Дифференциальные уравнения и оптимальное управление

Differential equations and optimal control

МЕТОД ПОСТРОЕНИЯ ОПТИМАЛЬНОЙ СТРАТЕГИИ УПРАВЛЕНИЯ В ЛИНЕЙНОЙ ТЕРМИНАЛЬНОЙ ЗАДАЧЕ

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Рассматривается задача оптимального управления линейной дискретной системой с неизвестными ограниченными возмущениями, которую требуется за конечное время перевести с гарантией на терминальное множество, обеспечивая при этом минимум гарантированного значения терминального критерия качества. Определяется оптимальная стратегия управления, учитывающая информацию о состоянии системы в один будущий момент времени, и предлагается эффективный метод ее вычисления. Результаты численных экспериментов демонстрируют улучшение качества управления на основе введенной оптимальной стратегии в сопоставлении с оптимальной гарантирующей программой при сравнимой трудоемкости их вычисления.

Ключевые слова: линейная система; возмущения; оптимальное управление; стратегия управления; алгоритм.

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A METHOD FOR CONSTRUCTING AN OPTIMAL CONTROL STRATEGY IN A LINEAR TERMINAL PROBLEM

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This paper deals with an optimal control problem for a linear discrete system subject to unknown bounded disturbances, where the control goal is to steer the system with guarantees into a given terminal set while minimising the terminal cost function. We define an optimal control strategy which takes into account the state of the system at one future time instant and propose an efficient numerical method for its construction. The results of numerical experiments show an improvement in performance under the optimal control strategy in comparison to the optimal open-loop worst-case control while maintaining comparable computation times.

Keywords: linear system; disturbance; optimal control; control strategy; algorithm.

Introduction

Optimal control problems for dynamical systems under uncertainty have been studied in the literature since late 1960s [1–3]. The simplest approach that guarantees constraints satisfaction and achieves the guaranteed value of the cost at the worst-case disturbance realisation is to find an optimal open-loop worst-case control. The optimal open-loop worst-case control is constructed before the control process starts and is not corrected during it; no information about possible future state measurements is used for its construction. It is well known that optimal open-loop worst-case controls underestimate the potential of the control process, i. e., they give a conservative estimate of the guaranteed optimal value of the problem and often cannot be constructed because of the constraints infeasibility (see, e. g., [4–6]). However, dynamic programming takes into account all future state realisations, but the practical derivation of the dynamic programming strategy is computationally intense with the exception of special cases of low dimensional systems and short control intervals.

Therefore, such control strategies are relevant that take into account some information about the future states of the system and at the same time the complexity of their construction is comparable to the complexity of calculating optimal open-loop worst-case controls. One of the possible approaches was proposed in papers [6–8]. In [6] linear terminal problems were considered [7] deals with linear-quadratic optimal control problems and [8] deals with problems of minimising the total momentum of the control input. All these papers assume that before control process starts, we can choose one or more time instants (closing time instants of the system according to [6; 8]), at which we can measure exactly the system state and make corrections in the control input.

This paper deals with the problem considered in [6]. In contrast to [6], where a complex iterative algorithm was used to construct an optimal control strategy with one closing instant, which requires sequential optimisation first in control inputs and then in a parameter, we use the ideas of [8] to reduce the problem under consideration to a single linear program, which allows to calculate the optimal control and the optimal parameter simultaneously.

Compared to [8], the problem studied in this paper has a terminal performance index and a discrete time system, while in [8] a Lagrange cost of a special type and continuous time systems are investigated. Further comparison of the results from [8] and the ones of this paper, the drawbacks and advantages of the two methods are discussed in example 2. Two more examples demonstrate the efficiency of the new approach.

Optimal open-loop worst-case control

Consider a linear discrete-time time-invariant control system with a disturbance

$$x(t+1) = Ax(t) + Bu(t) + Mw(t), \ x(0) = x_0, \ t = 0, 1, \ldots, T-1, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in U \subset \mathbb{R}^r$ is the control input, $w(t) \in W \subset \mathbb{R}^p$ is the unknown disturbance at time t, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $M \in \mathbb{R}^{n \times p}$ are given matrices; $U = \left\{ u \in \mathbb{R}^r : u_{\min} \le u \le u_{\max} \right\}$, $W = \left\{ w \in \mathbb{R}^p : \|w\|_{\infty} \le w_{\max} \right\}$, where $u_{\min}, u_{\max} \in \mathbb{R}^r$, $w_{\max} > 0$, $\|z\|_{\infty} = \max_i |z_i|$. A trajectory of system (1) generated by a feasible control input $u(\cdot) = \left(u(t) \in U, t = 0, 1, \ldots, T-1 \right)$ and a disturbance $w(\cdot) = \left(w(t) \in W, t = 0, 1, \ldots, T-1 \right)$ is denoted by $x(t | x_0, u(\cdot), w(\cdot))$, $t = 0, 1, \ldots, T-1$.

Given a terminal set $X_T = \left\{ x \in \mathbb{R}^n : g_{\min} \leq Hx \leq g_{\max} \right\}$, where $H \in \mathbb{R}^{m \times n}$, $g_{\min}, g_{\max} \in \mathbb{R}^m$, the control goal is to steer system (1) at time instant T to the terminal set with guarantees. The requirement of a guaranteed (robust) steering to the set X_T without any further assumptions yields the definition of a feasible open-loop worst-case control.

Definition 1. A control input $u(\cdot)$ is called a feasible open-loop worst-case control if for any possible realisation of the disturbance $w(\cdot)$ it steers system (1) at time instant T into the terminal set, i. e. the following inclusion holds

$$x\bigl(T \,|\, x_0,\, u(\cdot),\, w(\cdot)\bigr) \in X_T \ \ \forall\, w(t) \in W,\ t = 0,\, 1,\, \ldots,\, T-1. \tag{2}$$

The quality of a feasible open-loop worst-case control $u(\cdot)$ is measured by the value

$$J(u) = \max_{w(\cdot)} c' x(T),\ c \in \mathbb{R}^n, \tag{3}$$

that represents the terminal cost (Mayer's performance index) at the worst realisation of the disturbance and is called the guaranteed value of the terminal cost. Here the prime symbol denotes a transpose.

Definition 2. A feasible open-loop worst-case control $u^0(\cdot)$ is called optimal if it minimises the guaranteed value of the terminal cost (3): $J(u^0) = \min_u J(u)$.

