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КОНЕЧНЫЕ ГРУППЫ С ЗАДАННЫМИ СИСТЕМАМИ ОБОБЩЕННЫХ σ-ПЕРЕСТАНОВОЧНЫХ ПОДГРУПП

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Пусть $\sigma = \{\sigma_i | i \in I\}$ – разбиение множества всех простых чисел $\mathbb{P}$, а G – конечная группа. Множество $\mathcal{H}$ подгрупп группы G называется *полным холловым* σ-*множеством* группы G, если каждый член ≠1 из $\mathcal{H}$ является холловой σ_i-подгруппой группы G для некоторого $i \in I$ и $\mathcal{H}$ содержит ровно одну холлову σ_i-подгруппу группы G для всех i таких, что $\sigma_i \cap \pi(G) \neq \varnothing$. Группа считается σ-*примарной*, если она есть конечная σ_i-группа для некоторого i. Подгруппа A группы G называется σ-*перестановочной* в G, если G содержит полное холлово σ-множество $\mathcal{H}$ такое, что $AH^x = H^xA$ для любого $H \in \mathcal{H}$ и любого $x \in G$; σ-*субнормальной* в G, если существует подгруппа цепи $A = A_0 \leq A_1 \leq \ldots \leq A_t = G$ такая, что либо $A_{i-1} \trianglelefteq A_i$, либо $A_i / (A_i - 1)_{A_i}$ является σ-примарной для всех $i = 1, \ldots, t$; $\mathfrak{U}$-*нормальной* в G, если каждый главный фактор группы G между A_G и A^G циклический. Мы говорим, что подгруппа H группы G является: (i) *частично* σ-*перестановочной* в G, если существуют $\mathfrak{U}$-нормальная подгруппа A и σ-перестановочная подгруппа B из G такие, что $H = \langle A, B \rangle$; (ii) $(\mathfrak{U}, \sigma)$-*вложенной* в G, если существуют частично σ-перестановочная подгруппа S и σ-субнормальная подгруппа T из G такие, что $G = HT$

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и $H \cap T \le S \le H$. Мы изучаем G, предполагая, что некоторые подгруппы группы G являются частично σ-перестановочными или $(\mathfrak{U}, \sigma)$-вложенными в G. Некоторые известные результаты обобщены.

Ключевые слова: конечная группа; σ-разрешимые группы; σ-нильпотентная группа; частично σ-перестановочная подгруппа; $(\mathfrak{U}, \sigma)$-вложенная подгруппа; $\mathfrak{U}$-нормальная подгруппа.

FINITE GROUPS WITH GIVEN SYSTEMS OF GENERALISED σ-PERMUTABLE SUBGROUPS

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Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and G be a finite group. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall* σ*-set* of G if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $i \in I$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \varnothing$. A group is said to be σ*-primary* if it is a finite σ_i-group for some i. A subgroup A of G is said to be: σ*-permutable* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$; σ*-subnormal* in G if there is a subgroup chain $A = A_0 \le A_1 \le \ldots \le A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_i - 1)_{A_i}$ is σ-primary for all $i = 1, \ldots, t$; $\mathfrak{U}$*-normal* in G if every chief factor of G between A_G and A^G is cyclic. We say that a subgroup H of G is: (i) *partially* σ*-permutable* in G if there are a $\mathfrak{U}$-normal subgroup A and a σ-permutable subgroup B of G such that $H = \langle A, B\rangle$; (ii) $(\mathfrak{U}, \sigma)$*-embedded* in G if there are a partially σ-permutable subgroup S and a σ-subnormal subgroup T of G such that $G = HT$ and $H \cap T \le S \le H$. We study G assuming that some subgroups of G are partially σ-permutable or $(\mathfrak{U}, \sigma)$-embedded in G. Some known results are generalised.

Keywords: finite group; σ-soluble groups; σ-nilpotent group; partially σ-permutable subgroup; $(\mathfrak{U}, \sigma)$-embedded subgroup; $\mathfrak{U}$-normal subgroup.

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

A subgroup A of G is said to be $\mathfrak{U}$*-normal* in G [1] if either $A \trianglelefteq G$ or $A_G \neq A^G$ and every chief factor of G between A_G and A^G is cyclic.

Following L. Shemetkov [2], we use σ to denote some partition of $\mathbb{P}$. Thus $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \varnothing$ for all $i \neq j$. The symbol $\sigma(n)$ denotes the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \varnothing\}$; $\sigma(G) = \sigma(|G|)$.

The group G is said to be [3–5]: σ*-primary* if G is a σ_i-group for some $i \in I$; σ*-nilpotent* if $G = G_1 \times \ldots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$; σ*-soluble* if every chief factor of G is σ-primary.

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall* σ*-set* of G [6; 7] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $i \in I$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \varnothing$.

