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Sums of nearly Kähler f-structures on homogeneous Φ-spaces of order k

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Introduction. We continue investigation of *canonical f–structures* [5] on *homogeneous Φ–spaces of order k* ($\Phi^k = id$) [6], [12] (also known as *homogenous k–symmetric spaces* [9]) in *the generalized Hermitian geometry* field (see, for example, [7]).

Recent results of the investigations extend some facts for the well known almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ (see [13], [14]) on homogeneous 3–symmetric spaces in *Hermitian geometry* and for canonical f–structures on naturally reductive homogeneous 4– and 5–symmetric spaces (see [2]). For example, any base canonical f–structure belongs to *nearly Kähler f–structures* (**NKf**) on arbitrary homogeneous Φ–space of any order k ($k \geq 3$) with naturally reductive metric [4] (see [10] for $k = 6$) and for more general set of metrics [11]. The papers [10], [4], [11] also contain necessary and sufficient conditions under which the sum and difference of two base canonical f–structures belong to the class **NKf**.

Let us consider a sum of three or more base canonical f–structures. It is clear that if each pair from the sum is *NKf*–structure then the entire sum belongs to the class **NKf**. The converse is not true in general. Thus this article indicates appropriate necessary and sufficient conditions for a sum of three base canonical f–structures and describes some special cases of the pointed theorem.

Preliminaries. Let G be a connected Lie group with an automorphism Φ. Denote by G^Φ the fixed points subgroup of Φ and by G^Φ_o the identity component of G^Φ. If a closed subgroup H of G satisfies $G^\Phi_o \subset H \subset G^\Phi$ then G/H is called a *homogeneous Φ–space* [12], [6].

Homogeneous Φ–spaces include *homogeneous Φ–spaces of order k* ($\Phi^k = id$) [6], [9], [12] which contain well known homogeneous symmetric spaces ($k = 2$, $\Phi^2 = id$) and homogeneous 3–symmetric spaces ($k = 3$, $\Phi^3 = id$).

Let consider homogeneous Φ–spaces G/H of order k and point some facts for them. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ Lie algebras for G and H respectively and let

$\varphi = d\Phi_e$ be the automorphism in $\mathfrak{g}$ ($\varphi^k = id$). It's known [12] G/H is reductive and its canonical reductive decomposition is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Denote by $\theta = \varphi|_{\mathfrak{m}}$, $s = [\frac{k-1}{2}]$ (integer part), $u = [\frac{k}{2}]$ (i.e. $u = s$ if k is odd and $u = s+1$ otherwise). Recall the decomposition of $\mathfrak{m}$ corresponding to the automorphism φ [9]:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_u, \qquad (1)$$

where some of $\mathfrak{m}_i$ can be trivial. We also will denote a subspace $\mathfrak{m}_{k-(i+j)}$ by $\mathfrak{m}_{i+j}$ if $i + j > u$ in the next theorems.

Any canonical f–structure can be represented (see [3], the definition of canonical structures is in [5]) as

$$f = (\zeta_1 J_1, \dots, \zeta_s J_s),$$

where $J_1, \dots, J_s$ are specially defined almost complex structures ($J_i^2 = -1$) on $\mathfrak{m}_1, \dots, \mathfrak{m}_s$, $\zeta_i \in \{-1; 0; 1\}$, $i = \overline{1, s}$, $f|_{\mathfrak{m}_u} = 0$ for even k. If subspace $\mathfrak{m}_i$ isn't trivial, $\zeta_i = 1$ and other $\zeta_j = 0$ ($j \neq i$), then the structure f will be denoted by f_i (i.e. f_i is the base canonical f–structure).

We will use the next Theorem 1 to prove new results. Observe that for $k = 2$ it yields well–known commutator relations for homogeneous symmetric spaces [8]:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

Theorem 1. [4], [11] *Suppose that G/H is a homogeneous Φ space of order k ($k \geq 2$); $\mathfrak{m}$ is the corresponding canonical reductive complement with decomposition (1); $i, j = 0, 1, \dots, u$; $i \geq j$; and $\mathfrak{m}_{i+j}$ denotes $\mathfrak{m}_{k-(i+j)}$ if $i + j > u$. Then, the following commutator relations are valid:*

$$[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}.$$

Let consider now the set of G–invariant Riemannian metrics on a homogeneous Φ–spaces G/H of order k in the case of semisimple compact Lie algebra $\mathfrak{g}$ with Killing form B. Using the bijective correspondence [8] between the G–invariant metrics and the $Ad(H)$–invariant inner products on the canonical reductive complement $\mathfrak{m}$ let take the next family:

$$\langle X, Y \rangle = \lambda_1 B(X_1, Y_1) + ... + \lambda_u B(X_u, Y_u), \qquad (2)$$

where $X, Y \in \mathfrak{g}$, $i = \overline{1, u}$, $X_i, Y_i \in \mathfrak{m}_i$, while $\mathfrak{m}_i$ is a summand of the decomposition (1), $\lambda_i \in \mathbb{R}, \lambda_i < 0$.

