Секция 4. Анализ, геометрия и топология

Invariant nearly Kähler f-structures on regular Φ-spaces[1]

V.V. Balashchenko
Belarusian State University
balashchenko@bsu.by; vitbal@tut.by

Introduction. Homogeneous Φ-spaces [4] (generalized symmetric spaces [9]) form a remarkable class of homogeneous manifolds and play an important role in differential geometry and its applications. The important feature is that any homogeneous *regular* Φ-space $(G/H, \Phi)$ admits the commutative algebra $\mathcal{A}(\theta)$ [3] of *canonical affinor structures*. This algebra contains classical structures such as almost complex structures, almost product structures, f–structures of K. Yano ($f^3 + f = 0$) etc. (see [3], [7]).

It is a classical result [10] that *homogeneous k–symmetric spaces* are included into the class of regular Φ-spaces. Besides, it turned out that canonical f–structures on homogeneous k–symmetric spaces provide large classes of invariant examples for Hermitian and generalized Hermitian geometry (see [1], [7], [2] and some others). It should be mentioned that first results in this direction were obtained for homogeneous 3–symmetric spaces [12], [5].

Here we present new results for the most general case of arbitrary Riemannian regular Φ-spaces. More exactly, we indicate the base canonical f–structures on naturally reductive regular Φ-spaces, which are nearly Kähler f-structures. As a particular case, it follows the corresponding result for naturally reductive homogeneous k–symmetric spaces [2].

Canonical structures on regular Φ-spaces. Let G be a connected Lie group, Φ its (analytic) automorphism, G^Φ the subgroup of all fixed points of Φ, and G^Φ_o the identity component of G^Φ. Suppose a closed subgroup H of G satisfies the condition

$$G^\Phi_o \subset H \subset G^\Phi.$$

[1]This research was partially supported by the Belarus Republic Foundation for Basic Research (project F10R–132) in the framework of the joint BRFBR–RFBR project "Spaces with symmetries"and the Belarus State Research Program "Convergence sub-Program "Mathematical Methods".

Then G/H is called a *homogeneous* Φ*-space* [4].

Homogeneous Φ-spaces include homogeneous symmetric spaces ($\Phi^2 = id$) and, more general, *homogeneous* Φ*-spaces of order* k ($\Phi^k = id$) or, in the other terminology, *homogeneous* k*-symmetric spaces* [9].

For any homogeneous Φ–space G/H one can define the mapping

$$S_o = D\colon\ G/H \to G/H,\ xH \to \Phi(x)H.$$

It is known [10] that S_o is an analytic diffeomorphism of G/H. S_o is usually called a "symmetry"of G/H at the point $o = H$. It is evident that in view of homogeneity the "symmetry"S_p can be defined at any point $p \in G/H$.

The class of homogeneous Φ–spaces is very large and contains even non-reductive homogeneous spaces. Now we recall the definition of a regular Φ–space first introduced in [10]. Let G/H be a homogeneous Φ-space, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - id$ and the Fitting decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to A, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote 0- and 1-component of the decomposition respectively. It is clear that $\mathfrak{h} = Ker\, A$, $\mathfrak{h} \subset \mathfrak{g}_0$. A homogeneous Φ-space G/H is called a *regular* Φ*-space* if $\mathfrak{h} = \mathfrak{g}_0$.

It was proved in [10] that any homogeneous k-symmetric space is a regular Φ-space and any regular Φ-space is reductive. More exactly, the Fitting decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},\ \ \mathfrak{m} = A\mathfrak{g}$$

is a *canonical* reductive one. Besides, this decomposition is obviously φ–invariant. Denote by θ the restriction of φ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$.

An invariant affinor structure F (i.e. a tensor field of type $(1,1)$) on a homogeneous regular Φ-space G/H is called *canonical* if its value at the point $o = H$ is a polynomial in θ [3]. It follows that any canonical structure is invariant, in addition, with respect to the "symmetries"$\{S_p\}$ of G/H. Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on G/H. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. Moreover, $\dim \mathcal{A}(\theta) = deg\ \nu \ \leq\ \dim G/H$, where ν is a minimal polynomial of the operator θ. Note that the algebra $\mathcal{A}(\theta)$ for any symmetric Φ-space ($\Phi^2 = id$) consists of scalar structures only, i.e. it is isomorphic to $\mathbb{R}$.