A subgroup A of G is said to be [3]: σ*-permutable* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$; σ*-subnormal* in G if there is a subgroup chain $A = A_0 \le A_1 \le \ldots \le A_t = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_i - 1)_{A_i}$ is σ-primary for all $i = 1, \ldots, t$.

Note that in the classical case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, σ-permutable subgroups are also called *S-permutable* [8; 9], and in this case A is σ-subnormal in G if and only if it is subnormal in G.

The σ-permutable and σ-subnormal subgroups were studied by a lot of authors (see, in particular, the papers [3–6; 10–29]).

In this paper we consider some applications of the following generalisation of σ-subnormal and σ-permutable subgroups.

Definition 1. We say that a subgroup H of G is

(i) *partially* σ*-permutable* in G if there are a $\mathfrak{U}$-normal subgroup A and a σ-permutable subgroup B of G such that $H = \langle A, B \rangle$;

(ii) $(\mathfrak{U}, \sigma)$*-embedded* in G if there are a partially σ-permutable subgroup S and a σ-subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq S \leq H$.

Note that every $\mathfrak{U}$-normal subgroup $A = \langle A, 1 \rangle$ and every σ-permutable subgroup $B = \langle 1, B \rangle$ are partially σ-permutable in G. Moreover, every partially σ-permutable subgroup S is $(\mathfrak{U}, \sigma)$-embedded in G since in this case we have $G = SG$ and $S \cap G = S \leq S$, where G is a σ-subnormal subgroup of G by definition.

Now we consider the following examples, which allow you to get various applications of the introduced concepts.

Example 1. (i) A subgroup H of G is said to be *weakly* σ*-permutable* [30] or *weakly* σ*-quasinormal* [31] in G if there is a σ-subnormal subgroup T and a σ-permutable subgroup S of G such that $G = HT$ and $H \cap T \leq S \leq H$. Every weakly σ-quasinormal subgroup is $(\mathfrak{U}, \sigma)$-embedded in the group.

(ii) A subgroup H of G is said to be *weakly S-permutable* in G [32] if there are an S-permutable subgroup S and a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq S \leq H$. It is clear that every weakly S-permutable subgroup is $(\mathfrak{U}, \sigma)$-embedded for every partition σ of $\mathbb{P}$.

(iii) Recall that a subgroup M of G is called *modular* in G if M is a modular element (in the sense of Kurosh [33, p. 43]) of the lattice $\mathcal{L}(G)$ of all subgroups of G, that is (i) $\langle X, M \cap Z \rangle = \langle X, M \rangle \cap Z$ for all $X \leq G$, $Z \leq G$ such that $X \leq Z$, and (ii) $\langle M, Y \cap Z \rangle = \langle M, Y \rangle \cap Z$ for all $Y \leq G$, $Z \leq G$ such that $M \leq Z$.

A subgroup H of G is called *m*-σ*-permutable* in G [34] if there are a modular subgroup A and a σ-permutable subgroup B of G such that $H = \langle A, B \rangle$. In view of [33, theorem 5.1.9], every modular subgroup is $\mathfrak{U}$-normal in the group. Therefore, every m-σ-permutable subgroup is partially σ-permutable.

(iv) A subgroup H of G is called *weakly m*-σ*-permutable* in G [34] if there are an m-σ-permutable subgroup S and a σ-subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq S \leq H$. It is clear that every weakly m-σ-permutable subgroup is $(\mathfrak{U}, \sigma)$-embedded.

(v) A subgroup A of G is said to be *c-normal* in G [35] if for some normal subgroup T of G we have $AT = G$ and $A \cap T \leq A_G$. Every c-normal subgroup is $(\mathfrak{U}, \sigma)$-embedded.

Our first observation generalises corresponding results in [34; 35].

Theorem A. *(i) If every non-nilpotent maximal subgroup of G is* $(\mathfrak{U}, \sigma)$*-embedded in G, then G is* σ*-soluble.*

(ii) G is soluble if and only if every maximal subgroup of G is $(\mathfrak{U}, \sigma)$*-embedded in G and G possesses a complete Hall* σ*-set* $\mathcal{H}$ *whose members are soluble groups.*

In view of example 1 (iii), we get also from theorem A the following corollary.

Corollary 1 [34, theorem B]. *If every non-nilpotent maximal subgroup of G is weakly m*-σ*-permutable in G, then G is* σ*-soluble.*

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from theorem A (ii) the following know result.

Corollary 2 [35, theorem 3.1]. *If every maximal subgroup of G is c-normal in G, then G is soluble.*

Now, recall that if $M_2 < M_1 < G$ where M_2 is a maximal subgroup of M_1 and M_1 is a maximal subgroup of G, then M_2 is said to be a 2*-maximal subgroup* of G.

