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Canonical structures and distributions on spaces with symmetries of order k[1]

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Introduction. Idea of symmetry is very important and fruitful in natural sciences, specifically, in mathematics. In this respect, theory of symmetric spaces plays a remarkable role in many branches of mathematics. More general, among homogeneous manifolds of Lie groups there exists a wide and

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very interesting class of spaces with symmetries of order k, i.e *homogeneous k–symmetric spaces*, which are homogeneous spaces generated by Lie groups automorphisms Φ of order k ($\Phi^k = id$) [7].

Any homogeneous k–symmetric space $(G/H, \Phi)$ admits the commutative algebra $\mathcal{A}(\theta)$ [3] of *canonical affinor structures*. This algebra contains well-known classical structures such as almost complex structures, almost product structures, f–structures of K. Yano ($f^3 + f = 0$) etc. (see [3], [5]). The main feature of the canonical structures is their invariance with respect to the symmetries of order k of the k–symmetric space $(G/H, \Phi)$.

Here we present several new results on invariant distributions generated by canonical almost product structures on naturally reductive k–symmetric spaces. Besides, using canonical structures, we construct four left-invariant metric f–structures on the 6–dimensional generalized Heisenberg group and provide new invariant examples for the classes of nearly Kähler and Hermitian f–structures as well as almost Hermitian G_1–structures.

Canonical structures on k–symmetric spaces. Let G be a connected Lie group, Φ its (analytic) automorphism, G/H a *homogeneous Φ–space* [3], [4], i.e. G/H is generated by the Lie group automorphism Φ [11]. In the case $\Phi^k = id$ the pair $(G/H, \Phi)$ is a homogeneous Φ–space of *order k* or, in the other terminology, *homogeneous k–symmetric space* (see [7]). The special case $k = 2$ leads to homogeneous symmetric spaces.

For any homogeneous Φ–space G/H one can define [9] the analytic diffeomorphism $S_o\colon\ G/H \to G/H,\ xH \to \Phi(x)H$, which is usually called a "symmetry"of G/H at the point $o = H$. In view of homogeneity the "symmetry"S_p can be defined at any point $p \in G/H$. This implies that any homogeneous k–symmetric space is a space with *symmetries of order k*.

Let G/H be a homogeneous Φ–space of order k, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - id$. Recall [9] that G/H is a reductive space for which the corresponding *canonical reductive decomposition* is of the form: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},\ \mathfrak{m} = A\mathfrak{g}$. Besides, this decomposition is obviously φ–invariant. Denote by θ the restriction of φ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$.

Recall [3] that an invariant affinor structure F on a homogeneous Φ–space G/H of order k is called *canonical* if its value at the point $o = H$ is a polynomial in θ. It follows that any canonical structure is invariant, in addition, with respect to the "symmetries"$\{S_p\}$ of G/H. The set $\mathcal{A}(\theta)$ of all canonical affinor structures on $(G/H, \Phi)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. Evidently, the algebra $\mathcal{A}(\theta)$ for any symmetric Φ–space ($\Phi^2 = id$) is trivial, i.e. it is isomorphic to $\mathbb{R}$.

However, the algebra $\mathcal{A}(\theta)$ for homogeneous Φ–spaces of order k ($k \geq 3$) contains a rich collection of classical structures such as almost complex structures J ($J^2 = -1$), almost product structures P ($P^2 = 1$), f–structures ($f^3 + f = 0$), h–structures ($h^3 - h = 0$). All these canonical structures on homogeneous k–symmetric spaces were completely described [3], [5]. Note that the first and the most remarkable example of canonical structures is the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on homogeneous 3–symmetric spaces (N.A.Stepanov, J.A.Wolf-A.Gray).

Below we illustrate some new applications of canonical structures to the theory of Riemannian almost product structures as well as to Hermitian and generalized Hermitian geometry.

