Statistical fluctuations of cooperative radiation produced by nonisochronous electrons-oscillators

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Shot noise, intrinsic to the ensemble of nonisochronous electrons-oscillators, is the cause of statistical fluctuations in cooperative radiation generated by single-pass cyclotron-resonance masers (CRMs). Autophasing time — the time required for the cooperative radiation power to peak — is the critical parameter characterizing the dynamics of electrons-oscillators interacting via the radiation field. Shot-noise related fluctuations of the autophasing time imposes appreciable limitations on the possibility of coherent summation of electromagnetic oscillations from several single-pass CRMs.

Premodulation of charged particles leads to a considerable narrowing of the autophasing time distribution function. When the number of particles N_e exceeds a certain value that depends on the degree to which the particles have been premodulated, the relative root-mean-square deviation (RMSD) of the autophasing time δ_T changes from a logarithmic dependence on N_e ($\delta_T \sim 1/\ln N_e$) to square-root ($\delta_T \sim 1/\sqrt{N_e}$). As a result, there is an increased probability of coherent summation of electromagnetic oscillations from several single-pass generators.

A slight energy spread ($\sim 4\%$) results in a twofold drop of the maximum attainable power of cooperative radiation.

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I. INTRODUCTION

R. H. Dicke in his pioneering work [1] has shown that a system of N_a number of inverted two-level atoms can spontaneously transit to the ground state during the time inversely proportional to the number of atoms N_a^{-1}. Because the total energy emitted by the collection of atoms is proportional to N_a, the radiation intensity reveals a square dependence on on N_a. This type of radiation was termed the "collective spontaneous radiation", or the "Dicke superradiance". A remarkable progress has been achieved in the understanding of this phenomenon in [2]. The authors of [2] have developed a quantum theory of superradiance for a single-mode model. The simplicity of the model enables a detailed study of the kinetics and statistical properties of superradiance. Later, it has been found [3–5] that the radiation intensity undergoes appreciable statistical fluctuations and the RMSD of the superradiative instability time T slowly decreases according to a logarithmic law as the number of particles is increased: $\delta_T \approx 1.3/\ln N_a$.

The elementary unit of the system we have just described is a two-level atom, the concept widely used in quantum electronics and optics to describe physical processes. In the excited state, the two-level atom interacting with the radiation field is also involved in relaxation processes caused by inter-particle collisions. If the radiation growth time T is much greater than the relaxation time, then the pulse of superradiance is not formed, and the particles emit incoherently. This greatly restricts the selection of active medium for experimental investigation of superradiance and design of superradiance-based short-pulse electromagnetic sources [6, 7].

In this connection, a considerable attention has been paid recently to the generation of short superradiance pulses [8, 9], also termed as "self-amplified spontaneous emission", "superradiance", or "cooperative radiation" [10], using short electron beams propagating in complex electrodynamical structures (undulators, resonators, dielectric and corrugated waveguides, or photonic crystals) [8, 9, 11–13]. In such systems, the average relaxation time that depends on the collisions between charged particles exceeds manifold the time needed for the formation of the cooperative-radiation pulse.

Cooperative radiation is generated via a nonlinear interaction of charged particles and the electromagnetic field, causing the electrons to bunch. As a result, the whole of the electron beam becomes spatially modulated. Spatial modulation is accompanied by a coherent emission of electromagnetic waves.

The possibility to produce short pulses of cooperative radiation was first substantiated in [14, 15] for free electron lasers (FELs). However, the experimentally measured output characteristics of cooperative radiation (power, radiation spectra, and energy in a pulse) seemed really stochastic, which, according to theoretical [16, 17] and experimental [18–22] studies, was closely related to the presence of the shot noise inherent in charged-particle beams. It is noteworthy that high frequency and time stability of the radiation pulse is crucial for many applications. For this reason, electron beams are premodulated at the radiation frequency to reduce the influence of shot noise on cooperative radiation [23–27].

In a microwave range, single-pass CRMs [28–30] and Cherenkov generators are more commonly used [31–33].

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The generators of cooperative radiation operate in two regimes: traveling wave [31–33] and backward wave [34–38].