Our next theorem generalises a well-known Agrawal's result on supersolubility of groups with S-permutable 2-maximal subgroups.

Theorem B. *If every* 2*-maximal subgroup of G is partially* σ*-permutable in G and G possesses a complete Hall* σ*-set* $\mathcal{H}$ *whose members are supersoluble, then G is supersoluble.*

Corollary 3. *If every* 2*-maximal subgroup of G is* σ*-permutable in G and G possesses a complete Hall* σ*-set* $\mathcal{H}$ *whose members are supersoluble, then G is supersoluble.*

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from theorem B the following known results.

Corollary 4 [36; 37, chapter 1, theorem 6.5]. *If every* 2*-maximal subgroup of G is S-permutable in G, then G is supersoluble.*

Corollary 5 [38]. *If every* 2*-maximal subgroup of G is modular in G, then G is supersoluble.*

Recall that G is *meta*-σ*-nilpotent* [7] if G is an extension of a σ-nilpotent group by a σ-nilpotent group. An analysis of many open questions leads to the necessity of studying various classes of meta-σ-nilpotent groups (see, for example, the recent papers [3; 11–18; 30] and the survey [7]).

Out next result gives the following characterisation of meta-σ-nilpotent groups.

Theorem C. *(i) The following conditions are equivalent:*

(a) G possesses a complete Hall σ*-set* $\mathcal{H}$ *whose members are* $(\mathfrak{U}, \sigma)$*-embedded in G;*

(b) G is meta-σ-nilpotent;

(c) G is σ-soluble and every σ-Hall subgroup H of G (that is $\sigma(H) \cap \sigma(|G:H|) = \varnothing$*) is c-normal in G.*

(ii) If G possesses a complete Hall σ-set $\mathcal{H}$ *whose members are partially σ-permutable in G, then the derived subgroup G′ of G is σ-nilpotent.*

A group G is said to be: a D_π-*group* if G possesses a Hall π-subgroup E and every π-subgroup of G is contained in some conjugate of E; a *σ-full group of Sylow type* [3] if every subgroup E of G is a D_{σ_i}-group for each $\sigma_i \in \sigma(E)$.

In view of example 1 (ii) we get from theorem C the following corollary.

Corollary 6 [30, theorem 1.4]. *Let G be a σ-full group of Sylow type. If every Hall* σ_i*-subgroup of G is weakly σ-permutable in G for all* $\sigma_i \in \sigma(G)$*, then G is σ-soluble.*

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from theorem C the following known result.

Corollary 7 [39, chapter I, theorem 3.49]. *G is metanilpotent if and only if every Sylow subgroup of G is c-normal.*

Proof of theorem A

First we prove the following two lemmas.

Lemma 1. *Let A, B and N be subgroups of G, where A is partially σ-permutable in G and N is normal in G. Then:*

(1) AN/N is partially σ-permutable in G/N.

(2) If G is σ-full group of Sylow type and $A \le B$*, then A is partially σ-permutable in B.*

(3) If G is σ-full group of Sylow type, $N \le B$ *and B/N is partially σ-permutable in G/N, then B is partially σ-permutable in G.*

(4) If G is σ-full group of Sylow type and B is partially σ-permutable in G, then $\langle A, B\rangle$ *is partially σ-permutable in G.*

P r o o f. Let $A = \langle L, T\rangle$, where L is $\mathfrak{U}$-normal and T is σ-permutable subgroups of G.

(1) $AN/N = \langle LN/N, TN/N\rangle$, where LN/N is $\mathfrak{U}$-normal in G/N by [40, lemma 2.8 (2)] and TN/N is σ-permutable in G/N by [3, lemma 2.8 (2)]. Hence AN/N is partially σ-permutable in G/N.

(2) This follows from [3, lemma 2.8 (1); 40 lemma 2.8].

(3) Let $B/N = \langle V/N, W/N\rangle$, where V/N is $\mathfrak{U}$-normal in G/N and W/N is σ-permutable in G/N. Then $B = \langle V, W\rangle$, where V is $\mathfrak{U}$-normal in G by [40 lemma 2.8 (3)] and W is σ-permutable in G. Hence B is partially σ-permutable in G.

(4) Let $B = \langle V, W\rangle$, where V is $\mathfrak{U}$-normal and W is a σ-permutable subgroups of G. Then

$$\langle A, B\rangle = \langle\langle L, T\rangle, \langle V, W\rangle\rangle = \langle\langle L, V\rangle, \langle T, W\rangle\rangle,$$

where $\langle L, V\rangle$ is $\mathfrak{U}$-normal in G by [40, lemma 2.8 (1)] and $\langle T, W\rangle$ is σ-permutable in G by [3, lemma 2.8 (4)]. Hence $\langle A, B\rangle$ is partially σ-permutable in G.

