(v) = 0$, where μ_2 is the two-dimensional Lebesgue measure on the complex plane $\mathbb{C}$.

Mahler's problem can also be formulated in terms of simultaneous Diophantine approximation: let $S_n(t)$, $t > 0$, be the set of real numbers x such that the inequality

$$\max_{1 \le l \le n} |qx^l - p_l| < q^{-t},\ t > n^{-1}, \tag{5}$$

has infinitely many solutions in integer vectors $\overline{m} = (q, p_1, \ldots, p_n)$, then $\mu S_n(t) = 0$.

The two formulations are equivalent by Khintchine's transference principle [10]; the equality $\mu M_n(w) = 0$ implies that $\mu S_n(t) = 0$ and vice versa.

K. Mahler was able to prove that $\mu M_n(w) = 0$ for $w > 4n$, with other researchers offering consecutive improvements: $w > 3n$ by Jurjen Koksma [16], $w \le 2$ by William LeVeque [17]; $w \le 2 - \frac{2}{n}$ by Friedrich Kasch and Bodo Volkmann [18]; $w \le 2 - \frac{7}{3n}$ by Wolfgang Schmidt [19]; $w \le \frac{3}{2}$, $w \le \frac{4}{3}$ by B. Volkmann [20] using Davenport's lemma [21].

The first proof of Mahler's conjecture for $n = 2$ was obtained by academician J. Kubilius for the inequality (5) using the method developed by academician Ivan Vinogradov [1; 22].

From Minkowski's linear forms theorem we see that the system of Diophantine inequalities

$$\max\left(\|xq\|, \|x^2 q\|\right) < q^{-1/2}$$

has infinitely many solutions for all $x \in \mathbb{R}$ and positive integers $q \in \mathbb{N}$. However, if we slightly reduce the order of the right-hand side to obtain

$$\max\left(\|xq\|, \|x^2 q\|\right) < q^{-1/2-\varepsilon},$$

then this new inequality has, for an arbitrarily small positive ε, only a finite number of solutions for almost all x. This result was proved by J. Kubilius in 1949 [1], thus proving Mahler's conjecture in the quadratic case. Soon thereafter John William Scott Cassels [23] obtained an improvement of Kubilius's result, proving that the system of inequalities

$$\|xq\| < \varphi(q),\ \|x^2q\| < f(q)$$

has a finite number of solutions in $q \in \mathbb{N}$ for almost all x if the series

$$\sum_{q=1}^{\infty} \varphi(q) f(q)$$

converges and

$$f(q) \geq \max\left(\varphi(q),\ q^{-1/2} \log q \sigma(q)\right),$$

where $\sigma(q)$ is the number of positive integer divisors of q. In 1959 J. Kubilius improved this result by relaxing the requirements on $\max(\varphi(q),\ f(q))$, proving the following theorem [2].

Theorem 3. *The inequality*

$$\max\left(\|xq\|,\ \|x^2q\|\right) < \psi(q)$$

has only a finite number of solutions in $q \in \mathbb{N}$ *for a positive function* $\psi(q)$ *if* $q^{-1/2}\psi(q)$ *is non-increasing and the series*

$$\sum_{q=1}^{\infty} q^{-1/2}\psi(q)$$

converges.

In a private conversation with the V. Bernik, J. Kubilius mentioned that he had obtained the proof of Leveque's result prior to the publication of Leveque's paper, but J. Kubilius wasn't expedient in publishing this proof.

Mahler's problem was solved in 1964 by Belarusian mathematician V. Sprindžuk, a pupil of J. Kubilius. V. Sprindžuk proved Mahler's conjecture not only in the fields $\mathbb{R}$ and $\mathbb{C}$, but also reformulated and proved it for p-adic numbers and formal power series. V. Sprindžuk laid down the groundworks of metric theory of Diophantine approximation, publishing two monographs in Russian and English [4; 11].

The next few sections of this article will be devoted to solutions and generalisations of the problems that were posed in the 1950–60s and were related to Koksma's and Mahler's classifications, as well as the classical problems of Vladimir Sprindžuk, Alan Baker and Wolfgang Schmidt [24]. After that, we'll move on to applications of the methods of metric theory of Diophantine approximation to quantifying distributions of rational and algebraic numbers, as well as discriminants and resultants of integer polynomials.