In guaranteed (robust) optimal control problems, along with the disturbed system (1), one considers a so-called nominal system

$$x_0(t+1) = Ax_0(t) + Bu(t),\ x_0(0) = x_0,\ t = 0,\, 1,\, \ldots,\, T-1,$$

that is used to formulate a deterministic optimal control problem equivalent to the problem of minimising the cost (3) subject to system (1) and inclusion (2). The method for constructing this deterministic problem is well investigated in the literature (see, e. g., [6]). It uses the linearity of system (1) and estimates of the worst-case realisations of disturbances in the directions specified by the vector c and the rows h_i' of the matrix H:

$$\gamma_0(\tau) = w_{\max} \sum_{t=0}^{T-\tau-1} \left\| c' A^t M \right\|_1,\ \gamma_i(\tau) = w_{\max} \sum_{t=0}^{T-\tau-1} \left\| h_i' A^t M \right\|_1,\ i = 1,\, \ldots,\, m,\ \|z\|_1 = \sum_i |z_i|.$$

The vector of estimates $\gamma(0) = \bigl(\gamma_i(0),\ i = 1,\, \ldots,\, m\bigr)$ allows to define a «tightened» terminal set for the nominal system and to formulate the deterministic problem in the form

$$J(u) = \min_{u(\cdot)} c' x_0(T) + \gamma_0(0), \tag{4}$$

$$x_0(t+1) = Ax_0(t) + Bu(t),\ x_0(0) = x_0,\ u(t) \in U,\ t = 0,\, 1,\, \ldots,\, T-1,$$

$$g_{\min} + \gamma(0) \leq Hx_0(T) \leq g_{\max} - \gamma(0).$$

Using the formula $x_0(T) = A^T x_0 + \sum_{t=0}^{T-1} A^{T-t-1} Bu(t)$ for the terminal state of the nominal system and substituting it in problem (4) we conclude that the optimal open-loop worst-case control $u^0(\cdot)$ can be calculated as a solution to the linear program

$$\min_{u(\cdot)} \sum_{t=0}^{T-1} c' A^{T-t-1} Bu(t),$$

$$g_{\min} + \gamma(0) - HA^T x_0 \leq \sum_{t=0}^{T-1} HA^{T-t-1} Bu(t) \leq g_{\max} - \gamma(0) - HA^T x_0,$$

$$u_{\min} \leq u(t) \leq u_{\max},\ t = 0,\, \ldots,\, T-1,$$

where the constant $c' A^T x_0 + \gamma_0(0)$ in the cost is omitted.

The optimal open-loop worst-case control is the simplest solution of the problem under consideration, when system (1) has to be robustly steered to the terminal set while minimising the terminal cost (3). The open-loop control does not take into account the possibility of future state measurements of system (1), that allow to close the control loop and to make corrections to the planned control inputs (see, e. g., [7; 8]). In contrast to optimal open-loop worst-case controls, such a possibility is taken into account by the control strategies. One of such control strategies is introduced in the next section.

Optimal control strategy

Before the control process starts, we fix a time instant $T_1 \in \{1, 2, \ldots, T-1\}$ that is referred to as the closing instant of system (1) (see [6; 8]). Denote $\Delta_0 = \{0, 1, \ldots, T_1 - 1\}$, $\Delta_1 = \{T_1, T_1 + 1, \ldots, T-1\}$; $u_k(\cdot) = \big(u_k(t) \in U,\ t \in \Delta_k\big)$ is the control input on the interval Δ_k; $w_k(\cdot) = \big(w_k(t) \in W,\ t \in \Delta_k\big)$ is the disturbance on Δ_k; $U_k = \big\{u_k(\cdot): u_k(t) \in U,\ t \in \Delta_k\big\}$ is the set of feasible control inputs on Δ_k; $W_k = \big\{w_k(\cdot): w_k(t) \in W,\ t \in \Delta_k\big\}$ is the set of possible disturbances on Δ_k, $k = 0, 1$.

Assume that on the interval Δ_0 a control input $u_0(\cdot) = u_0(\cdot|x_0) \in U_0$ is chosen. At time T_1 system (1) reaches a state x_1 that belongs to the set

$$X(T_1|x_0, u_0(\cdot)) = \big\{x \in \mathbb{R}^n : x = x(T_1|x_0, u_0(\cdot), w_0(\cdot)),\ w_0(\cdot) \in W_0\big\}.$$

Following [7; 8], it is assumed that at time instant T_1 we can:

1) measure exactly the current state $x_1 = x(T_1|x_0, u_0(\cdot), w_0(\cdot))$;

2) choose a new control input $u_1(\cdot) = u_1(\cdot|x_1) \in U_1$ on Δ_1 taking into account the obtained state measurement x_1.

Taking into account 1) and 2) we look for a solution of the problem under consideration in terms of a control strategy (with the closing instant T_1):

$$\pi_1 = \big\{u_0(\cdot|x_0);\ u_1(\cdot|x_1),\ x_1 \in X(T_1|x_0, u_0(\cdot|x_0))\big\},$$

where the control input $u_0(\cdot) = u_0(\cdot|x_0)$ is referred to as an initial open-loop control.

A trajectory of control system (1), corresponding to a strategy π_1 and a disturbance $w(\cdot) = \big(w_0(\cdot), w_1(\cdot)\big)$, is defined as a sequential solution of two systems [7; 8]:

$$x(t+1) = Ax(t) + Bu_0(t) + Mw_0(t),\ x(0) = x_0,\ t \in \Delta_0,$$

$$x(t+1) = Ax(t) + Bu_1(t|x(T_1)) + Mw_1(t),\ x(T_1) = x(T_1|x_0, u_0(\cdot), w_0(\cdot)),\ t \in \Delta_1.$$

Now we discuss conditions for the strategy π_1 to be feasible, i. e. for the trajectory defined above to guarantee the terminal constraints satisfaction.

First, the control input $u_1(\cdot|x_1) \in U_1$ that is chosen at the time instant T_1, must satisfy the inclusion

$$X(T|x_1, u_1(\cdot|x_1)) \subseteq X_T, \tag{5}$$

where $X(T|x_1, u_1(\cdot)) = \big\{x \in \mathbb{R}^n : x = x(T|x_1, u_1(\cdot), w_1(\cdot)),\ w_1(\cdot) \in W_1\big\}$ is the set of possible terminal states $x(T|x_1, u_1(\cdot), w_1(\cdot))$ of system (1) with the initial condition $x(T_1) = x_1$, the control input $u_1(\cdot)$ and the disturbance $w_1(\cdot)$.

Secondly, the control input $u_0(\cdot)$ should be such that for all states x_1 from the set $X(T_1|x_0, u_0(\cdot))$ there exist a control $u_1(\cdot|x_1)$, satisfying (5). Summarising, we obtain the next definition.