The bilinear symmetric mapping $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ for the Nomizu function [8] α is determined (see [8]) from

$$2\langle U(X,Y), Z\rangle = \langle X, [Z,Y]_{\mathfrak{m}}\rangle + \langle [Z,X]_{\mathfrak{m}}, Y\rangle \quad \forall Z \in \mathfrak{m} \tag{3}$$

in case of the Levi–Civita connection ∇ for an invariant Riemannian metric $g = \langle\cdot,\cdot\rangle$ on the homogeneous reductive space G/H.

We establish in Theorem 2 that $U(X,Y)$ is determined by the commutator of $X, Y \in \mathfrak{m}$ in the case of homogeneous k–symmetric spaces with the metric (2).

Theorem 2. [11] *Consider a homogeneous Φ–space of order k ($k \geq 3$) $M = G/H$ with the metric* (2)*, and suppose that the Lie algebra $\mathfrak{g}$ of G is semisimple and compact. Take arbitrary elements X_i, Y_i, Y_j of the summands $\mathfrak{m}_i$ and $\mathfrak{m}_j$ in* (1) *for $i, j = \overline{1,u}$ with $i > j$. Then U satisfies*

$$U(X_i, Y_j)_{\mathfrak{m}_{i\pm j}} = \frac{\lambda_j - \lambda_i}{2\lambda_{i\pm j}}[X_i, Y_j]_{\mathfrak{m}_{i\pm j}}, \quad U(X_i, Y_i) = U(X_i, Y_j)_{\mathfrak{m}_n} = 0,$$

where $\mathfrak{m}_{i+j}$ with $i + j > u$ stands for $\mathfrak{m}_{k-(i+j)}$, while λ_{i+j} with $i + j > u$ stands for $\lambda_{k-(i+j)}$, and $\mathfrak{m}_n$ is an arbitrary summand of (1) *except for $\mathfrak{m}_{i-j}$ and $\mathfrak{m}_{i+j}$.*

Finally, let point defining property for *NKf*–structures [1]:

$$\nabla_{fX}(f)fX = 0, \tag{4}$$

where f is a *metric f–structure* on a (pseudo)Riemannian manifold (M, g), ∇ is the Levi-Civita connection of (M, g), $X, Y \in \mathfrak{X}(M)$.

New Results. The results are formulated for a sum $f_v + f_w + f_z$ of three base canonical f–structures f_v, f_w, f_z. Similar results can be received for f–structures $f_v + f_w - f_z$, $f_v - f_w + f_z$ and $f_v - f_w - f_z$.

Let us remind first the recent necessary and sufficient conditions for a sum of two canonical f–structures and class **NKf**.

Theorem 3. [11] *Consider a homogeneous Φ–space $M = G/H$ of order k with the metric* (2) *and arbitrary base canonical f–structures f_i and f_j on M, with $i > j$. The structure $f_i + f_j$ is of class* **NKf** *if and only if two conditions simultaneously hold:*

1) $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j}$ *or both* $i = 2j$ *and* $\lambda_i = 2\lambda_j$.

2) $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i-j}$ *or* $\lambda_i = \lambda_j$.

Using similar approach as for Theorem 3 (i.e. the expression 4 is analyzed taking into account Theorem 2, commutator and other helpful relations from [11] for the homogeneous k–symmetric spaces) we prove the theorem below for a sum of three base canonical f–structures.

Theorem 4. *Consider a homogeneous* Φ*–space* $M = G/H$ *of order* k *with the metric* (2) *and arbitrary base canonical* f*–structures* f_u*,* f_w*,* f_z *on* M*, with* $u > w > z$*. The structure* $f_u + f_w + f_z$ *is of class* **NKf** *if and only if for each triple* (i, j, t) *from the set* $\{(u, w, z), (u, z, w), (w, z, u)\}$ *two conditions simultaneously hold:*

1) $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j}$ *or both* $i = 2j$ *and* $\lambda_i = 2\lambda_j$
or both $t = i - j$ *and* $\lambda_t = \lambda_i - \lambda_j$*.*

2) $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i-j}$ *or* $\lambda_i = \lambda_j$*.*

So, the only new condition in Theorem 4 is $t = i - j$ and $\lambda_t = \lambda_i - \lambda_j$. It allows to additionally vary metrics (2) to find an *NKf*–structure among canonical f–structures.

For example, let consider order $k = 7$ or $k = 8$ in Theorem 4. We have only three base canonical f–structure f_1, f_2 and f_3 in these cases. If we take $\lambda_2 = 2\lambda_1$ and $\lambda_3 = 3\lambda_1$ then the first condition from Theorem 4 is automatically satisfied and only condition $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i-j}$ should be verified for the taken set of coefficients λ.

If we take a naturally reductive metric (i.e. $\lambda_i = \lambda_j$ for all i, j in the expression (2) and Theorem 4) then only condition $[\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j}$ should be verified. Moreover, the structure $f_u + f_w + f_z$ is of class **NKf** in this case if and only if each pair $f_u + f_w$, $f_u + f_z$, $f_w + f_z$ from the sum is *NKf*–structure.

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