List of bibliography


Canonical structures and distributions on spaces with symmetries of order $k$\footnote{This research was partially supported by the Belarus Republic Foundation for Basic Research (project F10R–132) in the framework of the joint BRFBR–RFBR project.}

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Introduction. Idea of symmetry is very important and fruitful in natural sciences, specifically, in mathematics. In this respect, theory of symmetric spaces plays a remarkable role in many branches of mathematics. More general, among homogeneous manifolds of Lie groups there exists a wide and...
very interesting class of spaces with symmetries of order $k$, i.e. homogeneous $k$–symmetric spaces, which are homogeneous spaces generated by Lie groups automorphisms $\Phi$ of order $k$ ($\Phi^k = id$) [7].

Any homogeneous $k$–symmetric space $(G/H, \Phi)$ admits the commutative algebra $\mathcal{A}(\theta)$ [3] of canonical affinor structures. This algebra contains well-known classical structures such as almost complex structures, almost product structures, $f$–structures of K. Yano ($f^3 + f = 0$) etc. (see [3], [5]). The main feature of the canonical structures is their invariance with respect to the symmetries of order $k$ of the $k$–symmetric space $(G/H, \Phi)$.

Here we present several new results on invariant distributions generated by canonical almost product structures on naturally reductive $k$–symmetric spaces. Besides, using canonical structures, we construct four left-invariant metric $f$–structures on the $6$–dimensional generalized Heisenberg group and provide new invariant examples for the classes of nearly Kähler and Hermitian $f$–structures as well as almost Hermitian $G_1$–structures.

Canonical structures on $k$–symmetric spaces. Let $G$ be a connected Lie group, $\Phi$ its (analytic) automorphism, $G/H$ a homogeneous $\Phi$–space [3], [4], i.e. $G/H$ is generated by the Lie group automorphism $\Phi$ [11]. In the case $\Phi^k = id$ the pair $(G/H, \Phi)$ is a homogeneous $\Phi$–space of order $k$ or, in the other terminology, homogeneous $k$–symmetric space (see [7]). The special case $k = 2$ leads to homogeneous symmetric spaces.

For any homogeneous $\Phi$–space $G/H$ one can define [9] the analytic diffeomorphism $S_p: G/H \to G/H$, $xH \to \Phi(x)H$, which is usually called a "symmetry"of $G/H$ at the point $o = H$. In view of homogeneity the "symmetry"$S_p$ can be defined at any point $p \in G/H$. This implies that any homogeneous $k$–symmetric space is a space with symmetries of order $k$.

Let $G/H$ be a homogeneous $\Phi$–space of order $k$, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for $G$ and $H$, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - id$. Recall [9] that $G/H$ is a reductive space for which the corresponding canonical reductive decomposition is of the form:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} = A\mathfrak{g}.$$  

Besides, this decomposition is obviously $\varphi$–invariant. Denote by $\theta$ the restriction of $\varphi$ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$.

Recall [3] that an invariant affinor structure $F$ on a homogeneous $\Phi$–space $G/H$ of order $k$ is called canonical if its value at the point $o = H$ is a polynomial in $\theta$. It follows that any canonical structure is invariant, in addition, with respect to the "symmetries"$\{S_p\}$ of $G/H$. The set $\mathcal{A}(\theta)$ of all canonical affinor structures on $(G/H, \Phi)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on $G/H$. Evidently, the algebra $\mathcal{A}(\theta)$ for any symmetric $\Phi$–space ($\Phi^2 = id$) is trivial, i.e. it is isomorphic to $\mathbb{R}$. 

Ломоносовские чтения на Алтае
However, the algebra $\mathcal{A}(\theta)$ for homogeneous $\Phi$-spaces of order $k$ ($k \geq 3$) contains a rich collection of classical structures such as almost complex structures $J$ ($J^2 = -1$), almost product structures $P$ ($P^2 = 1$), $f$-structures ($f^3 + f = 0$), $h$-structures ($h^3 - h = 0$). All these canonical structures on homogeneous $k$-symmetric spaces were completely described [3], [5]. Note that the first and the most remarkable example of canonical structures is the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on homogeneous 3-symmetric spaces (N.A.Steanov, J.A.Wolf-A.Gray).

Below we illustrate some new applications of canonical structures to the theory of Riemannian almost product structures as well as to Hermitian and generalized Hermitian geometry.