Definition 3. A strategy π_1 is called a feasible control strategy if

$$X(T|x_1, u_1(\cdot|x_1)) \subseteq X_T\ \ \forall x_1 \in X(T_1|x_0, u_0(\cdot)).$$

Obviously, an arbitrary feasible strategy with the initial open-loop control $u_0(\cdot|x_0)$ is not better than a feasible control strategy of the form

$$\pi_1 = \big\{u_0(\cdot|x_0);\ u_1^0(\cdot|x_1),\ x_1 \in X(T_1|x_0, u_0(\cdot|x_0))\big\}, \tag{6}$$

which on Δ_1 consists of the optimal open-loop worst-case controls $u_1^0(\cdot|x_1)$ for states x_1. Every open-loop control $u_1^0(\cdot|x_1)$ for a fixed x_1 is the solution of the problem

$$J_1(x_1) = \min_{u_1(\cdot) \in U_1} \max_{w_1(\cdot) \in W_1} c'x(T|x_1, u_1(\cdot), w_1(\cdot)) \tag{7}$$

subject to (5). The quality of strategy (6) is obviously measured by the value

$$V(\pi_1) = \max_{w_0(\cdot) \in W_0} J_1\big(x(T_1|x_0, u_0(\cdot), w_0(\cdot))\big).$$

Note that if problem (7) is infeasible, then we assume $J_1(x_1) = +\infty$. Therefore, if the strategy π_1 is not feasible, i. e. for some $x_1 \in X(T_1|x_0, u_0(\cdot))$ there is no control input $u_1(\cdot|x_1)$ that satisfies (5), then $V(\pi_1) = +\infty$.

Definition 4. A feasible control strategy

$$\pi_1^0 = \left\{ u_0^0(\cdot|x_0); \, u_1^0(\cdot|x_1), \, x_1 \in X(T_1|x_0, u_0^0(\cdot|x_0)) \right\}, \tag{8}$$

is called optimal, if $V(\pi_1^0) = \min V(\pi_1)$, where minimum is taken over all feasible strategies of the form (6). A control input $u_0^0(\cdot|x_0)$ is called the optimal initial open-loop control (within the optimal strategy π_1^0).

Hence, the control strategy (8) is optimal if $u_0^0(\cdot|x_0)$ is a solution of the minimax problem

$$V(\pi_1^0) = \min_{u_0(\cdot)\in U_0} \max_{w_0(\cdot)\in W_0} J_1(x(T_1)),$$

$$\tag{9}$$

$$x(t+1) = Ax(t) + Bu_0(t) + Mw_0(t), \, x(0) = x_0, \, t \in \Delta_0,$$

and $u_1^0(\cdot|x_1)$ are solutions of problems (7) for the states $x_1 \in X(T_1|x_0, u_0^0(\cdot|x_0))$.

Problem (9) implies that the optimal guaranteed value of the strategy π_1^0 is equal to

$$V(\pi_1^0) = \min_{u_0(\cdot)\in U_0} \max_{w_0(\cdot)\in W_0} \min_{u_1(\cdot)\in U_1} \max_{w_1(\cdot)\in W_1} c'x\big(T\,|\,x(T_1|x_0, u_0(\cdot), w_0(\cdot)), \, u_1(\cdot|x(T_1|x_0, u_0(\cdot), w_0(\cdot))), \, w_1(\cdot)\big),$$

while the optimal guaranteed value of the open-loop worst-case control $u^0(t)$ is calculated as

$$J(u^0) = \min_{u_0(\cdot)\in U_0} \min_{u_1(\cdot)\in U_1} \max_{w_0(\cdot)\in W_0} \max_{w_1(\cdot)\in W_1} c'x\big(T\,|\,x(T_1|x_0, u_0(\cdot), w_0(\cdot)), \, u_1(\cdot), \, w_1(\cdot)\big).$$

Taking into account the minimax inequality we conclude that $V(\pi_1^0) \le J(u^0)$. In the last section we will provide some examples where the optimal control strategy with one closing instant achieves a significant improvement in comparison to the optimal open-loop worst-case control.

Calculating the optimal initial open-loop control

Before the control process starts, we need to know only the optimal initial open-loop control $u_0^0(\cdot|x_0)$. The collection of the optimal open-loop worst-case controls $u_1^0(\cdot|x_1)$ is not calculated in advance. The control input $u_1^0(\cdot|x_1(T_1))$ is only found at the closing time instant T_1, when the current state $x(T_1)$ is measured. Therefore, the purpose of this section is to propose an efficient method for calculating the optimal initial open-loop control $u_0^0(\cdot|x_0)$, i. e. solving problem (9).

Problem (9) is the terminal control problem of the same type as the problem for calculating the optimal open-loop worst-case control $u^0(\cdot)$. It has no explicit terminal constraints; however, these are implicitly imposed by the condition $x(T_1) \in X_1 = \{x_1 : J_1(x_1) < +\infty\}$. The principal difficulty in solving problem (9) is that the function J_1 in the performance index is defined implicitly as the optimal value of problem (7). In this regard, for the purposes of further presentation, we reformulate problem (9) in an equivalent form (see [9]):

$$V(\pi_1^0) = \min_{u_0(\cdot),\, \alpha} \alpha,$$

$$x(t+1) = Ax(t) + Bu_0(t) + Mw_0(t), \, x(0) = x_0, \, u_0(t)\in U_0, \, t\in\Delta_0, \tag{10}$$

$$J_1(x(T_1)) \le \alpha \,\, \forall\, w_0(\cdot)\in W_0.$$

The function $J_1(x_1)$, $x_1 \in \mathbb{R}^n$, as the optimal value of a linear program (to which problem (7) is reduced), is a piecewise linear convex function (see [10, p. 180]), therefore for any fixed $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ the α-level set $X_1(\alpha) = \{x_1 \in X_1 : J_1(x_1) \le \alpha\}$ is a convex polyhedron. Here $\alpha_{\min} = \inf c'x + \gamma_0(T_1)$, $\alpha_{\max} = \sup c'x - \gamma_0(T_1)$, subject to $g_{\min} + \gamma(T_1) \le Hx \le g_{\max} - \gamma(T_1)$. Then in (10) the terminal constraint has the form $x(T_1) \in X_1(\alpha)$ $\forall\, w_0(\cdot) \in W_0$.