To conclude the article, we will touch upon applications of Diophantine approximation in mathematical physics and wireless communications.

Generalisations of the Mahler – Sprindžuk problem

Several results related to Mahler's conjecture have been improved and generalised. In particular, a full analogue of Khintchine's theorem was proved for the inequalities (2) and (3). Let $\mathcal{L}_n(\psi)$ denote the set of $x \in I$ such that the inequality

$$|P(x)| < H^{-n+1}\psi(H)$$

has infinitely many solutions in polynomials $P(x) \in \mathbb{Z}[x]$ of degree n and height $H = H(P)$.

Theorem 4. *The Lebesgue measure of the set* $\mathcal{L}_n(\psi)$ *is*

$$\mu\mathcal{L}_n(\psi) = 0 \text{ if } \sum_{H=1}^{\infty}\psi(H) < \infty, \tag{6}$$

$$\mu\mathcal{L}_n(\psi) = \mu I \text{ if } \sum_{H=1}^{\infty}\psi(H) = \infty. \tag{7}$$

The equality (6) was proved by Vasili Bernik in the paper [25], and (7) was proved by Victor Beresnevich in [26]. In the case of convergence (6), theorem 4 also holds if the monotonicity requirement on $\psi(H)$ is omitted [27], as will be discussed later on. Analogues of theorem 4 also hold in the complex case [28], the p-adic case [29; 30], and in the case of approximation by algebraic numbers [15; 31; 32]. Many of these results have become parts of monographs [4; 11; 33].

Let $f_1(x), \ldots, f_n(x) \in C^{n+1}(I)$ be $n+1$ times continuously differentiable functions of the real variable $x \in I$ such that their Wronskian is non-zero,

$$W(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ f_1''(x) & f_2''(x) & \ldots & f_n''(x) \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n)}(x) & f_2^{(n)}(x) & \ldots & f_n^{(n)}(x) \end{vmatrix} \neq 0, \tag{8}$$

for almost all $x \in I$. Let $\mathcal{K}_n(\psi)$ denote the set of $x \in I$ such that the inequality

$$|F(x)| = |a_n f_n(x) + \ldots + a_1 f_1(x) + a_0| < H^{-n+1}\psi_1(H)$$

has infinitely many solutions, where H is the «height» of the function $F(x)$ defined as $\max_{0 \le j \le n} |a_j|$. W. Schmidt [19] proved that for $\psi_1(H) = H^{-\gamma}$, $\gamma > 1$, we have the equality $\mu\mathcal{K}_2(\psi_1) = 0$. The paper [34] proves this result for $n = 3$.

V. Sprindžuk conjectured [4] that for an arbitrary n and $\psi_1(H) = H^{-\gamma}$, $\gamma > 1$, we have $\mu\mathcal{K}_n(\psi_1) = 0$. This conjecture was proved by Dmitry Kleinbock and Grigori Margulis [35]. Soon thereafter, the following theorem was proved for a curve $S(x) = (f_1(x), \ldots, f_n(x))$ satisfying the condition (8).

Theorem 5. *The measure of $\mathcal{K}_n(\psi)$ is given as*

$$\mu\mathcal{K}_n(\psi) = 0 \text{ if } \sum_{H=1}^{\infty} \psi(H) < \infty, \tag{9}$$

$$\mu\mathcal{K}_n(\psi) = \mu I \text{ if } \sum_{H=1}^{\infty} \psi(H) = \infty. \tag{10}$$

The equality (9) was independently proved by V. Beresnevich [36] and V. Bernik, D. Kleinbock and G. Margulis [37], and (10) was proved by the latter authors in [38]. Several related results in metric theory of Diophantine approximation are presented in monographs [4; 11; 15; 33; 39–41] and the article [42].

Diophantine approximation and the Hausdorff dimension

A natural next step in metric theory of Diophantine approximation was generalisation of the inequalities (1) to $|xq - p| < q^{-s}$, $s > 1$, and $|P(x)| < H^{-w}$, $w > n$.

For $s > 1$ and $w > n$ theorems 4 and 5 yield the equality $\mu\mathcal{K}_n(\psi) = 0$, i. e. the sets $\mathcal{K}_n(\psi)$ are indistinguishable in terms of the Lebesgue measure. This motivated researchers to study the Hausdorff dimension of these sets.