The lemma is proved.

Lemma 2. *Let A, B and N be subgroups of G, where A is* $(\mathfrak{U}, \sigma)$*-embedded in G and N is normal in G.*

(1) If either $N \le A$ *or* $(|A|, |N|) = 1$*, then AN/N is* $(\mathfrak{U}, \sigma)$*-embedded in G/N.*

(2) If G is σ-full group of Sylow type and $A \le B$*, then A is* $(\mathfrak{U}, \sigma)$*-embedded in B.*

(3) If G is σ-full group of Sylow type, $N \le B$ *and B/N is* $(\mathfrak{U}, \sigma)$*-embedded in G/N, then B is* $(\mathfrak{U}, \sigma)$*-embedded in G.*

P r o o f. Let T be a σ-subnormal subgroup of G such that $AT = G$ and $A \cap T \le S \le A$ for some partially σ-permutable subgroup S of G.

(1) First note that $NT \cap NA = (T \cap A)N$. Therefore $G/N = (AN/N)(TN/N)$ and

$$(AN/N) \cap (TN/N) = (AN \cap TN/N) = (A \cap T)N/N \le SN/N,$$

where SN/N is a partially σ-permutable subgroup of G/N by lemma 1 (1). Hence AN/N is $(\mathfrak{U}, \sigma)$-embedded in G/N.

(2) $B = A(B \cap T)$ and $(B \cap T) \cap A = T \cap A \le S \le A$, where S is partially σ-permutable in B by lemma 1 (2). Hence A is $(\mathfrak{U}, \sigma)$-embedded in B.

(3) See the proof of (1) and use lemma 1 (3).

The lemma is proved.

P r o o f o f t h e o r e m A. (i) Assume that this assertion is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G.

(1) G/R is σ-soluble. Hence R is not σ-primary and it is a unique minimal normal subgroup of G.

Note that if M/R is a non-nilpotent maximal subgroup of G/R, then M is a non-nilpotent maximal subgroup of G and so it is $(\mathfrak{U}, \sigma)$-embedded in G by hypothesis. Hence M/R is $(\mathfrak{U}, \sigma)$-embedded in G/R by lemma 2 (1). Therefore the hypothesis holds for G/R. Hence G/R is σ-soluble and so R is not σ-primary by the choice of G. Now assume that G has a minimal normal subgroup $N \neq R$. Then G/N is σ-soluble and N is not σ-primary. But, in view of the G-isomorphism $RN/R \simeq N$, the σ-solubility of G/R implies that N is σ-primary. This contradiction completes the proof of (1).

In view of claim (1), R is not abelian. Hence $|\pi(R)| > 1$. Let p be any odd prime dividing $|R|$ and R_p a Sylow p-subgroup of R.

(2) If G_p is a Sylow p-subgroup of G with $R_p = G_p \cap R$, then there is a maximal subgroup M of G such that $RM = G$ and $G_p \leq N_G(R_p) \leq M$.

It is clear that $G_p \leq N_G(R_p)$. The Frattini argument implies that $G = RN_G(R_p)$. Conversely, claim (1) implies that $N_G(R_p) \neq G$, so for some maximal subgroup M of G we have $RM = G$ and $G_p \leq N_G(R_p) \leq M$.

(3) M is not nilpotent and $M_G = 1$. Hence M is $(\mathfrak{U}, \sigma)$-embedded in G.

Assume that M is nilpotent, and let $D = M \cap R$. Then D is a normal subgroup of M and R_p is a Sylow p-subgroup of D since $R_p \leq G_p \leq M$. Hence R_p is characteristic in D and so it is normal in M. Therefore $Z(J(R_p))$ is normal in M. Claims (1) and (2) imply that $M_G = 1$. Hence $N_G(Z(J(R_p))) = M$ and so $N_R(Z(J(R_p))) = D$ is nilpotent. This implies that R is p-nilpotent by the Glauberman – Thompson theorem on the normal p-complements. But then R is a p-group, contrary to claim (1). Hence we have (3).

(4) There is a σ-subnormal subgroup T of G such that $MT = G$, $M \cap T = 1$ and p does not divide $|T|$.

By claim (3), there are a partially σ-permutable subgroup S and a σ-subnormal subgroup T of G such that $G = MT$ and $M \cap T \leq S \leq M$. Then $S = \langle A, B \rangle$ for some $\mathfrak{U}$-normal subgroup A and σ-permutable subgroup B of G. Moreover, from the definition $\mathfrak{U}$-normality and claim (1) it follows that, in fact, $S = B$ and $S_G = 1$. Suppose that $S \neq 1$. Then for every $\sigma_i \in \sigma(S)$ we have $S = O_{\sigma_i'}(S) \times O_{\sigma_i}(S)$ by [3, theorem B]. Therefore for every Hall σ_i-subgroup H of G from $SH = HS = O_{\sigma_i'}(S)H$ we get that $1 < O_{\sigma_i}(S) \leq H_G$, contrary to claim (1). Therefore $S = 1$, so $T \cap M = 1$. Hence $|T| = |G : M|$, so p does not divide $|T|$ since $G_p \leq M$ by claim (2).