Since the exact description of the polyhedra $X_1(\alpha)$ for all values of the parameter α is difficult, in [6; 8] it was proposed to replace $X_1(\alpha)$ with their outer polyhedral approximations with normals to the faces of these

polyhedra being independent of α. Let $p_j \in \mathbb{R}^n$, $j = 1, 2, \ldots, m_1$, $\|p_j\| = 1$, be a collection of vectors which represent the mentioned normals, and $P_1 \in \mathbb{R}^{m_1 \times n}$ be a matrix, which rows are the vectors p_j. Denote

$$f_j(\alpha) = \max p_j' x_1, \ x_1 \in X_1(\alpha). \tag{11}$$

Then the outer approximating polyhedron for $X_1(\alpha)$ is $\bar{X}_1(\alpha) = \left\{ x_1 \in \mathbb{R}^n : P_1 x_1 \leq f(\alpha) \right\}$, where $f(\alpha) = \left(f_j(\alpha), \ j = 1, \ldots, m_1 \right)$. With a sufficiently large set of vectors p_j, $j = 1, 2, \ldots, m_1$, $\bar{X}_1(\alpha)$ approximate the sets $X_1(\alpha)$, $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, quite accurately.

In [6], in order to solve problem (10), an iterative algorithm was proposed. Each iteration of the algorithm for the current value α_k refines the approximation $\bar{X}_1(\alpha_k)$ and calculates the control $u^0_{\alpha_k}(t)$, $t \in \Delta_0$, that guarantees steering the system to the set $\bar{X}_1(\alpha_k)$ at the time instant T_1. Thus, at each iteration k the algorithm solves $m_1(k)$ problems (11) and one control problem. The value α_{k+1} is found by any line search method.

In what follows we propose a method for solving problem (10), which does not require application of the iterative procedure described in [6].

Denote $G_1 = HA^{T-T_1}$, $c_1' = c'A^{T-T_1}$ and consider the case when $\mathrm{rank}\left(G_1' | c_1 \right) = m + 1 \leq n$. Simple arguments yield that in this case $\alpha_{\min} = -\infty$, $\alpha_{\max} = +\infty$, i. e. function (11) is defined on the entire real axis. Let us show that $f(\alpha)$, $\alpha \in \mathbb{R}$, is affine.

Assumption 1. For any $j = 1, 2, \ldots, m_1$ the vector p_j is such that equalities $\mathrm{rank}\left(G_1' | c_1 \right) = \mathrm{rank}\left(G_1' | c_1 | p_j \right) = m + 1$ hold.

Proposition. *Let assumption 1 hold. Then*

$$f(\alpha) = f_0 + \lambda \alpha, \tag{12}$$

where $f_0 = f(0)$, $\lambda = \left(\lambda_j, \ j = 1, \ldots, m_1 \right)$, λ_j *satisfies the conditions* $G_1' y + c_1 \lambda_j = p_j$, $\lambda_j \geq 0$.

P r o o f. Consider problem (11) for a fixed index j. The set $X_1(\alpha)$ consists of those and only those vectors x_1 for which the following system is feasible

$$x(t+1) = Ax(t) + Bu_1(t), \ x(T_1) = x_1, \ u_1(t) \in U, \ t \in \Delta_1,$$

$$g_{\min} + \gamma(T_1) \leq Hx(T) \leq g_{\max} - \gamma(T_1), \ c'x(T) \leq \alpha - \gamma_0(T_1).$$

Following the arguments that were used to reduce the problem for constructing the optimal open-loop worst-case control to problem (4), problem (11) can also be reduced to a linear program. We represent it in the form

$$f_j(\alpha) = \max_{x_1, u_1} p_j' x_1,$$

$$g_{\min} + \gamma(T_1) \leq G_1 x_1 + \sum_{t \in \Delta_1} G_1 D(t) u_1(t) \leq g_{\max} - \gamma(T_1),$$

$$c_1' x_1 + \sum_{t \in \Delta_1} c_1' D(t) u_1(t) \leq \alpha - \gamma_0(T_1),$$

$$u_{\min} \leq u_1(t) \leq u_{\max}, \ t \in \Delta_1, \tag{13}$$

where $D(t) = A^{T_1 - t - 1} B$, $t \in \Delta_1$.

The problem dual to (13) has the form

$$f_j(\alpha) = \min_{\substack{y^*, y_*, \lambda_j, \\ v_*(t), v^*(t), t \in \Delta_1}} \left(g_{\max} - \gamma(T_1) \right)' y^* - \left(g_{\min} + \gamma(T_1) \right)' y_* + \left(\alpha - \gamma_0(T_1) \right) \lambda_j + \sum_{t \in \Delta_1} \left(u_{\max}' v^*(t) - u_{\min}' v_*(t) \right),$$

$$G_1' y^* - G_1' y_* + c_1 \lambda_j = p_j,$$

$$D(t)' G_1' y^* - D(t)' G_1' y_* + D(t)' c_1 \lambda_j + v^*(t) - v_*(t) = 0,$$

$$\lambda_j \geq 0, \ y^* \geq 0, \ y_* \geq 0, \ v^*(t) \geq 0, \ v_*(t) \geq 0, \ t \in \Delta_1, \tag{14}$$

and the complimentary slackness conditions hold [9; 10]. Note that all the dual variables in (14), similarly to λ_j, depend on the index j that is omitted for simplicity of presentation.

Denote $y = y^* - y_*$. Then, according to assumption 1, the system of linear algebraic equations (14) has a unique solution (y, λ_j). If that solution has $\lambda_j < 0$, then the dual problem (14) is infeasible and the primal problem (13) is unbounded on $X_1(\alpha)$. Let $\lambda_j \geq 0$, then both problems (13), (14) are feasible. The second group of equality constraints in problem (14) can be represented as

$$v^*(t) - v_*(t) = -D(t)' p_j, \; t \in \Delta_1.$$

Taking into account non-negativeness of the dual variables and the complementary slackness conditions, it is clear that the dual variables are calculated according to the formulae

$$y_i^* = \begin{cases} y_i, & y_i \geq 0, \\ 0, & y_i < 0, \end{cases} \; y_{*i} = \begin{cases} 0, & y_i \geq 0, \\ -y_i, & y_i < 0, \end{cases} \; i = 1, \ldots, m,$$

$$v_*(t) = \left(v_{*k}(t), \; k = 1, \ldots, r\right), \; v^*(t) = \left(v_k^*(t), \; k = 1, \ldots, r\right), \quad (15)$$

$$v_{*k}(t) = \begin{cases} v_k(t), & v_k(t) \geq 0, \\ 0, & v_k(t) < 0, \end{cases} \; v_k^*(t) = \begin{cases} 0, & v_k(t) \geq 0, \\ -v_k(t), & v_k(t) < 0, \end{cases} \; t \in \Delta_1,$$

where $v(t) = \left(v_k(t), \; k = 1, \ldots, r\right) = D(t)' p_j, \; t \in \Delta_1.$

Problem (14) for $\alpha = 0$ has same optimal solution, therefore we can conclude that $f_j(\alpha) = f_j(0) + \lambda_j \alpha$, which proves formula (12).

