# Erratum to "Quantum phases for point-like charged particles and for electrically neutral dipoles in an electromagnetic field" [Ann. Phys. 392 (2018) 49-62]

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Three years ago, our paper [1] has been published, where we advanced the idea of explaining quantum phase effects for electric/ magnetic dipoles, expressed as the sum of four terms [1, 2]

$$\text{¢}_{\text{dipoles}} \approx \frac{1}{\hbar c}\int(\boldsymbol{m}\times\boldsymbol{E})\cdot d\boldsymbol{s} - \frac{1}{\hbar c}\int((\boldsymbol{p}\times\boldsymbol{B}))\cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int(\boldsymbol{p}\cdot\boldsymbol{E})\boldsymbol{v}\cdot d\boldsymbol{s} - \frac{1}{\hbar c^2}\int(\boldsymbol{m}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{s} \quad (1)$$

(Eq. (11) of [1]) through the superposition of the corresponding quantum phases for point-like charged particles. Hereinafter $\boldsymbol{m}$ ($\boldsymbol{p}$) denotes the magnetic (electric) dipole moment, $\boldsymbol{E}$($\boldsymbol{B}$) is the electric (magnetic) field, $\boldsymbol{v}$ is the velocity, and $d\boldsymbol{s}=\boldsymbol{v}dt$ is the path element.

Developing this idea, we disclosed two new quantum phases for point-like charges – next to the known electric and magnetic Aharonov-Bohm (A-B) phases [3] – which we named as complementary electric and magnetic A-B phases, correspondingly [1].

Further on, we found that complementary A-B phases for electric charges can be described via fundamental equations of quantum mechanics only in the case [1], where we abandon the customary definition of the momentum operator via the canonical momentum $\hat{\boldsymbol{P}}_c$ of a particle in an EM field, *i.e*.

$$\hat{\boldsymbol{P}}_c = \hat{\boldsymbol{p}} + \frac{e\hat{\boldsymbol{A}}}{c} \to \hat{\boldsymbol{P}}_c = -i\hbar\nabla \quad (1)$$

(Eq. (30) of [1]), and adopt its new definition via the sum of the mechanical and interactional EM momentum for a particle in an external EM field,

$$\hat{\boldsymbol{p}} + \hat{\boldsymbol{P}}_{EM} \to \hat{\boldsymbol{P}} = -i\hbar\nabla \quad (2)$$

(Eq. (35) of [1]). The proposed redefinition of the momentum operator (2) has a number of important implications, and their analysis essentially depends on the particular expression for the interactional EM field momentum $\boldsymbol{P}_{EM}$ for various physical problems (see, e.g., Ref. [3]).

Now, we would like to point out an unfortunate error committed in [1] under determination of $\boldsymbol{P}_{EM}$ for the system "point-like charged particle in an external EM field" as a function of the scalar ¨ and vector $\boldsymbol{A}$ potentials of the external EM field.

Namely, the obtained in Ref. [1] expression,

$$\boldsymbol{P}_{EM} = \frac{e\boldsymbol{A}}{c} + \frac{\boldsymbol{v}e\text{¨}}{c^2} + \frac{e\boldsymbol{v}(\boldsymbol{A}\cdot\boldsymbol{v})}{c^3}, \quad (3)$$

(Eq. (38) of [1]) must be corrected as

$$\boldsymbol{P}_{EM} = \frac{e\boldsymbol{A}}{c} + \frac{\boldsymbol{v}e\text{¨}}{c^2} - \frac{e\boldsymbol{v}(\boldsymbol{A}\cdot\boldsymbol{v})}{c^3}. \quad (3a)$$

Indeed, using the known expression for the interactional EM field momentum via the Poynting vector [5]

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$$\boldsymbol{P}_{EM}=\frac{1}{4\text{®}c}\int_V\left(\boldsymbol{E}\times\boldsymbol{B}_e\right)dV+\frac{1}{4\text{®}c}\int_V\left(\boldsymbol{E}_e\times\boldsymbol{B}\right)dV \qquad (4)$$

(Eq. (A.1) of [1]), where $\boldsymbol{E}$ ($\boldsymbol{B}$) denotes the external electric (magnetic) field, while $\boldsymbol{E}_e$ ($\boldsymbol{B}_e$) stands for the electric (magnetic) field of a charged particle. Then we obtain, respectively, the first and second integrals on the rhs of Eq. (4) as follows:

$$\frac{1}{4\text{®}c}\int_V\left(\boldsymbol{E}\times\boldsymbol{B}_e\right)dV=\frac{\boldsymbol{v}e\text{¨}}{c^2} \qquad (5)$$