The distinguishing feature of the traveling-wave regime is that the group velocity of electromagnetic waves and the velocity of charged particles are co-directional. This regime was applied in first experiments with single-pass CRMs [29, 30] and Cherenkov generators [31]. Theoretical consideration of radiation gain in short-pulse traveling-wave generators revealed an important peculiarity: the stability of the cooperative radiation parameters can be improved noticeably by injecting into the electrodynamical structure of a beam with a sharp front whose duration is comparable with the emission period [32, 33]. In this case, the Fourier transform of the beam current contains quite a significant spectral component at the radiation frequency. As a result, the generation of electromagnetic oscillations starts with coherent spontaneous emission of the whole beam instead of incoherent spontaneous emission from individual particles. As a result, the degree of fluctuations in cooperative radiation is decreased.

Though the early single-pass CRMs and Cherenkov generators operated in the traveling-wave regime, the most impressive results were obtained in experiments with single-pass backward-wave tubes generating cooperative radiation whose peak power was appreciably greater than the beam power [36–38]. For example, the peak radiated power of 1.2 GW was attained at 9.3 GHz for beam power of 0.87 GW [36]. The stability of output parameters of the cooperative-radiation pulse generated in the backward-wave regime, as in a single-pass traveling-wave generator, strongly depended on the beam front.

The analysis of theoretical and experimental works on generation of cooperative radiation reveals that the main cause of statistical spread of the output characteristics is the shot noise inherent in the electron beam. The impact it produces can be diminished by premodulating the beam at the radiation frequency [23–27]. Naturally, one wonders what degree of beam premodulation is required for certain applications. This problem would be most prevalent for high-current generators with electron beams composed of ectons, individual portions of electrons that contain up to 10^{11} elementary charge carriers [39, 40]. Taking into account the complex structure of the electron beam is a must for accurate estimations of the shot noise [41], which is essential in solving the problem of coherent summation of electromagnetic oscillations from several short-pulse sources of radiation [12, 13].

This paper considers the effect of shot noise on the generation of cooperative radiation from a premodulated short beam of particles. We shall consider this issue by the example of an ensemble of nonisochronous electrons-oscillators interacting with one another via the radiation field, used as one of the basic models for the description of nonlinear generation of electromagnetic waves in CRMs [42–47]. It has been shown in [44, 45] that in the absence of external action, the instability evolves in the ensemble of nonisochronous electrons-oscillators, accompanied at the initial stage by an exponential growth of the radiated power and autophasing of electrons-oscillators. This exponential growth is then suppressed due to nonlinearity, and the pulse of cooperative radiation is formed [44]. The influence of nonisochronism on the peak power of cooperative radiation and the autophasing time of electrons-oscillators was studied in detail by Vainstein and colleagues [45]. However, the authors of [45] assumed that the beam had been premodulated at the radiation frequency and left out the effects related to shot noise [47].

In our analysis of statistical properties of cooperative radiation, the peak radiated power and autophasing time serve as random variables. The peak power is the major output characteristic of short-pulse electromagnetic sources, and the autophasing time is the parameter defining the minimum time of particle passage through the generator needed for cooperative instability to evolve. Moreover, the RMSD in the autophasing time is of fundamental importance for coherent summation of oscillations from several cooperative-radiation sources.

The paper's outline is as follows. First, we derive a system of equations describing the interaction of nonisochronous oscillators via the radiation field by the example of electron ensemble circulating in a uniform magnetic field. Further comes a detailed consideration of statistical fluctuations of cooperative radiation in the presence of shot noise from the ensemble of nonisochronous electrons-oscillators with and without phase premodulation of charged particles. Particular attention is given to finding the autophasing time distribution function. Based on the described study, we shall then investigate in detail the statistical fluctuations in single-pass CRMs and discuss the limits imposed on the output characteristics thereof by the shot noise and the energy spread. Furthermore, we shall discuss the possibility of coherent summation of oscillations coming from several single-pass sources of cooperative cyclotron radiation.