Along with the proposition we derived a simple formula for calculating the components of the vector f_0 that allows to avoid solving linear programs (13) or (14):

$$f_{0j} = f_j(0) = \left(g_{\max} - \gamma(T_1)\right)' y^* - \left(g_{\min} + \gamma(T_1)\right)' y_* - \gamma_0(T_1) \lambda_j + \sum_{t \in \Delta_1} \left(u'_{\max} v^*(t) - u'_{\min} v_*(t)\right).$$

It also follows that if $\operatorname{rank}\left(G_1' | c_1\right) \neq \operatorname{rank}\left(G_1' | c_1 | p_j\right)$, then $f_j(\alpha) = +\infty$.

Considering the relation (12) the problem for constructing the optimal initial open-loop control (10) (case $\operatorname{rank}\left(G_1' | c_1\right) = m + 1 \leq n$) can be presented in the form

$$V\left(\pi_1^0\right) = \min_{u_0(\cdot), \alpha} \alpha,$$

$$x(t+1) = Ax(t) + Bu_0(t) + Mw_0(t), \; x(0) = x_0, \; u_0(t) \in U, \; t \in \Delta_0, \quad (16)$$

$$P_1 x(T_1) \leq f_0 + \lambda \alpha \; \forall w_0(\cdot) \in W_0,$$

and further reduces to the linear program

$$\min_{u_0(\cdot), \alpha} \alpha,$$

$$\sum_{t \in \Delta_0} P_1 A^{T_1 - t - 1} Bu_0(t) - \lambda \alpha \leq f_0 - \overline{\gamma} - P_1 A^{T_1} x_0, \quad (17)$$

$$u_{\min} \leq u_0(t) \leq u_{\max}, \; t \in \Delta_0,$$

where $\overline{\gamma} = \left(\overline{\gamma}_j, \; j = 1, \ldots, m_1\right) \colon \overline{\gamma}_j = w_{\max} \sum_{t \in \Delta_0} \left\| p_j' A^t M \right\|_1.$

Now let us consider the case when $\operatorname{rank}\left(G_1' | c_1\right) = n < m + 1$. In this case $-\infty < \alpha_{\min} \leq \alpha_{\max} < +\infty$, function $f(\alpha), \; \alpha \in [\alpha_{\min}, \alpha_{\max}]$, is piecewise affine and concave. Here in order to construct the optimal initial open-loop control $u_0^0(\cdot | x_0)$ we propose to replace the approximation of the set $X_1(\alpha)$ with the approximation of the set

$$\Xi_1(\alpha) = \left\{ (\xi_0, \xi) \in \mathbb{R}^{m+1} \colon \exists u_1(\cdot) \in U_1, \; g_{\min} + \gamma(T_1) \leq \xi + \sum_{t \in \Delta_1} G_1 D(t) u_1(t) \leq g_{\max} - \gamma(T_1), \right.$$

$$\xi_0 + \sum_{t \in \Delta_1} c_1' D(t) u_1(t) \le \alpha - \gamma_0(T_1) \},$$

by the polytope $\bar{\Xi}_1(\alpha) = \{ (\xi_0, \xi) \in \mathbb{R}^{m+1} : q_0 \xi_0 + Q_1 \xi \le \bar{f}(\alpha) \}$, where $q_0 = (q_{0j} \ge 0, \ j = 1, \dots, m_1)$, matrix $Q_1 \in \mathbb{R}^{m_1 \times m}$ consists of rows $q_j' \in \mathbb{R}^m$, $j = 1, \dots, m_1$, that are chosen in advance, and $\bar{f}(\alpha) = (\bar{f}_j(\alpha), \ j = 1, \dots, m_1)$, where $\bar{f}_j(\alpha)$ is the optimal value of the linear program

$$\bar{f}_j(\alpha) = \max_{\xi_0, \xi, u_1} q_{0j} \xi_0 + q_j' \xi,$$

$$g_{\min} + \gamma(T_1) \le \xi + \sum_{t \in \Delta_1} G_1 D(t) u_1(t) \le g_{\max} - \gamma(T_1), \ \xi_0 + \sum_{t \in \Delta_1} c_1' D(t) u_1(t) \le \alpha - \gamma_0(T_1), \tag{18}$$

$$u_{\min} \le u_1(t) \le u_{\max}, \ t \in \Delta_1.$$

Following the arguments of the proposition, we derive that $\bar{f}(\alpha)$, $\alpha \in \mathbb{R}$, is calculated by the formulae

$$\bar{f}(\alpha) = \bar{f}_0 + \lambda \alpha, \ \lambda = q_0 \ge 0,$$

$$\bar{f}_j(0) = (g_{\max} - \gamma(T_1))' y^* - (g_{\min} + \gamma(T_1))' y_* - \gamma_0(T_1) q_{0j} + \sum_{t \in \Delta_1} (u_{\max}' v^*(t) - u_{\min}' v_*(t)),$$

where $y_*, \ y^*, \ v_*(t), \ v^*(t), \ t \in \Delta_1$, are found from (15) with the following adjustment: $y = q_j$, $v(t) = D'(t) G_1' q_j + D'(t) c_1 q_{0j}$, $t \in \Delta_1$.

The problem for constructing the optimal initial open-loop control (10) (case $\operatorname{rank}(G_1' | c_1) = n < m + 1$) has the form

$$V(\pi_1^0) = \min_{u_0(\cdot), \alpha} \alpha,$$

$$x(t+1) = Ax(t) + Bu_0(t) + Mw_0(t), \ x(0) = x_0, \ u_0(t) \in U, \ t \in \Delta_0, \tag{19}$$

$$\bar{P}_1 x(T_1) \le \bar{f}_0 + q_0 \alpha \ \forall w_0(\cdot) \in W_0$$

with $\bar{P}_1 = q_0 c_1' + Q_1 G_1$ and can be reduced to the linear program similarly to the reduction of problem (16) to problem (17).

The resulting linear program of the form (17) has $T_1 + 1$ variables and m_1 constraints. Depending on the required accuracy of approximation of the set $X_1(\alpha)$ or the set $\Xi_1(\alpha)$ the number of constraints can be quite large. However, in contrast to the method from [6], where first problems (13), (18) are solved and then the problem of the dimension comparable with the dimension of problems (17), (19) for a fixed parameter α is solved, problem (17) is solved only once and its solution immediately yields the optimal value of the parameter $\alpha^0 = V(\pi_1^0)$ and the optimal initial open-loop control $u_0^0(\cdot | x_0)$.