The final contradiction for (i). Let L be a minimal σ-subnormal subgroup of G contained in T. Then L is a simple group. If L is a σ_i-group for some i, then $L \leq O_{\sigma_i}(G)$ by [12, lemma 2.2 (10)], which is impossible by claim (1).

Hence L is non-abelian, so it is subnormal in G by [12, lemma 2.2 (7)]. Suppose that $L \nleq R$. Then $L \cap R = 1$. Conversely, $R \leq N_G(L)$ by [41, chapter A, theorem 14.3]. Hence $LR = L \times R$, so $L \leq C_G(R)$. But claim (1) implies that $R \nleq C_G(R)$ and so $C_G(R) = 1$, a contradiction. Hence L is a minimal normal subgroup of R. It follows that p divides $|L|$ and hence p divides $|T|$, contrary to claim (4). Therefore assertion (i) is true.

(ii) In view of theorem A, it is enough to show that if G is soluble, then every maximal subgroup M of G is $(\mathfrak{U}, \sigma)$-embedded in G. If $M_G \neq 1$, then M/M_G is $(\mathfrak{U}, \sigma)$-embedded in G/M_G by induction, so M is $(\mathfrak{U}, \sigma)$-embedded in G by lemma 2 (3). Conversely, if $M_G = 1$ and R is a minimal normal subgroup of G, then R is abelian and so $G = R \rtimes M$. Hence M is $(\mathfrak{U}, \sigma)$-embedded in G.

The theorem is proved.

Proof of theorem B

Lemma 3 [6, theorem A]. *If G is σ-soluble, then G is a σ-full group of Sylow type.*

Lemma 4. *If G is σ-soluble and G possesses a complete Hall σ-set whose members are p-soluble, then G is p-soluble.*

P r o o f. Suppose that this lemma is false and let G be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$ be a complete Hall σ-set of G. Then H_i is p-soluble by lemma 3 for all i.

First show that if R is minimal normal subgroup of G, then G/R is p-soluble. It is enough to show that the hypothesis holds for G/R.

Note that for every chief factor $(H/R)/(K/R)$ of G/R we have that $(H/R)/(K/R) \simeq_G H/K$, where H/K is a chief factor of G and H/K is σ-primary since G is σ-soluble. So $(H/R)/(K/R)$ is σ-primary, hence G/R is σ-soluble.

Note also that $\{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R, where $H_iR/R \simeq H_i/(H_i \cap R)$ is p-soluble since H_i is p-soluble. Therefore the hypothesis holds for G/R, so G/R is p-soluble by the choice of G.

Now show that R is p-soluble. Since G is σ-soluble, R is σ-primary, that is, σ_i-group for some i. Also, for every Hall σ_i-subgroup H of G we have $R \leq H$. So, R is p-soluble by the hypothesis, hence G is p-soluble.

The lemma is proved.

P r o o f o f t h e o r e m B. Suppose that this theorem is false and let G be a counterexample of minimal order. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$. Then $t > 1$ since H_1 is supersoluble by hypothesis.

(1) If R is minimal normal subgroup of G, then G/R is supersoluble. Hence R is the unique minimal normal subgroup of G, R is not cyclic and $R \nleq \Phi(G)$.

It is enough to show that the hypothesis holds for G/R. First note that $\{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R, where $H_iR/R \simeq H_i/(H_i \cap R)$ is supersoluble since H_i is supersoluble by hypothesis.

Now assume that statement (1) is false. Then G/R is not nilpotent, so every Sylow p-subgroup in G/R is proper. Then for every Sylow p-subgroup P of G/R it follows that P is contained in some maximal subgroup of G/R. Hence R is contained in some 2-maximal subgroup T of G and so T is partially σ-permutable in G by the hypothesis. But then T/R is 2-maximal of G/R and partially σ-permutable in G/R by lemma 1 (1). Therefore the hypothesis holds for G/R, so G/R is supersoluble by the choice of G. Then we have a contradiction.

Moreover, it is well-known that the class of all supersoluble groups is a saturated formation [42 chapter VI, definition 8.6]. Hence the choice of G implies that R is the unique minimal normal subgroup of G, R is not cyclic and $R \nleq \Phi(G)$. Hence we have (1).

(2) G is soluble.

Every 2-maximal subgroup in G is partially σ-permutable and so, partially σ-subnormal in G by [3, theorem B]. Then, in view of theorem in [40], G is σ-soluble. Hence, from lemma 4 it follows that G is soluble. So we have (2).