II. COOPERATIVE CYCLOTRON RADIATION

Let us consider the behavior of a nonrelativistic electron beam in a uniform magnetic field $\vec{H}$ directed along the OZ axis in the presence of the radiative energy loss. (The behavior of the relativistic electron beam is discussed in Appendix A.) Particle velocity components perpendicular and parallel to the magnetic-filed vector are denoted by $\vec{v}_{\perp k}$ and $\vec{v}_{zk}$, respectively. Then the behavior of particles is described by the equations of motion in the form

$$\dot{\vec{p}}_{\perp k} = \frac{e}{c}\vec{v}_{\perp k} \times \vec{H} + \vec{F}_{\perp k},$$
$$\dot{p}_{zk} = F_{zk}. \tag{1}$$

Here, $\vec{F}_k$ is the force acting on the kth particle from all particles and $\vec{p}_k$ is its momentum related to the veloc-

ity $\vec{v}_k$ as

$$\vec{p}_k = \frac{m\vec{v}_k}{\sqrt{1 - v_k^2/c^2}} \approx m\vec{v}_k\left(1 + \frac{v_{\perp k}^2 + v_{zk}^2}{2c^2}\right). \quad (2)$$

If the beam size is less than the radiation wavelength $\lambda = 2\pi mc/eH$, and the Coulomb repulsion force and induction fields can be neglected (see Appendix B), then the force acting on each particle is given by the sum (see Appendix C)

$$\vec{F}_k = e\sum_j \frac{2e}{3c^3}\dddot{\vec{v}}_j, \quad (3)$$

where $\frac{2e}{3c^3}\dddot{\vec{v}}_j$ is the radiation field induced by the jth particle.

Let us pay attention to an essential circumstance [50]: the expression for $\vec{F}_k$ is true if the radiative friction force is appreciably less than the Lorentz force $\frac{e}{c}\vec{v}_k \times \vec{H}$ acting on each particle. Otherwise, unphysical self-accelerated solutions may arise.

The requirement that the radiative friction force should be much less than the Lorentz force imposes limitation on the magnetic field strength (in the opposite case the considered theory is invalid). For single particle [50]:

$$H \ll \frac{m^2c^4}{e^3} \approx 6 \cdot 10^{15} \text{ Gs.} \quad (4)$$

But in the case of a dense beam of coherently emitting particle with N_e electrons, mass $M = N_e m$, and charge $Q = N_e e$ we can write by analogy with (4)

$$H \ll H_{cr} = \frac{M^2c^4}{Q^3} = \frac{m^2c^4}{N_e e^3}. \quad (5)$$

With the present-day acceleration facilities, dense beams with $N_e \sim 10^{10}$ electrons are available; substitution of $N_e \sim 10^{10}$ into (5) gives $H \ll 60$ kGs. If this condition is violated, the equation set (1) with the force (3) is inapplicable.

Thus, if the radiation reaction force is much less than the Lorentz force and the beam size is less than the radiation wavelength, then the equations of motion describing the interaction between charged particles have the form:

$$\dot{p}_{\perp k} = \frac{e}{c}\vec{v}_k \times \vec{H} + \frac{2e^2 N_e}{3c^3}\dddot{\vec{v}}_\perp,$$
$$\dddot{\vec{v}}_\perp = \frac{1}{N_e}\sum_k \dddot{\vec{v}}_{\perp k}. \quad (6)$$

In the absence of energy losses through emission, the particles are in circular motion with cyclic frequencies [50]

$$\Omega_k = \frac{eH}{mc}\sqrt{1 - \frac{v_k^2}{c^2}} \approx \frac{eH}{mc}\left(1 - \frac{v_{\perp k}^2 + v_{zk}^2}{2c^2}\right), \quad (7)$$

depending on $\vec{v}_{\perp k}$, which is responsible for nonisochronism of oscillations.

Using the approximate relation

$$\dddot{\vec{v}}_{\perp k} \approx -\Omega_k^2\vec{v}_{\perp k} \approx -\Omega^2\vec{v}_{\perp k}, \quad (8)$$

where

$$\Omega = \frac{eH}{mc}, \quad (9)$$

we shall write vector equations (6) in a component-wise fashion

$$\dot{v}_{xk} = \Omega\left(1 - \frac{1}{2}\frac{\vec{v}_{\perp k}^2 + \vec{v}_{zk}^2}{c^2}\right)v_{yk} - \frac{2e^2\Omega^2 N_e}{3mc^3}v_x,$$
$$\dot{v}_{yk} = -\Omega\left(1 - \frac{1}{2}\frac{\vec{v}_k^2 + \vec{v}_{zk}^2}{c^2}\right)v_{xk} - \frac{2e^2\Omega^2 N_e}{3mc^3}v_y, \quad (10)$$
$$\dot{v}_{zk} = 0,$$

where

$$\vec{v}_x = \frac{1}{N_e}\sum_k \vec{v}_{xk},$$
$$\vec{v}_y = \frac{1}{N_e}\sum_k \vec{v}_{yk}. \quad (11)$$