Note that in the second case $(\operatorname{rank}(G_1' | c_1) = n < m + 1)$ the space dimension where we approximate the set $\Xi_1(\alpha)$ is higher than the state space dimension n, which can be undesirable and lead to a significant increase in the number of constraints in problem (19). To avoid this problem one has to explore the piecewise affine structure of the function $f(\alpha)$, $\alpha \in [\alpha_{\min}, \alpha_{\max}]$. This will be the focus of a future work.

It is also worth mentioning that the idea of approximating the set $\Xi_1(\alpha)$ can be applied in the case $\operatorname{rank}(G_1' | c_1) = m + 1$, if the number of terminal constraints m is less than the number of states n of the control system. Such an approach reduces the dimension of the space where approximations are constructed and is applied to solve examples 2 and 3 in the next section.

Examples

Let us illustrate the proposed method for constructing the optimal control strategy by three examples. The first example is a discrete analogue of the problem from [6], the second is the problem of minimising the total momentum of the control input from [8], and the third is a modification of the latter. Discrete systems for the examples are obtained by discretisation of continuous systems with the sampling period $h = 0, 1$.

Example 1. Consider a discretised problem from [6]:

$$x_2(T) \to \max,$$

$$x(t+1) = \begin{pmatrix} 0.9950 & 0.0998 \\ -0.0998 & 0.9950 \end{pmatrix} x(t) + \begin{pmatrix} 0.0050 \\ 0.0998 \end{pmatrix} u(t) + \begin{pmatrix} 0.0050 \\ 0.0998 \end{pmatrix} w(t), \ x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{20}$$

$$x(T) \in X_T = \left\{ x \in \mathbb{R}^2 : x_* \le x_1 \le x^* \right\}, \ |u(t)| \le 1, \ |w(t)| \le 0.5, \ t = 0, \ldots, T-1.$$

Let us choose the control horizon $T = 120$ and the closing instant $T_1 = 80$. In [6] $x_* = 2, \ x^* = 7$; however, in this case there exists no feasible open-loop worst-case control (in both continuous problem from [6] and discrete problem (20)). Therefore, we choose $x_* = 2, \ x^* = 10$. This modification does not affect the optimal control strategy since the constraint $x_1(T) \le 10$ is not active, but it allows to compare the optimal open-loop worst-case control and the optimal strategy. In particular, problem (20) has the optimal open-loop worst-case control $u^0(\cdot)$ that achieves the optimal guaranteed value equal to $J(u^0) = 1.501102$, while the optimal control strategy (8) has the optimal guaranteed value $V(\pi_1^0) = \alpha^0 = 2.754\,215$. Calculating the optimal open-loop worst-case control takes 0.0151 s, while to obtain the optimal strategy we needed 0.0186 s. Figures 1 and 2 illustrate the results. The obtained solutions correspond to results from [6] (with provision for discretisation), but allow to avoid a computationally intense iterative procedure (see table 1 in [6]).

Let us explain in more detail fig. 1, which shows the state trajectories of the nominal system corresponding to (20) under the optimal open-loop worst-case control $u^0(t)$, $t = 0, \ldots, 119$ (dashed line 1), and under the optimal initial open-loop control $u_0^0(t|x_0)$, $t = 0, \ldots, 79$ (solid line 2). A dotted line represents the set $X(T|x_0, u(\cdot))$ of possible states of system (20) under the optimal open-loop worst-case control. This set lies entirely in the terminal set X_T (grey area), which illustrates that constraints (2) are satisfies with guarantees. The optimal initial open-loop control $u_0^0(\cdot|x_0)$ generates the set $X(T_1|x_0, u_0^0(\cdot|x_0))$ of possible states of system (20) at the closing instant T_1. This set belongs to the set $X_1(\alpha^0)$ (dotted lines at the bottom of fig. 1), i. e. for any $x_1 \in X(T_1|x_0, u_0^0(\cdot|x_0))$ the inequality $J_1(x_1) \ge \alpha^0$ holds. For a satisfactory representation of the set $X_1(\alpha^0)$, 83 vectors were required, i. e. $m_1 = 83$. Point x_1^* corresponds to the extremal value of the function J_1, i. e. $J_1(x_1^*) = \alpha^0$. Despite the approximation of the set $X_1(\alpha^0)$, the last equality holds exactly. The state trajectory that corresponds to the optimal open-loop worst-case control $u_1^0(\cdot|x_1^*)$ for the state $x(T_1) = x_1^*$ is shown by dash-dotted line 3. Note that geometrically

$$J(u^0) = \min x_2, \ x \in X(T|x_0, u^0(\cdot)), \ V(\pi_1^0) = \alpha^0 = \min x_2, \ x \in X(T|x_1^*, u_1^0(\cdot|x_1^*)).$$

Figure 2 represents the optimal open-loop worst-case control $u^0(t)$, $t = 0, \ldots, 119$ (dashed line), optimal initial open-loop control $u_0^0(t|x_0)$, $t = 0, \ldots, 79$ (solid line before the closing instant), optimal open-loop worst-case control $u_1^0(t|x_1^*)$, $t = 80, \ldots, 119$ (solid line after the closing instant), and the trajectories that correspond to the worst-case disturbance. The latter here is the disturbance that delivers the exact optimal value. In the example under consideration, the worst disturbances for the optimal open-loop worst-case control and for the optimal strategy coincide and are equal to $w^*(t) = w_{\max} \mathrm{sign}\left(c' A^{T-t-1} M\right)$, $t = 0, \ldots, T-1$.