(Eq. (A.7) of [1]), and

$$\frac{1}{4\text{®}c}\int_V\left(\boldsymbol{E}_e\times\boldsymbol{B}\right)dV=-\frac{1}{4\text{®}c}\int_V\left(\boldsymbol{A}\times\left(\nabla\times\boldsymbol{E}_e\right)\right)dV+\frac{e\boldsymbol{A}}{c} \qquad (6)$$

(Eq. (A.11) of [1]). Further manipulations with the remaining integral on the rhs of Eq. (6) yield:

$$-\frac{1}{4\text{®}c}\int_V\left(\boldsymbol{A}\times\left(\nabla\times\boldsymbol{E}_e\right)\right)dV=-\frac{e\boldsymbol{v}\left(\boldsymbol{A}\cdot\boldsymbol{v}\right)}{c^3}-\frac{\boldsymbol{v}}{4\text{®}c^2}\int_V\left(\boldsymbol{A}\cdot\left(\left(\boldsymbol{v}\cdot\nabla\right)\boldsymbol{E}_e\right)\right)dV \qquad (7)$$

(Eq. (A.20) of Ref. [1]), and we show in Ref. [1] that for the system “point-like charge in an external EM field”, the second integral on the rhs of Eq. (7) vanishes. Hence, combining Eqs. (4)-(7), we arrive at Eq. (3a), instead of the sign wise erroneous equation (3) reported in Ref. [1].

Therefore, in the Coulomb gauge, the corresponding Hamiltonian with the momentum operator (2) and the interactional EM field momentum (3a) takes the form

$$\hat{H}=\frac{\left(-i\hbar\nabla-\boldsymbol{P}_{EM}\right)^2}{2M}+e\text{¨}=-\frac{\hbar^2}{2M}\Delta+e\text{¨}-\frac{e\boldsymbol{A}\cdot\boldsymbol{v}}{c}-\frac{e\text{¨}v^2}{c^2}+\frac{ev^2\left(\boldsymbol{A}\cdot\boldsymbol{v}\right)}{c^3}= \\ -\frac{\hbar^2}{2M}\Delta+\left(1-\frac{v^2}{c^2}\right)\left(e\text{¨}-\frac{e\boldsymbol{A}\cdot\boldsymbol{v}}{c}\right). \qquad (8)$$

The quantum phase of charged particle in an EM field is defined as [6]

$$\text{¢}_{\mathrm{EM}}=-\frac{1}{\hbar}\int\left(\hat{H}-\hat{H}_0\right)dt, \qquad (9)$$

where $\hat{H}_0=-\frac{\hbar^2}{2M}\Delta$ stands for the Hamiltonian of a particle outside an EM field. Substituting Eq. (8) into Eq. (9), we obtain the quantum phase of a charged particle in an EM field as follows:

$$\text{¢}_{EM}=-\frac{1}{\hbar}\int e\text{¨}dt+\frac{1}{\hbar c}\int e\boldsymbol{A}\cdot d\boldsymbol{s}+\frac{1}{\hbar c^2}\int e\text{¨}\boldsymbol{v}\cdot d\boldsymbol{s}-\frac{1}{\hbar c^3}\int e\left(\boldsymbol{A}\cdot\boldsymbol{v}\right)\boldsymbol{v}\cdot d\boldsymbol{s}, \qquad (10)$$

where we designated $d\boldsymbol{s}=\boldsymbol{v}dt$ the element of the path of particle.

The first and the second terms on the rhs of Eq. (10) correspond to the known electric and magnetic A-B phases [3], while the third and the firth terms stand respectively for the complementary electric and complementary magnetic A-B phases.

The reported corrections do not influence the conclusions of Ref. [1] and open a way to some new insights in the physical meaning of quantum phases.