Vojtěch Jarnik [43] and Abram Besicovitch [44] independently proved that

$$\dim \mathcal{K}_1(q^{-s}) = \frac{2}{s+1}.$$

In the paper [24] A. Baker and W. Schmidt considered, in addition to the set $\mathcal{K}_n(\psi)$, the set $\mathcal{T}_n(v)$, $v > n+1$, of real numbers $x \in I$ such that the inequality

$$|x - \alpha| < H^{-v}, \ v > n+1,$$

has infinitely many solutions in algebraic numbers α of degree at most n and height at most $H = H(\alpha)$. They introduced the concept of a regular system and proved the following theorem.

Theorem 6. *The following equalities hold*

$$\dim \mathcal{T}_n(v) = \frac{n+1}{v},$$

$$\frac{n+1}{w+1} \le \dim \mathcal{K}_n(w) \le 2\frac{n+1}{w+1}. \tag{11}$$

Theorem 6 was strengthened in [45], where the upper estimate in (11) was replaced by $\frac{n+1}{w+1}$, thus proving the equality

$$\dim \mathcal{K}_n(w) = \frac{n+1}{w+1}.$$

In the paper [46], Yuri Melnichuk obtained estimates for the Hausdorff dimension of the set of points in the unit circle and the unit sphere with a given order of approximation by rational numbers.

Theorem 6 was generalised for the field of p-adic numbers [47]. Later, regular systems have been constructed in the space $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ from the result [48] and its generalisations [49].

The concept of regular systems of points introduced by A. Baker and W. Schmidt in [24] led to lower bounds for the Hausdorff dimensions of sets of real numbers where an integer polynomial and its derivative provide a given order of approximation of zero. Note that problems related to the Hausdorff measure are closely related to Wirsing's conjecture [50; 51].

A collection of papers by Detta Dickinson, Maurice Dodson, Victor Beresnevich, Vasily Bernik and Sanju Velani [33; 52; 53] introduced the concept of ubiquitous systems, which are in many aspects similar to regular systems. This concept was used to prove lower bounds for the Hausdorff dimension of the sets of real points related to theorem 5.

Interesting facts related to Hausdorff measure of Diophantine sets were obtained by Bryan Rynne [54]. It turns out that in certain cases the Hausdorff dimension is independent of the measure of solutions of Diophantine inequalities.

Let $\overline{\tau} = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_+^n$ and $\tau_1 \geq \ldots \geq \tau_n$, $\sum\limits_{i=1}^{n} \tau_i \geq n$.

Let $W_n(\tau) = \left\{x \in \mathbb{R}^n : |qx_i - p_i| < q^{-\tau_i},\ 1 \leq i \leq n, \text{ for infinitely many } (p, q) \in \mathbb{Z}^n \times \mathbb{N}\right\}$.

Then

$$\dim W_n(\tau) = \min_{1 \leq j \leq n} \frac{n + 1 + \sum\limits_{i=j+1}^{n} (\tau_j - \tau_i)}{\tau_j + 1}.$$

For $n = 2$, Rynne's theorem was generalised for rational approximation of the points of the curve $f \in C^{(3)}(I_0)$ defined on an interval I_0:

$$C_f = \left\{(x, f(x)) : x \in I_0\right\}.$$

Let $\overline{\tau} = (\tau_1, \tau_2)$, where τ_1 and τ_2 are positive numbers such that $0 < \min(\tau_1, \tau_2) < 1$ and $\tau_1 + \tau_2 \geq 1$. Assume that

$$\dim\left\{x \in I_0 : f''(x) = 0\right\} \leq \frac{2 - \min(\tau_1, \tau_2)}{1 + \max(\tau_1, \tau_2)}.$$

Then

$$\dim W_2(\overline{\tau}) \cap C_f = \frac{2 - \min(\tau_1, \tau_2)}{1 + \max(\tau_1, \tau_2)}.$$

Similar results have been obtained for approximation by real algebraic numbers [15; 55].

Victor Beresnevich and Evgeni Zorin proved the following fact [56].