Example 2. The following problem was solved in [8]:

$$\int_0^{t_f} |u(t)| \to \min,$$

$$\dot{x}_1 = x_2, \ \dot{x}_2 = -x_1 + u + w, \ x(0) = x_0, \tag{21}$$

$$|x(t_f)| \le x^*, \ |u(t)| \le 1, \ |w(t)| \le w^*, \ t \in [0, t_f].$$

Let $u(t) = u_1(t) - u_2(t)$, $0 \le u_j(t) \le 1$, $j = 1, 2$, $x_3(t) = \int_0^t u_1(s) + u_2(s)\,ds$, $t \in \left[0, t_f\right]$, and suppose that the control and the disturbance are discrete with the sampling period equal to $h = 0, 1$. In this case we obtain the discrete problem with $n = 3$, $r = 2$:

$$x_4(T) \to \min,$$

$$x(t+1) = \begin{pmatrix} 0.9950 & 0.0998 & 0 \\ -0.0998 & 0.9950 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0.0050 & -0.0050 \\ 0.0998 & -0.0998 \\ 0.1 & 0.1 \end{pmatrix} u(t) + \begin{pmatrix} 0.0005 \\ 0.1051 \\ 0 \end{pmatrix} w(t), \qquad (22)$$

$$x(0) = \bar{x}_0, \ \left|x_i(T)\right| \le x^*, \ i = 1, 2, \ 0 \le u_j(t) \le 1, \ j = 1, 2, \ \left|w(t)\right| \le w^*, \ t = 0, \ldots, T - 1,$$

where $T = \dfrac{t_f}{h}$, $\bar{x}_0 = \left(x_0, 0\right)$. We assume that parameters are as in [8]: $t_f = 10$ resulting in $T = 100$, $\bar{x}_0 = (5, 0, 0)$, $x^* = 2$, $w^* = 0,3$. The closing instant $T_1 = 80$ is chosen.

The optimal open-loop worst-case control $u^0(\cdot)$ of problem (22) gives the optimal guaranteed value equal to $J\left(u^0\right) = 5.722\,047$. The optimal control strategy has the optimal guaranteed value $V\left(\pi_1^0\right) = \alpha^0 = 5.039\,103$. Compared to [8], where $J\left(u^0\right) = 5.656\,317$ and $V\left(\pi_1^0\right) = 5.013\,186\,5$, slightly worse performance is due to discrete disturbance, while in [8] disturbance was assumed piecewise continuous. We are not presenting the optimal controls and trajectories here since they visually coincide with the results in [8].

The principal difference in solving problem (22) and applying the method from [8] to solve problem (21) is that the function $f(\alpha)$ as defined by (12) for problem (22) is linear in the parameter α, while for problem (21) this function is piecewise linear. This results in lower dimension of problem (19). The latter has $T_1 + 1$ variables, while the resulting problem in [8] has T variables. As a result, to obtain the optimal strategy by solution of problem (19) we needed only $0.081\,2$ s (0.75 s in [8]). The disadvantage of the method proposed in this paper compared to [8] is that we approximate the set $\Xi_1(\alpha)$ in $\mathbb{R}^3$ instead of $\mathbb{R}^2$ in [8].

Example 3. Consider a modification of problem (21)

$$\int_0^{t_f} \left|u(t)\right| \to \min,$$

$$\dot{x}_1 = x_2, \ \dot{x}_2 = -x_1 + x_3 + w, \ \dot{x}_3 = u, \ x(0) = x_0,$$

$$\left|x_i(t_f)\right| \le x^*, \ i = 1, 2, \ \left|u(t)\right| \le 1, \ \left|w(t)\right| \le w^*, \ t \in \left[0, t_f\right].$$

Here a modification concerns the so-called indirect control of system (21).

Introducing $u(t) = u_1(t) - u_2(t)$, $x_4(t) = \int_0^t u_1(s) + u_2(s)\,ds$, $t \in \left[0, t_f\right]$, as in example 2, we obtain the discrete problem with $n = 4$, $r = 2$:

$$x_4(T) \to \min,$$

$$x(t+1) = \begin{pmatrix} 0.9950 & 0.0998 & 0.0050 & 0 \\ -0.0998 & 0.9950 & 0.0998 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0.0002 & -0.0002 \\ 0.0050 & -0.0050 \\ 0.1 & -0.1 \\ 0.1 & 0.1 \end{pmatrix} u(t) + \begin{pmatrix} 0.0005 \\ 0.1051 \\ 0 \\ 0 \end{pmatrix} w(t), \quad (23)$$

$$x(0) = \bar{x}_0, \ \left|x_i(t_f)\right| \le x^*, \ i = 1, 2, \ 0 \le u_j(t) \le 1, \ j = 1, 2, \ \left|w(t)\right| \le w^*, \ t = 0, \ldots, T - 1.$$

The closing instant $T_1 = 60$ is chosen. We illustrate the solution for the initial condition $\bar{x}_0 = (7, 0, 0, 0)$, and the parameters $x^* = 1.5$, $w^* = 0.2$.
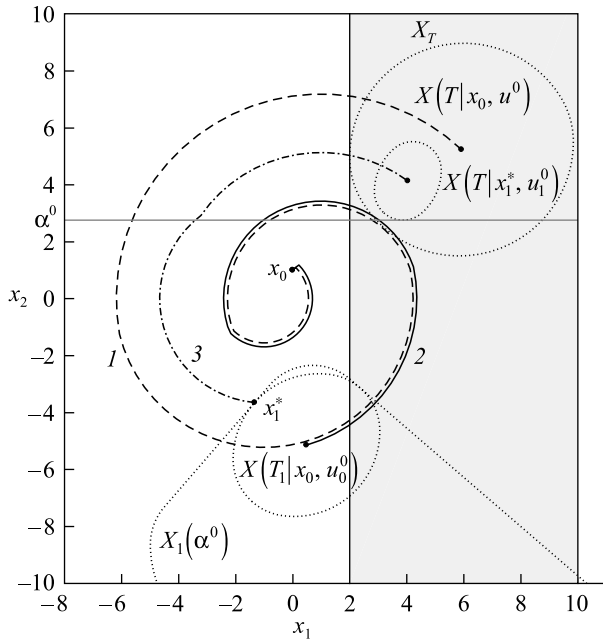
Fig. 1. Phase-plane solution representation of example 1.
Trajectories under the optimal open-loop worst-case
control (line 1), under the optimal initial
open-loop control (line 2), and the optimal open-loop
worst-case control on the interval after the closing
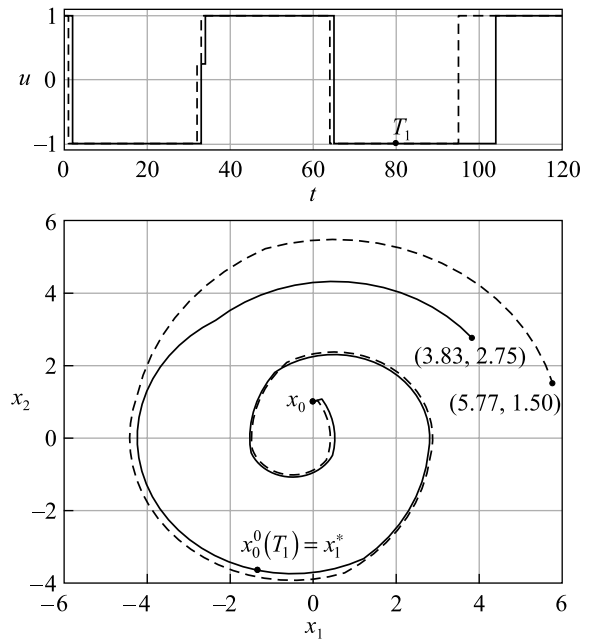instant for a sample state x_1^* (line 3)