The obtained Eq. (10) allows presenting the resultant quantum phase of a charged particle in an EM field in general form,

$$\text{¢}_{\mathrm{EM}}=-\frac{1}{\hbar}\left(1-\frac{v^2}{c^2}\right)\int e\text{¨}dt+\left(1-\frac{v^2}{c^2}\right)\frac{1}{\hbar c}\int e\left(\boldsymbol{A}\cdot\boldsymbol{v}\right)dt=-\frac{1}{\hbar\text{¥}^2}\int L_{\mathrm{int}}dt=\frac{1}{\hbar\text{¥}^2}S_{\mathrm{int}}, \qquad (11)$$

where we have used the equality $d\boldsymbol{s}$=$\boldsymbol{v}dt$, and designated

$$L_{\mathrm{int}}=-e\text{¨}+\frac{e}{c}\boldsymbol{A}\cdot\boldsymbol{v} \qquad (12)$$

the component of the Lagrangian of a charged particle, responsible for its interaction with the EM field. Then, we have used the definition of the interactional component of the action

$$S_{\text{int}} = \int L_{\text{int}} dt .$$

Hence, in the non-relativistic limit, where we can put ¥≈1, the total phase of a charged particle occurs proportional to the total action $S_{\text{total}}$ for a charged particle, which is composed as the sum of the mechanical $S_{\text{M}}$ and the interactional field component $S_{\text{int}}$, *i.e.*,

$$\text{¢}_{\text{total}} = \frac{1}{\hbar}\left(S_{\text{M}} + S_{\text{int}}\right) = \frac{1}{\hbar} S_{\text{total}} . \qquad (13)$$

Thus, the known semi-classical limit for the wave function of a freely moving particle [6]

$$\cdot = \cdot_0 e^{iS_M/\hbar}$$

keeps its shape for charged particle in an EM field, too, with the replacement of mechanical component of action $S_{\text{M}}$ by the total action $S_{\text{total}}$, *i.e.*,

$$\cdot = \cdot_0 e^{iS_{total}/\hbar} .$$

Next, we re-address Eq. (10) for the phase of a charged particle in an EM field and consider only the velocity-dependent phase components expressed by the last three terms of this equation. Using Eq. (3a), we can write for these phase components

$$\text{¢}_{EM}(\boldsymbol{v}) = \frac{1}{\hbar c}\int e\boldsymbol{A}\cdot d\boldsymbol{s} + \frac{1}{\hbar c^2}\int e\,\text{¨}\,\boldsymbol{v}\cdot d\boldsymbol{s} - \frac{1}{\hbar c^3}\int e\left(\boldsymbol{A}\cdot\boldsymbol{v}\right)\boldsymbol{v}\cdot d\boldsymbol{s} = \frac{1}{\hbar}\int \boldsymbol{P}_{EM}\cdot d\boldsymbol{s} . \qquad (14)$$

Concurrently we remind that the phase for a freely moving particle is given by the equation

$$\text{¢}_{\text{free}}(\boldsymbol{v}) = \frac{1}{\hbar}\int \boldsymbol{P}_M \cdot d\boldsymbol{s} \qquad (15)$$

due to the de Broglie relationship, where $\boldsymbol{P}_M$ denotes the mechanical momentum of a particle.

The total velocity-dependent phase of charged particle is given as the sum of Eqs. (14) and (15), and is equal to

$$\text{¢}_{\text{total}}(\boldsymbol{v}) = \frac{1}{\hbar}\int\left(\boldsymbol{P}_M + \boldsymbol{P}_{EM}\right)\cdot d\boldsymbol{s} . \qquad (16)$$

Taking also into account that the total phase $\text{¢}_{\text{total}}(\boldsymbol{v})$ of a moving charged particle can be presented via the corresponding wave vector $\boldsymbol{k}$ as

$$\text{¢}_{\text{total}}(\boldsymbol{v}) = \frac{1}{\hbar}\int \boldsymbol{k}\cdot d\boldsymbol{s} , \qquad (17)$$

we obtain through comparison of Eqs. (16) and (17)

$$\boldsymbol{k} = \left(\boldsymbol{P}_M + \boldsymbol{P}_{EM}\right)/\hbar .$$

Hence the wavelength of charged particle moving in the EM field is equal to

$$\lambda\!\!\!^{-} = \hbar/\left|\boldsymbol{P}_M + \boldsymbol{P}_{EM}\right| . \qquad (18)$$

Eq. (18) shows that the de Broglie wavelength of a charged particle moving in the presence of an EM field depends not only on its mechanical momentum, but rather on the modulus of the vector sum of mechanical $\boldsymbol{P}_M$ and interactional electromagnetic $\boldsymbol{P}_{EM}$ momenta.

Detailed analysis of the physical implications of Eqs. (13) and (18) will be done elsewhere.

**References**


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