Let M be a twice continuously differentiable submanifold of $\mathbb{R}^n$ of codimension m, and let

$$\frac{1}{n} \leq \tau \leq \frac{1}{m}.$$

Then

$$\dim S_n(\tau) \cap M \geq s := \frac{n+1}{\tau+1} - m.$$

Furthermore,

$$H^s(S_n(\tau) \cap M) = H^s(M).$$

Recently, Victor Beresnevich, Robert Vaughan, Sanju Velani and Evgeni Zorin [57] worked on finding an upper bound on the quantity of rational points within a ψ-neighbourhood of manifolds. Using this result, they proved the following theorem.

Theorem 7. *Let $M_f \subset \mathbb{R}^n$ be a manifold defined on an open subset $U \subset \mathbb{R}^d$, and let*

$$H^s\left(\left\{\alpha \in U : \left|\det\left(\frac{\partial^2 f_j}{\partial\alpha_i \partial\alpha_j}\right)_{1 \le i,\, j \le n}\right| = 0\right\}\right) = 0$$

for $s = \frac{n+1}{\tau+1} - m.$

If $d > \frac{n+1}{2}$ *and* $\frac{1}{n} \le \tau \le \frac{1}{2m+1}$, *then*

$$\dim S_n(\tau) \cap M_f \le s.$$

Generalisations of this result have been obtained by David Simmons and Jing-Jing Huang [58; 59].

Theorem 8. *Let* $M := \left\{(x,\, f(x)) : x \in U \subset \mathbb{R}^d\right\}$, *where* $f : U \to \mathbb{R}^m,\ f \in C^{(2)}$. *Let* $\overline{\tau} = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{R}^n_{>0}$ *with*

$$\tau_1 \ge \ldots \ge \tau_d \ge \max_{d+1 \le i \le n}\left\{\tau_i,\ \frac{1 - \sum_{j=1}^{n-m}\tau_{j+d}}{d}\right\} \text{ and } \sum_{i=1}^{m}\tau_{d+i} < 1.$$

Then

$$\dim\left(W_n(\tau) \cap M\right) \ge \min_{1 \le j \le d}\left\{\frac{n + 1 + \sum_{i=j}^{m}\left(\tau_j - \tau_i\right)}{\tau_j + 1} - m\right\}.$$

Along with the result of V. Beresnevich and E. Zorin, this yields the exact value of $\dim\left(W_n(\tau) \cap M\right)$. The proofs are based on the mass transference principle developed by V. Beresnevich and S. Velani [60].

Let us note several other notable results on Hausdorff dimension of sets of numbers with various Diophantine properties. Irina Morotskaya generalised the Baker – Schmidt – Bernik theorem for the field of p-adic numbers [47], and Dickinson and Dodson proved a generalisation of this theorem for non-degenerate curves [53]. In 2017 Victor Beresnevich, Jung-Jo Lee and Robert Vaughan proved a sharp lower bound for a set of τ-approximable numbers in a C^2 submanifold of $\mathbb{R}^n$ for $\frac{1}{n} \le \tau < \frac{1}{m}$. Rynne's result was improved in a paper by Konstantin Yavid [61].

Results of this section were strengthened and generalised over the last few years [54; 59; 60; 62–71].

Diophantine approximation in complex and *p*-adic cases

In Sprindžuk's monograph [4], Mahler's problem was generalised to the fields of complex and p-adic numbers. Generalisations of Khintchine's theorem for complex numbers were obtained by Denis Vasilyev [28]. Diophantine approximation in the field $\mathbb{C}$ was studied in the papers by Irina Morozova [72], Dmitry Kleinbock and George Tomanov [73], Natalia Sakovich [74]. An in-depth discussion of simultaneous Diophantine approximation can be found in [62]. The main results of these papers are complete analogues of Khintchine's theorem and the Jarnik – Besicovitch theorem for weighted simultaneous Diophantine approximation in the p-adic case, as well as a lower bound for the Hausdorff dimension of weighted simultaneously approximable points on p-adic curves [75].

Simultaneous Diophantine approximation

The results that gave rise to metric theory of Diophantine approximation were the Khintchine – Groshev theorem and Mahler's problem. J. Kubilius obtained a complete solution of Mahler's problem for $n = 2$ by first considering simultaneous approximation of a parabola by rational points in $\mathbb{R}^2$ and applying Khintchine's transference principle [10].

The next major problem in simultaneous approximation was posed by V. Sprindžuk [3; 4].