(3) $R = O_p(G) \nleq \Phi(G)$ *for some prime* $p \in \sigma_i$. *Hence for some maximal subgroup M of G we have* $G = R \rtimes M$ *and* $M \neq M_G = 1$.

By claim (2), G is soluble and so R is a p-group for some $p \in \sigma_i$. Hence the choice of G and claim (1) imply that R is a unique minimal normal subgroup of G. Moreover, $R \nleq \Phi(G)$ by claim (1), so $R = C_G(R) = O_p(G)$ by [41, chapter A, lemma 15.2]. Hence for some maximal subgroup M of G we have $G = R \rtimes M$ and $M \neq M_G = 1$ by claim (1).

(4) If $1 < H \leq M$, *then H is not* $\mathfrak{U}$*-normal in G.*

Indeed, if H is $\mathfrak{U}$-normal in G, then $H^G/H_G \leq Z_{\mathfrak{U}}(G/H_G)$, where $H_G = 1$ by claim (3). Hence $R \leq H^G \leq Z_{\mathfrak{U}}(G)$ by claim (1). But then R is cyclic, contrary to claim (1). This contradiction completes the proof of the claim.

(5) M is not a group of prime order.

Suppose that $|M| = q$ for some prime q. Hence $|M| = |G:R|$ is a prime and so R is a maximal subgroup of G. Then every maximal subgroup V of R is 2-maximal in G, so V is partially σ-permutable in G by hypothesis. So $V = <A, B>$, where A is $\mathfrak{U}$-normal and B is σ-permutable in G. Assume $A \neq 1$. Note $A_G = 1$ by the minimality of R. Then $R \leq A^G \leq Z_{\mathfrak{U}}(G)$ and so R is cyclic, contrary to claim (1). Hence $V = B$ is σ-permutable in G. Therefore every maximal subgroups of R is σ-permutable in G.

Note that $R \leq H_i$ since R is σ_i-group by claim (3) and $H_i = R \rtimes (H_i \cap M)$, again by claim (3). Since H_i is supersoluble by hypothesis, some maximal subgroup W of R is normal in H_i. In addition, W is σ-permutable in G since it is a maximal subgroup of R. Hence for each $j \neq i$ we have $WH_j = H_jW$, which implies that $H_j \leq N_G(W)$ since $R \cap WH_j = W(R \cap H_j) = W$. Therefore W is normal in G, so the minimality of R implies that $W = 1$ and hence $|R| = p$, which is impossible by claim (3). Hence we have (5).

(6) If T is a maximal subgroup of M, then T^G *is a* σ_i*-subgroup of G.*

Indeed, T is partially σ-permutable in G by hypothesis, so $T = <A, B>$ for some $\mathfrak{U}$-normal subgroup A and some σ-permutable subgroup B of G. Note that $T \neq 1$ by claim (5). Conversely, $A = 1$ by claim (4) and so $T = B$ is σ-permutable in G. Therefore T^G/T_G is σ-nilpotent [3, theorem B (ii)]. We have $T_G \leq M_G = 1$, so $T^G/T_G \simeq T^G/1 \simeq T^G$ is σ-nilpotent group. Hence the subgroup $O_{\sigma_k}(T^G)$ is characteristic in T^G, so it is normal in G. By claim (3) we have that $O_{\sigma_k}(T^G) = 1$ for all $k \neq i$. Hence $T^G = O_{\sigma_i}(T^G)$ is a σ_i-subgroup of G.

(7) M is not $\sigma_{i'}$*-group.*

Suppose that this is false and let T be a maximal subgroup of M. Then $T \neq 1$ by claim (5). Conversely, T is a $\sigma_{i'}$-group by the hypothesis and T^G is a σ_i-subgroup of G by claim (6). Then we have a contradiction. Hence, M is not a $\sigma_{i'}$-group.

(8) M is not σ_i*-group* (this follows from the facts that $t > 1$ and R is a σ_i-group).
Final contradiction.
Let T be a maximal subgroup of M, containing a Hall $\sigma_{i'}$-subgroup of M. Then T^G is σ_i-group by claim (6). Therefore, a Hall $\sigma_{i'}$-subgroup of M is the identity group. Hence M is σ_i-group, contrary to claim (8). This contradiction completes the proof of the result.

