Quasi–Kähler and Hermitian f–structures on homogeneous Φ–spaces of order k[1]

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Introduction. The well known almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ (see [11], [12]) on homogeneous 3–symmetric spaces belongs [12] to the classes in *Hermitian geometry* such as *quasi–Kähler structures* (class **QK**) and *nearly Kähler* (**NK**) structures in the naturally reductive case.

An extension of these results is investigation of so called *canonical f–structures* [3] on *homogeneous Φ–spaces of order k* ($\Phi^k = id$) [4], [10] (in other terminology, *homogenous k–symmetric spaces* [7]) in *the generalized Hermitian geometry* field (see, for example, [5]). For example [1], the base canonical f–structures are *Hermitian f–structures* (class **Hf**) on naturally reductive homogeneous 4– and 5–symmetric spaces. For order $k = 6$ the necessary and sufficient conditions are known [8] under which the base canonical f–structures belong to **Hf** and *nearly Kähler f–structures* (**NKf**). Finally, the pointed results were generalized for the base canonical f–structures on arbitrary homogeneous Φ–spaces of any order k ($k \geq 3$) with naturally reductive metric [2] and for more general set of metrics [9].

Now we continue similar investigations of the canonical f–structures on arbitrary homogeneous k–symmetric spaces. This article contains new results concerning algebraic sum of the base canonical f–structures and the class **Hf**. The structures are considered also with the restriction they are almost complex structures and it's proved they belong to the class **QK** in this case.

Preliminaries. Let G/H be a homogeneous Φ–space of order k with an automorphism Φ ($\Phi^k = id$, $k \geq 3$) [4], [7], [10]. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras and $\varphi = d\Phi_e$ is automorphism in $\mathfrak{g}$ ($\varphi^k = id$). It's known [10] G/H is reductive and its canonical reductive decomposition is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Denote by $\theta = \varphi|_{\mathfrak{m}}$, $s = [\frac{k-1}{2}]$ (whole part), $u = s$ (if k is odd), $u = s + 1$ (if k is even number). Recall the decomposition of $\mathfrak{m}$ corresponding to the automorphism φ [7]:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_u, \tag{1}$$

where some of $\mathfrak{m}_i$ can be trivial. If $i + j > u$ we will denote also subspace $\mathfrak{m}_{k-(i+j)}$ by $\mathfrak{m}_{i+j}$.

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Any canonical f–structure can be represented (see [3]) as $f = (\zeta_1 J_1, \ldots, \zeta_s J_s)$, where $J_1, \ldots, J_s$ are specially defined almost complex structures ($J_i^2 = -1$) on $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$, $\zeta_i \in \{-1; 0; 1\}$, $i = \overline{1, s}$, $f|_{\mathfrak{m}_u} = 0$ for even k. If subspace $\mathfrak{m}_i$ isn't trivial, $\zeta_i = 1$ and other $\zeta_j = 0$ $(j \neq i)$, then the structure f will be denoted by f_i (i.e. f_i is the base canonical f–structure).

Observe that for $k = 2$ the next Theorem 1 yields well–known commutator relations for homogeneous symmetric spaces [6]:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h},\ [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m},\ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

Theorem 1. [2], [9] *Suppose that G/H is a homogeneous Φ space of order k $(k \geq 2)$; $\mathfrak{m}$ is the corresponding canonical reductive complement with decomposition* (1)*; $i, j = 0, 1, \ldots, u$; $i \geq j$; and $\mathfrak{m}_{i+j}$ denotes $\mathfrak{m}_{k-(i+j)}$ if $i + j > u$. Then, the following commutator relations are valid:* $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}$.

Let us specify now the set of G–invariant Riemannian metrics on the homogeneous k–symmetric spaces. Using the bijective correspondence [6] between the G–invariant metrics and the $Ad(H)$–invariant inner products on the canonical reductive complement $\mathfrak{m}$, consider the following family of invariant metrics in the case of a semisimple compact Lie algebra $\mathfrak{g}$ with Killing form B:

$$\langle X, Y \rangle = \lambda_1 B(X_1, Y_1) + \ldots + \lambda_u B(X_u, Y_u), \tag{2}$$

where $X, Y \in \mathfrak{g}$, $i = \overline{1, u}$, $X_i, Y_i \in \mathfrak{m}_i$, while $\mathfrak{m}_i$ is a summand of the decomposition (1), $\lambda_i \in \mathbb{R}, \lambda_i < 0$.

In case of the Levi–Civita connection ∇ for an invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ on the homogeneous reductive space G/H the bilinear symmetric mapping $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ for the Nomizu function [6] α is determined (see [6]) from

$$2\langle U(X, Y), Z \rangle = \langle X, [Z, Y]_{\mathfrak{m}} \rangle + \langle [Z, X]_{\mathfrak{m}}, Y \rangle \quad \forall Z \in \mathfrak{m}. \tag{3}$$

In the next theorem we establish that $U(X, Y)$ is determined by the commutator of $X, Y \in \mathfrak{m}$ in the case of homogeneous k–symmetric spaces with the metric (2).