For a fixed vector $\overline{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $1 \le k \le n$, let $w \le w_n(x)$ be the exact upper bound of positive $w_1 > 0$ such that the system of inequalities

$$\max_{1 \le j \le k}\left(\left|P\left(x_j\right)\right| < H^{-w_1}\right)$$

has infinitely many solutions in polynomials $P(t)\in\mathbb{Z}[x]$ of degree $\deg P = n$ and height $H(P) = H$. V. Sprindžuk conjectured that $w = \frac{n-k+1}{k}$, which was proved in [76]. In 1980 V. Sprindžuk [5] posed a problem of approximating points in the space $\Omega = \mathbb{R}\times\mathbb{C}\times\mathbb{Q}_p = \{(x, z, \omega)\}$ by algebraic numbers from $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}_p$. He assumed that the following statement was true. Let μ_1 be the Lebesgue measure in $\mathbb{R}$ and $\mathbb{C}$, and μ_2 be the Haar measure in $\mathbb{Q}_p$. For almost all $\bar{u} = (x, z, \omega)$ (with respect to the product measure $\mu_1\times\mu_2$) the system of inequalities

$$|P(x)| < H^{-v_1},\ |P(z)| < H^{-v_2},\ |P(\omega)| < H^{-v_3},$$

where $v_j \geq -1$, $j = 1, 2$, $v_3 \geq 0$, has only a finite number of solutions in polynomials $P(t)\in\mathbb{Z}[t]$.

This conjecture was proved by Frantz Želudevič [77]. The next major step was generalisation of the Želudevič's results to systems of inequalities where the right-hand sides are arbitrary functions $\psi(H)$. Analogues of Khintchine's theorem were proved for both convergence and divergence cases [78–80].

Of particular interest is the paper [81] that made use of Kleinbock and Margulis's method [49; 68; 70; 71; 76; 78; 79; 82–92].

A number of papers in theory of Diophantine approximations are devoted to studying the distances between conjugate algebraic numbers. The papers [88; 92] can be considered as an introduction to this topic. In the paper [93] Victor Beresnevich, Vasili Bernik and Friedrich Götze applied methods of metric theory of Diophantine approximation to such problems. It was proved that the inequality $|\alpha_1 - \alpha_2| \ll Q^{(n+1)/3}$ not only has a solution in algebraic conjugate numbers α_1 and α_2 of degree n and height $H(\alpha_1) = H(\alpha_2) = Q$, but also the number of the respective minimal polynomials was estimated from below by $c(n)Q^{(n+1)/3}$. This method was generalised for the distribution of polynomial root clusters [93; 94].

Distribution of rational points close to curves and surfaces

In [1; 2] J. Kubilius showed how estimates of the number of rational points close to a parabola $G_2 = (x, x^2)$ lead to a proof of Mahler's conjecture for $n = 2$. The proof was based on Khintchine's transference principle [10]. It turned out that metric theorems concerned with estimation of a dot product between an integer vector $\bar{a} = (a_n, \ldots, a_1, a_0)$ and a vector-function $F(x) = (f_n(x), \ldots, f_1(x), 1)$ are stronger than metric theorems in simultaneous Diophantine approximation, which is implied by certain results from geometry of numbers and dynamical system theory [35]. In 1994 Martin Huxley [95] proved a theorem on distribution of rational numbers close to smooth curves, estimating the number $N_f(Q, \delta, J) := \#\left\{\left(\frac{p_1}{q}, \frac{p_2}{q}\right)\in\mathbb{Q}^2 : \frac{p_1}{q}\in J, \left|f\left(\frac{p_1}{q}\right) - \frac{p_2}{q}\right| \leq \delta, 0 < q < Q\right\}$.

Let $I\subset\mathbb{R}$ be a compact interval, c_1 and c_2 be positive constants, and let $F(I; c_1, c_2)$ be the set of functions $f: I\to\mathbb{R}$, $f\in C^{(2)}$, such that

$$c_1 \leq |f''(x)| \leq c_2\ \forall x\in I.$$

M. Huxley proved that

$$N_f(Q, \delta, I) \ll_\delta c^{10/3}\delta^{1-\delta}Q^2 + c^{1/3},\ c = \max\left(c_2, c_1^{-1}\right).$$

For $\delta > Q^{2/3}$, Huxley's result was improved in [52; 55; 56], where it was shown that for any $f\in F(I; c_1, c_2)$, any $Q > 1$ and $0 < \delta < \frac{1}{2}$, we have

$$N_f(Q, \delta, I) \ll \delta Q^2 + \delta^{-1/2}Q.$$

Later the quantity $N_f(Q, \delta, I)$ was estimated from below, and from above and below in the non-homogeneous case [1; 52; 53; 81].