Fig. 2. Optimal control and trajectories in example 1
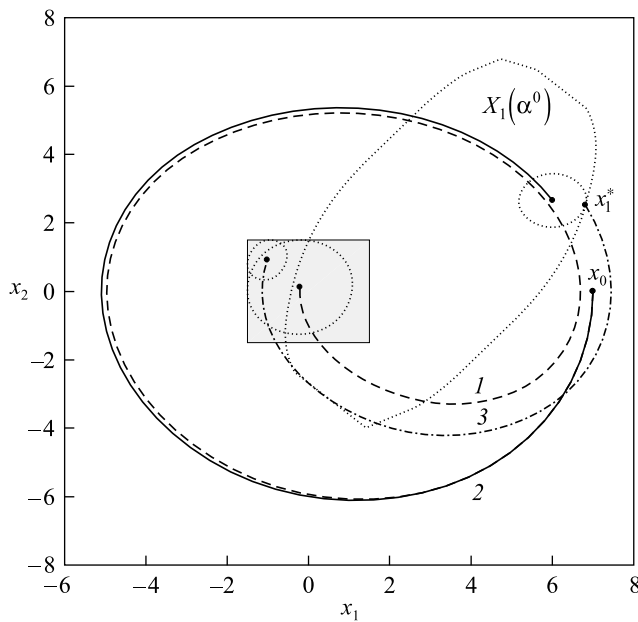under the worst-case disturbance $w^*(\cdot)$

Fig. 3. Phase-plane solution representation of example 2.
Trajectories under the optimal open-loop worst-case
control (line 1), under the optimal initial
open-loop control (line 2), and the optimal open-loop
worst-case control on the interval after the closing
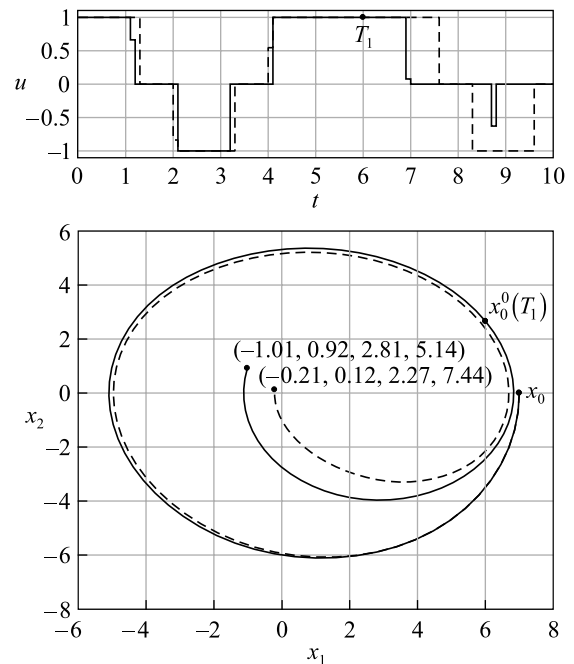instant for a sample state x_1^* (line 3)

Fig. 4. Optimal control and trajectories in example 2
under the worst-case disturbance $w^*(\cdot)$

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The optimal open-loop worst-case control $u^0(\cdot)$ in problem (23) has the optimal guaranteed value equal to $J\left(u^0\right) = 7.438\,263$. The optimal strategy gives $V\left(\pi_1^0\right) = \alpha^0 = 6.657\,643$. The time spent to construct the optimal open-loop worst-case control was 0.015 s, while for the optimal control strategy 0.139 s were spent.

Figures 3 and 4 show results for problem (23). In fig. 3 projections of the state trajectories on the phase plane $x_1 x_2$, the terminal set, sets of possible states and the intersection of the set $X_1\left(\alpha^0\right)$ by a plane $\left\{x_3 = 1.966\,4, \; x_4 = 4.166\,4\right\}$, where the point $x_0^0\left(T_1\right)$ and the set $X\left(T_1 \middle| x_0, \; u_0^0\left(\cdot \middle| x_0\right)\right)$ lie, are shown. To approximate sets $X_1(\alpha)$ we used $m_1 = 1385$ vectors. In the neighbourhood of the point $x_1^* = \left(6.813\,7, \; 2.519\,2, \; 1.966\,3, \; 4.166\,4\right)$ the approximation accuracy is $J_1\left(x_1^*\right) - \alpha^0 = 6.8 \cdot 10^{-5}$.

In fig. 4 the optimal open-loop worst-case control $u^0(\cdot)$, the optimal initial open-loop control $u_0^0\left(\cdot \middle| x_0\right)$ and the realisations of the particular optimal strategy and the corresponding trajectory in the process with the disturbance defined as in example 1, are presented. The mentioned disturbance is the worst for the optimal open-loop worst-case control $u^0(\cdot)$. The optimal strategy in the same process has the value equal to 5.136 926.

Conclusion

This paper considers a guaranteed terminal cost minimisation problem for linear discrete systems with unknown bounded disturbance. We study two types of control inputs that achieve the guaranteed constraint satisfaction and minimise the cost in the problem under consideration. The first is the optimal open-loop worst-case control that is constructed entirely before the control process starts, is not corrected during the process, and ignores any possible information about the system's future behaviour. The second is the optimal control strategy with one closing instant, where closure means taking into account a state measurement at one future time instant. The optimal control strategy consists of the optimal initial open-loop control, defined at time instances before the closure, and a collection of optimal open-loop worst-case controls, defined after the closing instant for all possible (due to disturbance and initial control) states at that closing instant. Practical application of the optimal strategy implies using the optimal initial open-loop control before the closing instant and then choosing optimal open-loop worst-case control depending on the state measurement in a particular control process.

While optimal control strategies with one closing instant for linear terminal problems were introduced in [6], the main contributions of this paper consist both in the new formulation of the problem for constructing the optimal initial open-loop control and the numerical method for its solution. The proposed formulation is a minimax optimal control problem with a cost function that is implicitly defined as the optimal value of another optimal control problem. We thoroughly elaborated the structure of this problem using the duality theory, which allowed us to reduce it to an equivalent linear program and significantly simplify the method for optimal strategy construction compared to the algorithm introduced in [6]. Numerical experiments demonstrate effectiveness of the proposed approach and superiority of the optimal control strategy over the optimal open-loop worst case control.

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