Canonical distributions on k–symmetric spaces. Any Riemannian almost product manifold (M, g, P) naturally admits two complementary mutually orthogonal distributions corresponding to the eigenvalues 1 and -1 of P. They are usually called *vertical* **V** and *horizontal* **H** respectively. In accordance with the Naveira classification [8] there are 36 classes of Riemannian almost product structures (8 types for each of distributions). It was proved [5] that, in accordance with the classification, there are exactly three classes of invariant naturally reductive almost product structures. They are (**TGF**, **TGF**), (**TGF**, **AF**), (**AF**, **AF**), where ***TGF*** is a *totally geodesic foliation*, ***AF*** is an *anti-foliation*.

Let G/H be a homogeneous k–symmetric space. Denote by $s = [\frac{k-1}{2}]$ (integer part), $u = s$ (if k is odd), $u = s + 1$ (if k is even number). Consider the corresponding canonical reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_u,$$

where subspaces $\mathfrak{m}_i$ ($i = \overline{1, u}$) are determined by the spectrum of the operator θ. Denote by P_i the *base* canonical almost product structure, which is *id* on the $\mathfrak{m}_i$ and $-id$ on the other subspaces.

The following results were obtained:

Theorem 1. *Let $(G/H, g)$ be a naturally reductive Φ–space of order $k = 2n, n \geq 2$ such that a subspace $\mathfrak{m}_n$ corresponding to the eigenvalue -1 of the operator θ is non-trivial. Then a canonical invariant distribution on G/H generated by the subspace $\mathfrak{m}_n$ is of type **TGF**. In other words, the canonical almost product structure P_n belongs to the class (**TGF**, **AF**).*

Theorem 2. *Let $(G/H, g)$ be a naturally reductive homogeneous k–symmetric space. Suppose P_i, $i = \overline{1, u}$ is a base canonical almost product structure such that for index i the following system of conditions is satisfied for any $j \neq i$:*

*$k = 3i, \quad 2i \neq k - j, \quad 2i \neq j$. Then the structure P_i belongs to the class (**TGF**, **AF**).*

All the canonical structures P for orders $k = 5, 6, 7$ were characterized in this sense (the case $k = 4$ was already studied [5]).

Canonical f–structures on the 6–dimensional generalized Heisenberg group. Canonical f–structures on homogeneous k–symmetric spaces play an important role in the *generalized Hermitian geometry* [6]. More exactly, these structures provide a wealth of invariant examples for main classes of metric f–structures (see, e.g., [5], [2]).

In this respect, the 6–dimensional generalized Heisenberg group (N, g) is of especial interest. Specifically, (N, g) can be simultaneously represented as Riemannian homogeneous k–symmetric spaces for $k = 3, 4$, and 6, where the metric g is not naturally reductive (see, e.g., [10], [5], [1]). We concentrate on the four left-invariant metric canonical f–structures on the Riemannian homogeneous 6–symmetric space (N, g). Two of them, f_1 and f_2, are base metric f–structures, the other two $f_3 = f_1 + f_2 = J$ and $f_4 = f_1 - f_2 = \tilde{J}$ are almost Hermitian structures. We notice that the structure $\tilde{J}$ is just the canonical almost complex structure for 3–symmetric space (N, g) [10]. Besides, the structure f_1 coincides with the canonical f–structure for the corresponding 4–symmetric space (see [5]). Thus, these structures were investigated before. Here we formulate the results for the structures f_2 and J.

Theorem 3. *Let (N, g) be the 6–dimensional generalized Heisenberg group considered as the Riemannian homogeneous 6–symmetric space. Then the canonical structure f_2 is a non-integrable nearly Kähler and Hermitian f–structure on the manifold N, but f_2 is not a Killing f–structure.*

Theorem 4. *The 6–dimensional generalized Heisenberg group (N, g) is a G_1–manifold with respect to the left-invariant canonical almost Hermitian structure $J = f_3$ of the Riemannian homogeneous 6–symmetric space (N, g, Φ). Besides, the structure J is neither nearly Kähler nor Hermitian structure on the manifold (N, g).*

We note that more detailed and some additional information can be found in [1].

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