**Canonical distributions on $k$-symmetric spaces.** Any Riemannian almost product manifold $(M, g, P)$ naturally admits two complementary mutually orthogonal distributions corresponding to the eigenvalues 1 and $-1$ of $P$. They are usually called vertical $V$ and horizontal $H$ respectively. In accordance with the Naveira classification [8] there are 36 classes of Riemannian almost product structures (8 types for each of distributions). It was proved [5] that, in accordance with the classification, there are exactly three classes of invariant naturally reductive almost product structures. They are $(\mathrm{TGF}, \mathrm{TGF})$, $(\mathrm{TGF}, \mathrm{AF})$, $(\mathrm{AF}, \mathrm{AF})$, where $\mathrm{TGF}$ is a totally geodesic foliation, $\mathrm{AF}$ is an anti-foliation.

Let $G/H$ be a homogeneous $k$-symmetric space. Denote by $s = \lfloor \frac{k+1}{2} \rfloor$ (integer part), $u = s$ (if $k$ is odd), $u = s + 1$ (if $k$ is even number). Consider the corresponding canonical reductive decomposition

$$g = h \oplus m = m_0 \oplus m = m_0 \oplus m_1 \oplus \ldots \oplus m_u,$$

where subspaces $m_i$ ($i = 1, u$) are determined by the spectrum of the operator $\theta$. Denote by $P_i$ the base canonical almost product structure, which is $id$ on the $m_i$ and $-id$ on the other subspaces.

The following results were obtained:

**Theorem 1.** Let $(G/H, g)$ be a naturally reductive $\Phi$-space of order $k = 2n, n \geq 2$ such that a subspace $m_n$ corresponding to the eigenvalue $-1$ of the operator $\theta$ is non-trivial. Then a canonical invariant distribution on $G/H$ generated by the subspace $m_n$ is of type $\mathrm{TGF}$. In other words, the canonical almost product structure $P_n$ belongs to the class $(\mathrm{TGF}, \mathrm{AF})$.

**Theorem 2.** Let $(G/H, g)$ be a naturally reductive homogeneous $k$-symmetric space. Suppose $P_i$, $i = 1, u$ is a base canonical almost product structure such that for index $i$ the following system of conditions is satisfied for any $j \neq i$: 
$k = 3i, \quad 2i \neq k - j, \quad 2i \neq j$. Then the structure $P_i$ belongs to the class $(TGF, AF)$.

All the canonical structures $P$ for orders $k = 5, 6, 7$ were characterized in this sense (the case $k = 4$ was already studied [5]).

**Canonical $f$–structures on the 6–dimensional generalized Heisenberg group.** Canonical $f$–structures on homogeneous $k$–symmetric spaces play an important role in the generalized Hermitian geometry [6]. More exactly, these structures provide a wealth of invariant examples for main classes of metric $f$–structures (see, e.g., [5], [2]).

In this respect, the 6–dimensional generalized Heisenberg group $(N, g)$ is of especial interest. Specifically, $(N, g)$ can be simultaneously represented as Riemannian homogeneous $k$–symmetric spaces for $k = 3, 4, 6$, where the metric $g$ is not naturally reductive (see, e.g., [10], [5], [1]). We concentrate on the four left-invariant metric canonical $f$–structures on the Riemannian homogeneous 6–symmetric space $(N, g)$. Two of them, $f_1$ and $f_2$, are base metric $f$–structures, the other two $f_3 = f_1 + f_2 = J$ and $f_4 = f_1 - f_2 = \tilde{J}$ are almost Hermitian structures. We notice that the structure $J$ is just the canonical almost complex structure for 3–symmetric space $(N, g)$ [10]. Besides, the structure $f_1$ coincides with the canonical $f$–structure for the corresponding 4–symmetric space (see [5]). Thus, these structures were investigated before. Here we formulate the results for the structures $f_2$ and $J$.

**Theorem 3.** Let $(N, g)$ be the 6–dimensional generalized Heisenberg group considered as the Riemannian homogeneous 6–symmetric space. Then the canonical structure $f_2$ is a non-integrable nearly Kähler and Hermitian $f$–structure on the manifold $N$, but $f_2$ is not a Killing $f$–structure.

**Theorem 4.** The 6–dimensional generalized Heisenberg group $(N, g)$ is a $G_1$–manifold with respect to the left-invariant canonical almost Hermitian structure $J = f_3$ of the Riemannian homogeneous 6–symmetric space $(N, g, \Phi)$. Besides, the structure $J$ is neither nearly Kähler nor Hermitian structure on the manifold $(N, g)$.

We note that more detailed and some additional information can be found in [1].

**List of bibliography**