Then we shall multiply the second equation (10) by $-i$ and add it to the first one:

$$\dot{v}_{xk} - i\dot{v}_{yk} = i\Omega\left(1 - \frac{1}{2}\frac{\vec{v}_{\perp k}^2 + \vec{v}_{zk}^2}{c^2}\right)(v_{xk} - iv_{yk})$$
$$- \frac{2e^2\Omega^2 N_e}{3mc^3}(v_x - iv_y). \quad (12)$$

Assuming that all $v_{zk} = v_z$ are equal, let us introduce the following notation: $a_k = e^{-i\Omega(1-v_z^2/2c^2)t}(v_{xk} - iv_{yk})/v_{\perp 0}$ ($v_{\perp 0} = \sum_k v_{\perp k}(0)/N_e$ is the average tangential speed of particles). Then (12) takes the form [44]

$$\frac{da_k}{d\tau} + i\theta|a_k|^2 a_k = -a,$$
$$a = \frac{1}{N_e}\sum_k a_k,$$
$$\theta = \frac{3mv_{\perp 0}^2 c}{4e^2\Omega N_e}, \quad (13)$$
$$\tau = \frac{2e^2\Omega^2 N_e}{3mc^3}t.$$

The equation set (13) fully describes the behavior of the ensemble of electrons-oscillators moving in a uniform magnetic field in the presence of the radiative energy loss.

The kinetic energy E_{rad} of transverse motion of the oscillators, the radiation power P_{rad}, and the time t are related to the dimensionless quantities by

$$E_{rad}/E_U = \frac{1}{N_e}\sum_k |a_k|^2, \quad (14)$$

$$P_{rad}/P_U = 2|a|^2, \qquad (15)$$

and

$$t/T_U = \tau, \qquad (16)$$

where

$$E_U = \frac{N_e m c^2}{2}\frac{v_{\perp 0}^2}{c^2}, \qquad (17)$$

$$P_U = \frac{2e^4 H^2 N_e^2}{3m^2 c^2}\frac{v_{\perp 0}^2}{c^2}, \qquad (18)$$

and

$$T_U = \frac{3m^3 c^5}{2e^4 H^2 N_e} = \frac{2\theta c^2}{\Omega v_{\perp 0}^2}. \qquad (19)$$

Let $H = 6.4$ kGs, $v/c = 0.38$, and $N_e = 10^9$, then $E_U = 6$ μJ, $P_U = 0.5$ kW, $T_U = 13$ ns, and $\Omega = 18$ GHz. In this case, the nonisochronism parameter is

$$\theta = \frac{3m^2 c^4}{4e^3 H N_e}\frac{v_{\perp 0}^2}{c^2} = 100. \qquad (20)$$

Let us note that by adding the term of the form $-i\nu a_k$ to the left-hand side of (13), we have that under the same initial conditions, all complex amplitudes a_k are multiplied by the additional phase factor $e^{i\nu\tau}$ having no effect either on the radiation power or on the autophasing time. This enables analyzing different systems from a common standpoint, and so the equation set describing the behavior of the ensemble of weakly nonlinear electrons-oscillators oscillating in anharmonic potential has the form [45]:

$$\frac{da_k}{d\tau} + i\theta(|a_k|^2 - 1)a_k = -a,$$
$$a = \frac{1}{N_e}\sum_k a_k. \qquad (21)$$

Here, the nonisochronism parameter θ is selected as ν. The normalized radiation power and electron energy are given by formulas [45]

$$P = 2|a|^2 \qquad (22)$$

and

$$E = \frac{1}{N_e}\sum_k |a_k|^2, \qquad (23)$$

respectively. In the absence of the energy spread, the absolute values of the amplitudes $|a_k(0)|$ equal unity at the initial time and the phases $\phi_k(0) = \arg(a_k(0))$ are uniformly distributed in the interval $[0; 2\pi)$.