Proof of theorem C

We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups.
Lemma 5 [3, corollary 2.4 and lemma 2.5]. *(1) The class* $\mathfrak{N}_\sigma$ *is closed under taking products of normal subgroups, homomorphic images and subgroups.*
(2) If G/N and G/R are σ*-nilpotent, then* $G/(N \cap R)$ *is* σ*-nilpotent.*
(3) If E is a normal subgroup of G and $E/(E \cap \Phi(G))$ *is* σ*-nilpotent, then E is* σ*-nilpotent.*
Recall that $G^{\mathfrak{N}_\sigma}$ denotes the σ-*nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N. In view of [43, proposition 2.2.8], we get from lemma 5 (1) the following result.
Lemma 6. *If N is a normal subgroup of G, then* $(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma}N/N$.
The next lemma is proved by the direct verifications on the basis of lemmas 5 and 6.
Lemma 7. *(1) G is meta-*σ*-nilpotent if and only if* $G^{\mathfrak{N}_\sigma}$ *is* σ*-nilpotent.*
*(2) If G is meta-*σ*-nilpotent, then every quotient G/N of G is meta-*σ*-nilpotent.*
*(3) If G/N and G/R are meta-*σ*-nilpotent, then* $G/(N \cap R)$ *is meta-*σ*-nilpotent.*
(4) If E is a normal subgroup of G and $E/(E \cap \Phi(G))$ *is meta-*σ*-nilpotent, then E is meta-*σ*-nilpotent.*
Lemma 8. *Let A, B and N be subgroups of G, where A is c-normal in G and N is normal in G.*
(1) If either $N \leq A$ *or* $(|A|, |N|) = 1$*, then AN/N is c-normal in G/N.*
(2) If $N \leq B$ *and B/N is c-normal in G/N, then B is c-normal in G.*
P r o o f. See the proof of lemma 2.
A natural number n is said to be a Π-*number* if $\sigma(n) \subseteq \Pi$. A subgroup A of G is said to be: a *Hall* Π-*subgroup* of G [6; 7] if $|A|$ is a Π-number and $|G : A|$ is a Π′-number; a σ-*Hall subgroup* of G if A is a Hall Π-subgroup of G for some $\Pi \subseteq \sigma$.
Recall also that a normal subgroup E of G is called *hypercyclically embedded* in G [33, p. 217] if every chief factor of G below E is cyclic.
P r o o f o f t h e o r e m C. Let $D = G^{\mathfrak{N}_\sigma}$ be the σ-nilpotent residual of G.
(i) (a) ⇒ (b). Assume that this is false and let G be a counterexample of minimal order. Then D is not σ-nilpotent since G/D is σ-nilpotent by lemma 5 (2). Let $\mathcal{H} = \{H_1, \ldots, H_t\}$. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$. Let S_i be a partially σ-permutable subgroup and T_i be a σ-subnormal subgroup of G such that $S_i \leq H_i$, $H_iT_i = G$ and $H_i \cap T_i \leq S_i$ for all $i = 1, \ldots, t$. Then, for every i, $S_i = \langle A_i, B_i \rangle$ for some $\mathfrak{U}$-normal subgroup A_i and σ-permutable subgroup B_i of G.
(1) If R is a σ*-primary minimal normal subgroup of G, then G/R is meta-*σ*-nilpotent and so G is* σ*-soluble. Moreover, R is a unique minimal normal subgroup of G,* $C_G(R) \leq R$ *and R is not cyclic.*
First we show that G/R is meta-σ-nilpotent. In view of the choice of G, it is enough to show that the hypothesis holds for G/R. Since R is σ-primary, for some i we have $R \leq H_i$ and $(|R|, |H_j|) = 1$ for all $j \neq i$. Therefore $\{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R whose members are $(\mathfrak{U}, \sigma)$-embedded in G/R by lemma 2 (1). Hence the hypothesis holds for G/R, so G/R is meta-σ-nilpotent and G is σ-soluble. Hence every minimal normal subgroup of G is σ-primary, so R is a unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by lemma 7 (4). Hence $C_G(R) \leq R$ by [41, chapter A, lemma 15.6]. Finally, note that in the case when R is cyclic we have $|R| = p$ for some prime p and so $G/C_G(R) = G/R$ is cyclic, which implies that G is metanilpotent and so it is meta-σ-nilpotent. Therefore we have (1).
(2) For some i, i = 1 say, we have $S_i = S_1 \neq 1$*. Moreover, if for some k we have* $S_k = 1$*, then* T_k *is a normal complement to* H_k *in G.*
Assume that $S_i = 1$. Then $H_i \cap T_i = 1$, so T_i is a σ-subnormal Hall σ_i'-subgroup of G. Hence T_i is a normal complement to H_i in G by [3, lemma 2.6 (10)]. Moreover, $G/T_i \simeq H_i$ is σ-nilpotent. Suppose that $S_i = 1$ for all $i = 1, \ldots, t$. Then $T_1 \cap \ldots \cap T_t = 1$ by [41, chapter A, theorem 1.6 (b)], so

$$G \simeq G/1 = G/(T_1 \cap \ldots \cap T_t)$$

is σ-nilpotent by lemma 5 (2). Then we have a contradiction. Hence for some i we have $S_i \neq 1$.