The most remarkable example of canonical structures is the canonical almost complex structure $J = (\theta - \theta^2)/\sqrt{3}$ on a homogeneous 3-symmetric space (see [11], [12] [5] and many others). It turns out that it is not an exception. In other words, the algebra $\mathcal{A}(\theta)$ contains many affinor structures of classical types, namely, *almost complex structures* J ($J^2 = -1$), *almost product structures* P ($P^2 = 1$), f*-structures* ($f^3 + f = 0$) and some others.

All the canonical structures of classical types above mentioned on regular Φ-spaces were completely described in [3], [7]. In particular, for homogeneous k-symmetric spaces, precise computational formulae were indicated. For future reference we formulate here the result about canonical f-structures only.

Denote by $\tilde{s}$ (respectively, s) the number of all irreducible factors (respectively, all irreducible quadratic factors) over $\mathbb{R}$ of a minimal polynomial ν. A regular Φ-space G/H admits a canonical f-structure if and only if $s \neq 0$. In this case $\mathcal{A}(\theta)$ contains [3] $3^s - 1$ different f-structures. Suppose $s = \tilde{s}$. Then 2^s f-structures are almost complex and the remaining $3^s - 2^s - 1$ have non-trivial kernels.

Nearly Kähler f-manifolds. Recall that an f–structure on a (pseudo)Riemannian manifold $(M, g = \langle\cdot,\cdot\rangle)$ is called a *metric f–structure*, if $\langle fX, Y\rangle + \langle X, fY\rangle = 0,\ \ X, Y \in \mathfrak{X}(M)$ (see [8]). In this case the triple (M, g, f) is called a *metric f–manifold*. It is easy to see that the particular cases $def\ f = 0$ and $def\ f = 1$ of metric f–structures lead to almost Hermitian structures and almost contact metric structures respectively.

Let M be a metric f–manifold. Then $\mathfrak{X}(M) = \mathcal{L} \oplus \mathcal{M}$, where $\mathcal{L} = Im\ f$ and $\mathcal{M} = Ker\ f$ are mutually orthogonal distributions, which are usually called the *first* and the *second fundamental distributions* of the f–structure respectively. Denote by ∇ the Levi-Civita connection of a (pseudo)Riemannian manifold (M, g), $\ X, Y \in \mathfrak{X}(M)$. Recall that a metric f-structure on (M, g) is called a *nearly Kähler f-structure* (briefly, NKf-structure) if $\nabla_{fX}(f)fX = 0$ for any smooth vector field X on M [1]. The class of all NKf-structures is denoted by **NKf**. Besides, **Kill f** means the class of all *Killing f-structures*, which are defined by the stronger condition $\nabla_X(f)X = 0$ [6]. It is important to note that in the case $f = J$ the classes **Kill f** and **NKf** coincide with the well-known class **NK** of *nearly Kähler structures*.

Canonical nearly Kähler f-structures. Let G/H be a regular Φ-space. In accordance with the structure of the minimal polynomial ν of the operator θ, we have the following canonical reductive decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{m}_1 \oplus \sum_{\alpha \in spec\ \theta} \mathfrak{m}_\alpha.$$

Theorem 1. *For this decomposition we have*

$$[\mathfrak{m}_\alpha, \mathfrak{m}_\beta] \subset \mathfrak{m}_{\alpha\beta} + \mathfrak{m}_{\overline{\alpha}\beta},$$

where the subspace $\mathfrak{m}_{\alpha\beta}$ (respectively, $\mathfrak{m}_{\overline{\alpha}\beta}$) is trivial if $\alpha\beta$ (respectively, $\overline{\alpha}\beta$) doesn't belong to spec θ (over $\mathbb{C}$).