III. SHOT NOISE

Shot noise in the ensemble of the electrons-oscillators is due to a random distribution of the initial phases $\phi_k(0)$. Each set of $\phi_k(0)$ corresponds to different initial conditions of the equation set (21). Fluctuations in the initial conditions lead to fluctuations in the output characteristics of cooperative radiation which we study by a numerical experiment.

As follows from (21), the behavior of the ensemble of nonisochronous oscillators in the absence of the energy spread is determined by two fixed parameters θ and N_e and a random set of initial phases $\phi_k(0)$. Because the distribution of initial phases $\phi_k(0)$ is random, the numerical experiment with each pair of values of θ and N_e must have many runs. This procedure will give information about statistical characteristics of cooperative radiation, the most important of which are peak power P_0, autophasing time T_0, and their relative RMSD — δ_P and δ_T.
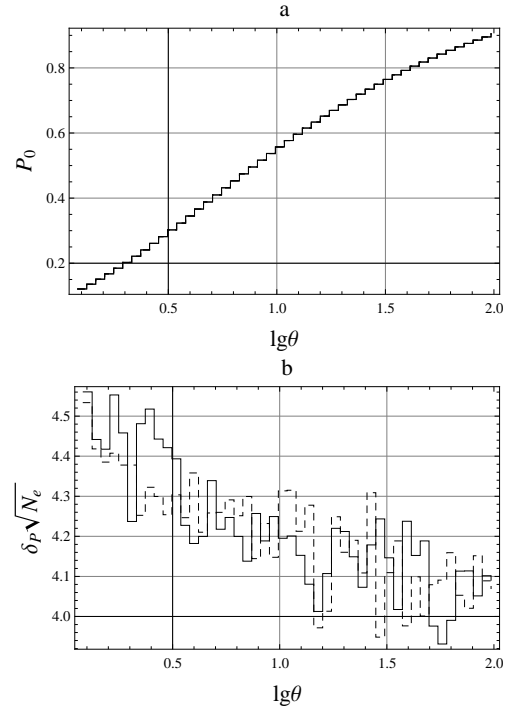

FIG. 1: Peak radiated power (a) and its RMSD (b) versus the nonisochronism parameter θ [black curve — $N_e = 6.75 \cdot 10^4$ and dashed curve — $1.08 \cdot 10^6$].

In numerical analysis of statistical fluctuations of cooperative radiation in the presence of shot noise, instead of the number N_e of real electrons we took the number $N = 36 \ll N_e$ of simulated electrons with the charge eN_e/N and initial phases

$$\phi_k(0) = \frac{2\pi k}{N} + \sqrt{\frac{12N}{N_e}}r_k, \quad k = 1..N, \qquad (24)$$
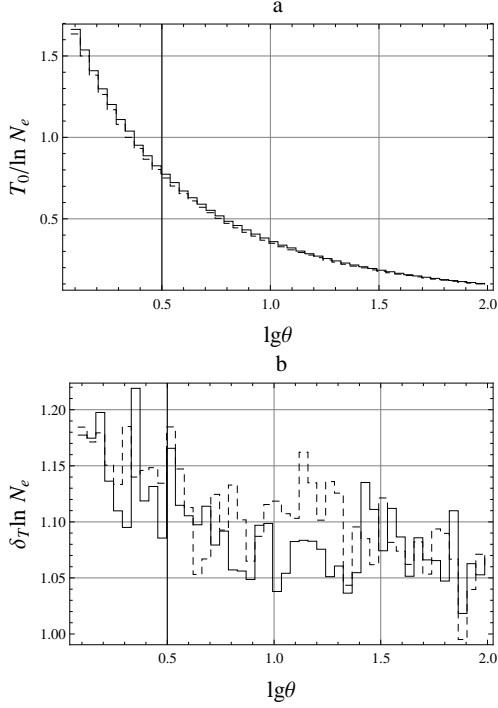
a

b

FIG. 2: Autophasing time (a) and its RMSD (b) versus the nonisochronism parameter θ [black curve — $N_e = 6.75 \cdot 10^4$ and dashed curve — $1.08 \cdot 10^6$].

where r_k are random variables uniformly distributed over the interval $[0;1)$. It has been shown in [49] that this procedure, boosting the performance of the program, correctly simulates the shot noise in the absence of energy spread. We selected the following values of controlling parameters: $N_e = 6.75 \cdot 10^4, 1.08 \cdot 10^6$, $\theta = 1$–107. The numerical experiment with each N_e–θ pair was repeated one hundred times.