Distribution of algebraic numbers

Distribution of rational numbers in real intervals is quite well-studied and can be described through Farey sequences. However, until recently, not much was known about distribution of algebraic numbers, even if their degrees were small. In 1985 V. Bernik was shown a letter from K. Mahler to V. Sprindžuk, where K. Mahler expressed his surprise about the many unanswered questions related to distribution of algebraic numbers both on the real line and in the complex plane. It wasn't until the 2010s that Denis Koleda began to solve the problems posed by K. Mahler [96–100]. Let us discuss the main results that he obtained.

Let A_n be the set of real algebraic numbers of degree n, and let the counting function of such algebraic numbers of height at most Q in the interval I be defined as

$$\Phi_n(Q, I) := \#\{\alpha \in A_n \cap I : H(\alpha) \le Q\}.$$

Theorem 9 [99; 100]. *There exists a continuous positive function* $\varphi_n(x)$ *such that for any interval* $I \subseteq \mathbb{R}$, *we have*

$$\Phi_n(Q, I) = \frac{Q^{n+1}}{2\zeta(n+1)} \int_I \varphi_n(x)\, dx + O\left(Q^n (\ln Q)^{l(n)}\right),$$

where $l(n) = 0$ *for* $n \ge 3$, $l(n) = 1$ *for* $n = 2$ *and the implicit constant in the big-O notation only depends on n.*

The function $\varphi_n(x)$ can be defined explicitly. The remainder term in theorem 9 was shown to be sharp [96], up to a constant, for all n. Further studies of the distribution of algebraic numbers by Denis Koleda, Friedrich Götze and Dmitri Zaporozhets [97] led to the following description of the density of points with algebraically conjugate coordinates in the space $\mathbb{R}^k \times \mathbb{C}^l$.

Let $A_n(k, l)$ be the set of points in the space $\mathbb{R}^k \times \mathbb{C}^l$ such that their coordinates are roots of the same irreducible integral polynomial of degree n (i. e. k real and l complex conjugate algebraic numbers of degree n over $\mathbb{Q}$). Let us define a function

$$\Phi_{k,l}(Q, B) := \{\alpha \in A_n(k, l) \cap B : H(\alpha) \le Q\}$$

for $Q \ge 1$ and $B \subset \mathbb{R}^k \times \mathbb{C}^l$.

Theorem 10 [97]. *Let* $B \subset \mathbb{R}^k \times \mathbb{C}^l$ *be a region such that its boundary* ∂B *is contained in a finite union of Lipschitz transformations of the cube* $[0, 1]^{k+2l-1}$. *Then*

$$\left| \frac{\Phi_{k,l}(Q, B)}{Q^{n+1}} - \frac{\mathrm{Vol}(B_H)}{2\zeta(n+1)} \int_B \rho_{k,l}(v)\, dv \right| \le \begin{cases} cQ^{-1} \log Q, \text{ if } n = 2 \text{ and } l = 0, \\ cQ^{-1} \text{ otherwise,} \end{cases}$$

where $\rho_{k,l} : \mathbb{R}^k \times \mathbb{C}^l \to \mathbb{R}$ *is an explicitly defined continuous non-negative function, dv is a volume element in the space* $\mathbb{R}^k \times \mathbb{C}^l$ *(considered as* $\mathbb{R}^{k+2l}$*),* $\zeta(\cdot)$ *is the Riemann zeta function, and* $\mathrm{Vol}(B_H)$ *is the volume of an* $(n+1)$*-dimensional region* B_H *which is the cube* $[-1, 1]^{n+1}$ *in the case when H is the naive height.*