(3) If $S_i \neq 1$*, then* $(H_i)_G \neq 1$.

Assume that this is false. Then every non-identity subgroup L of H_i is not σ-permutable in G since otherwise for every $x \in G$ we have $LH_i^x = H_i^x L = H_i^x$ which implies that $1 < L \leq (H_i)_G = 1$.

Therefore $B_i = 1$ and so $S_i = A_i$ is a $\mathfrak{U}$-normal subgroup of G with $(S_i)_G = 1$. But then we have $1 < (S_i)^G$ is hypercyclically embedded in G by the definition $\mathfrak{U}$-normality and so R is cyclic, contrary to claim (1). Hence we have (3).

(4) G possesses a σ-primary minimal normal subgroup, R say.

Claims (2) and (3) imply that $(H_1)_G \neq 1$. Therefore, if R is a minimal normal subgroup of G contained in $(H_1)_G$, then R is σ-primary.

The final contradiction for the implication (a) ⇒ (b). Claims (1) and (4) imply that G is σ-soluble, so R is a unique minimal normal subgroup of G by claim (1). Hence claims (2) and (3) imply that $T_2, \ldots, T_t$ are normal subgroups of G and $G/T_k \simeq H_k$ for all $k = 2, \ldots, t$. Hence $G/(T_2 \cap \ldots \cap T_t)$ is σ-nilpotent by lemma 5 (2). Conversely, $T_2 \cap \ldots \cap T_t = H_1$ by [41, chapter A, theorem 1.6 (b)] and so G is meta-σ-nilpotent, contrary to the choice of G. This contradiction completes the proof of the (a) ⇒ (b).

(b) ⇒ (c). The subgroup D is σ-nilpotent by lemma 7 (1). Let $\Pi = \sigma(H)$. Then H is a Hall Π-subgroup of G.

Suppose that $H_G \neq 1$. Then H/H_G is c-normal in G by induction since the hypothesis holds for G/H_G by lemma 7 (2). Hence H is c-normal in G by lemma 8 (2).

Now assume that $H_G = 1$. Then, since D is σ-nilpotent, it follows that $D \cap H = 1$. Conversely, G/D is σ-nilpotent by lemma 5 (2) and $H \simeq HD/D$ is a Hall Π-subgroup of G/D, so HD/D has a normal complement T/D in G/D. Then T is a normal subgroup of G such that $HT = G$ and $T \cap H \leq T \cap HD \cap H \leq D \cap H = 1$. Hence H is c-normal in G. Therefore the implication (b) ⇒ (c) holds.

(c) ⇒ (b). In view of example 1 (v), this application is a corollary of the implication (a) ⇒ (b).

(ii) Suppose that this assertion is false and let G be a counterexample of minimal order. Then G is not σ-nilpotent, so $|\sigma(G)| > 1$. Moreover, from part (i) we know that D is σ-nilpotent and so G is σ-soluble. Let $\mathcal{H} = \{H_1, \ldots, H_t\}$.

Let R be a minimal normal subgroup of G. Then R is a σ_i-group for some i, so the hypothesis holds for G/R by lemma 1 (1). Hence $(G/R)' = G'R/R = G'/(G' \cap R)$ is σ-nilpotent by the choice of G. Therefore $R \leq G'$ and $R \nleq \Phi(G)$ by lemma 5 (3). Moreover, if G has a minimal normal subgroup $N \neq R$ of G, then $N \leq G'$ and $G' \simeq G'/1 = G'/(R \cap N)$ is σ-nilpotent, contrary to the choice of G. Therefore R is a unique minimal normal subgroup of G and $C_G(R) \leq R$ by [41, chapter A, lemma 16.5]. We can assume without loss of generality that $R \leq H_1$.

Let M be a maximal subgroup of G such that $R \nleq M$. Then $M_G = 1$ and $|G : M|$ is a σ_i-number. Therefore for some $x \in G$ we have $H = H_2^x \leq M$. Then $H = \langle A, B \rangle$ for some $\mathfrak{U}$-normal subgroup A and σ-permutable subgroup B of G. Moreover, $A_G \leq M_G = 1$, so A^G is hypercyclically embedded in G by the definition $\mathfrak{U}$-normality. If $A \neq 1$, then $R \leq A^G$ and so $|R| = p$ for some prime p. But then $C_G(R) = R$ and $G/R = G/C_G(R)$ is cyclic. Hence G' is nilpotent. This contradiction shows that $A = 1$, so $H = B$ is σ-permutable in G. But then $HH^x = H^xH = H$ for all $x \in G$ since H is a Hall σ_i-subgroup of G for some i. Hence H is normal in G, so $1 < H \leq M_G$, a contradiction. Therefore assertion (ii) is true.

The theorem is proved.

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