Fig. 1 and 2 show the results of our computation from which we can draw some very important conclusions [47]. First, the relative RMSD of the dimensionless peak radiated power weakly dependent on the number of particles N_e (Fig.1) decreases as $\delta_P \approx 4.3/\sqrt{N_e}$. Second, the autophasing time decreasing as the nonisochronism parameter θ is increased depends logarithmically on the number of particles $T_0 \sim \ln N_e$. Third, δ_T reduces according to the approximate formula $\delta_T = q/\ln N_e$, where $q(\theta) \approx 1.1$ slowly decreases as $\lg(\theta)$ varies from 0 to 2. Extrapolating the obtained dependence of $\delta_P(N_e)$ and $\delta_T(N_e)$ to the region with large number of particles, $N_e = 10^9$–10^{12}, (typical number of electrons in single-pass generators), we get the estimates $\delta_T = 0.04$–0.05 and $\delta_P < 10^{-4}$. From this we can deduce that shot noise leads to a 4–5% fluctuation of the autophasing time at insignificant fluctuation of the peak radiated power.

Let us note here that phase premodulation of electrons-

oscillators:

$$\phi_k(0) = \frac{2\pi k}{N} + \sqrt{\frac{12N}{N_e}}r_k + \delta_\phi \cos\left(\frac{2\pi k}{N}\right), k = 1..N, \quad (25)$$

having practically no effect on P_0, T_0, and δ_P, leads to a noticeable decrease in the fluctuation of autophasing time: δ_T no longer decreases logarithmically ($\delta_T \sim 1/\ln N_e$), but as $1/\sqrt{N_e}$ (Fig. 3). In the expression (25), $\delta_\phi \ll 1$ is the premodulation parameter.
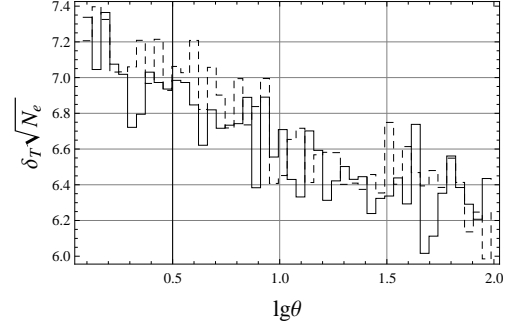

FIG. 3: RMS fluctuations of autophasing time versus the nonisochronism parameter θ at $\delta_\phi = 0.05$ [black curve — $N_e = 6.75 \cdot 10^4$ and dashed curve — $1.08 \cdot 10^6$].

IV. AUTOPHASING TIME

In the previous section we have shown that fluctuations in the cooperative radiation caused by shot noise depend on the degree to which the electrons-oscillators are premodulated. It has been found, in particular, that premodulation results in that the relative RMSD of the autophasing time δ_T becomes square-root dependent on N_e ($\delta_T \sim 1/\sqrt{N_e}$) instead of logarithmically dependent ($\delta_T \sim 1/\ln N_e$). Let us show how this important result can be obtained analytically.

According to (21) without premodulation, the contribution coming from each particle to the average oscillation amplitude $a_{in} = a(0)$ is $a_k(0) = e^{i\phi_k}/N_e$. The contribution coming from a premodulated particle $a_k(0) = e^{i\phi_k+i\delta_\phi \cos(\phi_k)}$ (the initial phases ϕ_k are uniformly distributed over the interval $[0;2\pi)$). By averaging over ϕ_k, we shall find the average value of a_{in} as well

as the RMSD thereof at $N_e \gg 1$ and $\delta_\phi \ll 1$:

$$< \mathrm{Im} a_{in} > = J_1(\delta_\phi),$$

$$< \mathrm{Re} a_{in} > = 0,$$

$$| < \mathrm{Im} a_{in} >^2 - < \mathrm{Im}^2 a_{in} > |^{1/2} = \sqrt{\frac{1}{2N_e}(1 + J_2(2\delta_\phi))}$$