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function on a finite interval $J \subset \mathbb{R}$, and let $\alpha = (\alpha_1, \alpha_2)$ be a point with algebraically conjugate coordinates such that the minimal polynomial of α_1, α_2 is of degree $\le n$ and height $\le Q$. Denote by $M_\varphi^n(Q, \gamma, J)$ the set of points α such that $|\varphi(\alpha_1) - \alpha_2| \le c_1(n) Q^{-\gamma}$. We show for $0 < \gamma < 1$ and any sufficiently large Q there exist positive values $c_2 < c_3$, where $c_i = c_i(n)$, $i = 2, 3, \ldots$, that are independent of Q and the following estimate holds [22; 24; 94; 98–103]

$$c_2 Q^{n+1-\gamma} < \# M_\varphi^n(Q, \gamma, J) < c_3 Q^{n+1-\gamma}.$$

The Mahler – Sprindžuk problem with a non-monotonic right-hand side

It is well known that in the case of convergence, the Khintchine – Groshev theorem also holds for non-monotonic functions $\psi(q)$. A number of researchers have posed other problems in Diophantine approximation with non-monotonic functions in the right-hand side. Generally, the solution then depends on convergence or divergence of the Khintchine-type series with $\psi(q)$ replaced by a different function. For the Mahler – Sprindžuk problem, V. Beresnevich showed in 2005 [27] that the theorem holds for a non-monotonic function $\psi(q)$.

In the papers [104; 105] Natalia Budarina generalised Beresnevich's result for the fields of complex and p-adic numbers, as well as non-degenerate curves. She used Sprindžuk's method, imposing progressively lower bounds on the size of the derivative, and finally applying Kleinbock and Marguli's method for the smallest derivatives [106].

Inhomogeneous Diophantine approximation

The results of the previous sections may be viewed as approximation of zero by values of linear combinations of the form $F(x) = a_n f_n(x) + \ldots + a_1 f_1(x) + a_0 \cdot 1$ that are homogeneous with respect to functions $f_n(x), \ldots, f_1(x), f_0(x) \equiv 1$. Minkowski's theorems on linear forms and successive minima provide a powerful mechanism for solving this type of problems [10; 14; 15; 19]. However, these methods have very limited utility for derivation of upper and lower bounds if the function $F(x)$ assumes the form $P_1(x) = a_n x^n + \ldots + a_1 x + a_0 + \pi$ or $P_2(x) = a_n x^n + \ldots + a_1 x + a_0 + \sin x$. These latter problems are known as inhomogeneous Diophantine approximation. A number of metric theorems related to a wide range of functions of the form $P_1(x)$ and $P_2(x)$ have been proved to date [60; 106–110].

Applications of metric theory of Diophantine approximation

The earlies applications of Diophantine approximation to celestial mechanics are probably due to Karl Siegel, who mentions them in his lecture notes. Vladimir Arnold [111] used the Khintchine – Groshev theorem during the development of Kolmogorov – Arnold – Moser theory to prove that almost all celestial systems similar to the Solar system are stable. Systematic application of metric theory of Diophantine approximation in small denominator problems in equations of mathematical physics is described in Ptashnik's monograph [112].

In 2021 a textbook dedicated to applications of metric theory of Diophantine approximation in wireless communications is due to be published by Springer Verlag. Let us point out the articles [31; 113–116] which can be regarded as an introduction to this field of applications, as well as the article by Victor Beresnevich and Sanju Velani, Faustin Adiceam, Jason Levesley and Evgeni Zorin [31] explaining the main ideas behind applications of Diophantine approximation to radio engineering [117; 118].

Biography

Academician J. Kubilius was born in Fermos village, Jurbarkas district of Lithuania. In 1940 he graduated from a grammar school in Raseiniai, in 1946 graduated from Vilnius University, and finished his postgraduate studies at Leningrad University in 1951 under scientific supervision of academician Yuri Linnik. At the age of 36 J. Kubilius successfully defended his doctor of science thesis. Between 1958 and 1992 he was the rector of Vilnius University.

J. Kubilius was the scientific advisor of Belarusian mathematicians Vladimir Sprindžuk (full member of the Academy of Sciences of BSSR) and Nikolai Lazakovich (doctor of science). The academic school of number theory in Belarus, which has 5 doctor of science degree holders and over 40 PhDs degree holders, owes a lot to the work of J. Kubilius. He was the scientific advisor of Ramunė Sliesoraitiene [119], whose PhDs thesis was devoted to metric theory of Diophantine approximation. He was an opponent during thesis defenses of Belarusian mathematicians Vasili Bernik, Ella Kovalevskaya, Vladimir Mashanov.

A full list of scientific degrees and honors of academician J. Kubilius takes up an entire printed page. For more details, we refer the reader to the article [120].

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