Using this result and some other facts, we obtain

Theorem 2. *Let G/H be a regular Φ-space with naturally reductive metric g, f_α the base canonical f-structure (i.e. $\mathfrak{m}_\alpha$ is an image of f_α). Suppose α satisfies any of two conditions:*
1) $mod\ \alpha = 1$; 2) $\overline{\alpha}\alpha \notin spec\,\theta$.
Then f_α is a nearly Kähler f-structure.

As a particular case, it immediately follows the result from [2]:

Corollary 1. *Any base canonical f-structure f_i on naturally reductive homogeneous k-symmetric space is a nearly Kähler f-structure.*

Библиографический список

[1] Balashchenko V.V. Invariant nearly Kähler f-structures on homogeneous spaces // Contemporary Mathematics. — 2001. — Vol. 288. — P. 263–267.

[2] Balashchenko V.V., Samsonov A.S. Nearly Kahler and Hermitian f–structures on homogeneous k–symmetric spaces // Doklady Mathematics. — 2010. — Vol. 81, no. 3. — P. 386–389. — URL: http://elib.bsu.by/handle/123456789/19333.

[3] Balashchenko V.V., Stepanov N.A. Canonical affinor structures of classical type on regular Φ–spaces // Sbornik: Mathematics. — 1995. — Vol. 186, no. 11. — P. 1551–1580.

[4] Fedenko A.S. Spaces with symmetries. — Minsk : Belarusian State University, 1977.

[5] Gray A. Riemannian manifolds with geodesic symmetries of order 3 // J. Diff. Geom. — 1972. — Vol. 7, no. 3-4. — P. 343–369.

[6] Gritsans A.S. Geometry of Killing f-manifolds // Russian Math. Surveys. — 1990. — Vol. 45, no. 4. — P. 168–169.

[7] Homogeneous spaces: theory and applications: monograph / Balashchenko V.V., Nikonorov Yu.G., Rodionov E.D., Slavsky V.V. — Hanty-Mansijsk : Polygrafist, 280. — 280 p. — http://elib.bsu.by/handle/123456789/9818.

[8] Kirichenko V.F. Methods of generalized Hermitian geometry in the theory of almost contact manifolds // J. Soviet Math. — 1988. — Vol. 42, no. 5. — P. 1885–1919.

[9] Kowalski O. Generalized symmetric spaces. – LN in Math. — Berlin, Heidelberg, New York : Springer-Verlag, 1980. — Vol. 805.

[10] Stepanov N.A. Basic facts of the theory of φ–spaces // Izv. Vyssh. Uchebn. Zaved. Mat. — 1967. — no. 3. — P. 88–95.

[11] Stepanov N.A. Homogeneous 3-cyclic spaces // Izv. Vyssh. Uchebn. Zaved. Mat. — 1967. — no. 12. — P. 65–74.

[12] Wolf J.A., Gray A. Homogeneous spaces defined by Lie group automorphisms // J. Diff. Geom. — 1968. — Vol. 2, no. 1,2. — P. 77–159.

On nonlocal equations, fluids and strings[1]

E. G. Reyes
Departamento de Matemática y Ciencia de la Computación,
Universidad de Santiago de Chile.
Casilla 307 Correo 2, Santiago, Chile
ereyes@fermat.usach.cl ; e_g_reyes@yahoo.ca

Abstract

We present results on the existence, uniqueness and regularity of solutions to the modified Camassa-Holm equation (which depends on the inverse of a second order differential operator) and to the generalized bosonic string equation (which depends on an exponential of a second order differential operator).

Camassa-Holm and geometry

The Camassa-Holm equation, [2],

$$2\,u_x\,u_{xx} + u\,u_{xxx} = u_t - u_{xxt} + 3\,u_x\,u\ , \tag{1}$$

is a bi-hamiltonian equation describing shallow water waves. It possesses classical solutions that blow-up in finite time for some regular data (i.e., it admits wave breaking, see [4]) and it also admits traveling wave solutions which are not always smooth [8].

The Camassa-Holm equation also has a geometric interpretation: it describes pseudo-spherical surfaces (see [3] and the recent review [12]).

[1]Paper issued with the financial support of Fondo Nacional de Desarrollo Científico y Tecnológico grant #1111042.