Theorem 2. [9] *Consider a homogeneous Φ–space of order k $(k \geq 3)$ $M = G/H$ with the metric* (2)*, and suppose that the Lie algebra $\mathfrak{g}$ of G is semisimple and compact. Take arbitrary elements X_i, Y_i, Y_j of the summands $\mathfrak{m}_i$ and $\mathfrak{m}_j$ in* (1) *for $i, j = \overline{1, u}$ with $i > j$. Then U satisfies*

$$U(X_i, Y_j)_{\mathfrak{m}_{i\pm j}} = \frac{\lambda_j - \lambda_i}{2\lambda_{i\pm j}} [X_i, Y_j]_{\mathfrak{m}_{i\pm j}}, \quad U(X_i, Y_i) = U(X_i, Y_j)_{\mathfrak{m}_n} = 0,$$

where $\mathfrak{m}_{i+j}$ with $i+j>u$ stands for $\mathfrak{m}_{k-(i+j)}$, while λ_{i+j} with $i+j>u$ stands for $\lambda_{k-(i+j)}$, and $\mathfrak{m}_n$ is an arbitrary summand of (1) *except for $\mathfrak{m}_{i-j}$ and $\mathfrak{m}_{i+j}$.*

New Results. The next theorems are proved taking into account Theorem 2, commutator and other useful relations (see [9]) for the homogeneous k–symmetric spaces. Note, that the theorems are formulated for the metrics (2) where Lie algebra $\mathfrak{g}$ is semisimple and compact. However, if we take arbitrary homogeneous naturally reductive k–symmetric space then the theorems are also applicable without semisimpleness and compactness requirements. Let us consider the class **Hf** defined by the condition $T(X,Y)=0$ where T is *composition tensor* and [5]

$$T(X,Y)=\frac{1}{4}f\left(\nabla_{fX}(f)fY-\nabla_{f^2X}(f)f^2Y\right),$$

where ∇ is the Levi–Civita connection of a (pseudo)Riemannian manifold (M,g), $X,Y\in\mathfrak{X}(M)$. For this class we have

Theorem 3. [9] *Let $M=G/H$ be a homogeneous Φ–space of order k with the metric* (2)*. Then for every base canonical f–structure f_i on M the following statements hold:*

if $3i\neq k$ then f_i is of the class **Hf***;*

*if $3i=k$ then $f_i\in$ **Hf** $\Leftrightarrow[\mathfrak{m}_i,\mathfrak{m}_i]\subset\mathfrak{h}$.*

Theorem 4. *Let $M=G/H$ be a homogeneous Φ–space of order k with the metric* (2) *and f_i, f_j are arbitrary base canonical f–structures on M with $i>j$. The structure $f_i-f_j\in$ **Hf** iff both conditions are satisfied:*

1) *The structures f_i and f_j are of the class* **Hf**.

2) *$i\neq 2j$ or both $[\mathfrak{m}_j,\mathfrak{m}_j]\subset\mathfrak{h}$ and $[\mathfrak{m}_i,\mathfrak{m}_j]\subset\mathfrak{m}_{i+j}$.*

Theorem 5. *Let $M=G/H$ be a homogeneous Φ–space of order k with the metric* (2) *and f_i, f_j are arbitrary base canonical f–structures on M with $i>j$. The structure $f_i+f_j\in$ **Hf** iff all next conditions are satisfied:*

1) *The structures f_i and f_j are of the class* **Hf**.

2) *$2i+j\neq k$ or both $[\mathfrak{m}_i,\mathfrak{m}_i]\subset\mathfrak{h}$ and $[\mathfrak{m}_i,\mathfrak{m}_j]\subset\mathfrak{m}_{i-j}$.*

3) *$i+2j\neq k$ or both $[\mathfrak{m}_j,\mathfrak{m}_j]\subset\mathfrak{h}$ and $[\mathfrak{m}_i,\mathfrak{m}_j]\subset\mathfrak{m}_{i-j}$.*

For the class **QK** of Hermitian geometry defined for an *almost Hermitian structure J* by well–known condition

$$\nabla_X(J)Y-\nabla_{JX}(J)JY=0,$$

where ∇,X,Y are the same as for composition tensor T, we have

Theorem 6. *Let $M=G/H$ be a homogeneous Φ–space of order k with the metric* (2)*, f_i is a base canonical f–structure on M and f_i is almost complex structure* (*i.e. all subspaces of decomposition* (1) *are trivial except $\mathfrak{m}_i$ and, probably, $\mathfrak{m}_0=\mathfrak{h}$*)*. Then $f_i\in$* **QK**.

It's easy to conclude from "almost complex structure" condition and Theorem 1 that $M = G/H$ is a locally symmetric space (i.e. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$) if $3i \neq k$ in Theorem 6. However, we don't have this conclusion if $3i = k$ and the base canonical f–structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ from Introduction is an example illustrating it.

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Ограниченно однородные по Клиффорду-Вольфу римановы многообразия[1]

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Введение

Начнем с определения важных классов изометрий и соответствующих метрических пространств.

Определение 1 ([9], [4]). *Пусть (X, d) — метрическое пространство. Изометрия f пространства (X, d) на себя называется переносом Клиффорда-Вольфа, кратко КВ-переносом, если f смещает все точки (X, d) на одно и то же расстояние, т. е. $d(y, f(y)) = d(x, f(x))$ для всех $x, y \in X$.*

Определение 2 ([9], [8]). *(Полное) метрическое пространство (X, d) называется (ограниченно) однородным по Клиффорду-Вольфу, кратко (ограниченно) КВ-однородным, если для всех $x, y \in X$ (соответственно, всех x, y из открытого шара $B(z, r(z))$, $r(z) > 0$, с произвольным центром $z \in X$) существует перенос Клиффорда-Вольфа пространства (X, d), перемещающий x в y.*

В статье [8] доказана следующая классификационная теорема.

Теорема 1. *Односвязное (связное) риманово многообразие КВ-однородно тогда и только тогда, когда оно изометрично прямому метрическому произведению некоторого евклидова пространства, нечетномерных сфер постоянной кривизны и односвязных компактных простых*

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