$$\approx \frac{1}{\sqrt{2N_e}},$$

$$| < \mathrm{Re} a_{in} >^2 - < \mathrm{Re}^2 a_{in} > |^{1/2} = \sqrt{\frac{1}{2N_e}(1 - J_2(2\delta_\phi))}$$

$$\approx \frac{1}{\sqrt{2N_e}},$$

$$(26)$$

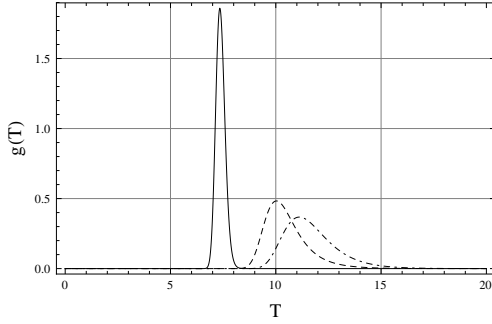where $J_2(2\delta_\phi)$ is the Bessel function. Since $N_e \gg 1$,

FIG. 4: Autophasing time distribution functions at $\tau_0 = 1$ and $N_e = 1.08 \cdot 10^6$ [solid curve — $\delta_\phi = 0.05$, dashed curve — $\delta_\phi = 0.01$, dot-dashed curve — $\delta_\phi = 0$].
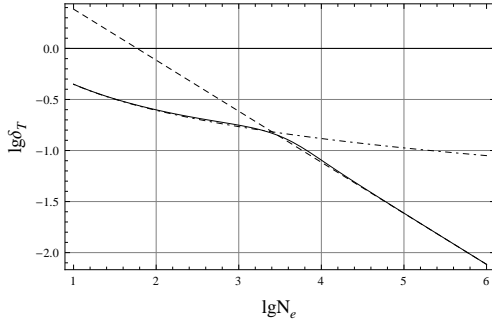

FIG. 5: RMSD as a function of the number of particles at $\delta_\phi = 0.05$ [solid curve corresponds to (30), dashed curve corresponds to the asymptotic expansion at $\delta_\phi^2 N_e \gg 1$, dot-dashed curve — to the asymptotic expansion at $\delta_\phi^2 N_e \ll 1$].

then in view of the central limiting theorem, $\mathrm{Re} a_{in}$ and $\mathrm{Im} a_{in}$ have a Gaussian distribution:

$$f(\mathrm{Re} a_{in}, \mathrm{Im} a_{in}) = \frac{N_e}{\pi} e^{-N(\mathrm{Re}^2 a_{in} + \mathrm{Im}^2 a_{in})}, \qquad (27)$$

leading to that the radiation power distribution $P_{in} = 2|a_{in}|^2$ at the initial time has the form:

$$f(P_{in}) = N_e e^{-N_e P_{in}/2 - N_e \alpha} I_0(N_e \sqrt{2P_{in}\alpha}). \qquad (28)$$

where I_0 is the modified Bessel function, $\alpha = J_1^2(\delta_\phi) \approx \delta_\phi^2/4$.

Let us estimate the autophasing time by formula

$$T = \tau_0 \ln(P_0/P_{in}) = \tau_0 \ln(2/P_{in}). \qquad (29)$$

Here, $\tau_0 = 1/\mathrm{Re}(-1 + \sqrt{1 + 4i\theta})$ [45] is the time period needed for the radiation power to increase by a factor of e in the linear instability stage. Equation (29) implies, first, that the linear stage is much longer than the time during which nonlinear effects are essential, and, second, the radiated power in the saturation stage is $P_0 = 2|a|^2 \sim 2$. The latter condition means coherent summation of oscillations from all the particles.

Using (28) and (29), we can find the distribution function $g(T)$ related to $f(P_{in})$ by

$$g(T) = f(P_{in})|dP_{in}/dT|. \qquad (30)$$

Thus we have

$$g(T) = \frac{N}{\tau_0} I_0(2N_e \sqrt{\alpha e^{-T/\tau_0}})$$
$$\times \exp(-N_e e^{-T/\tau_0} - N_e \alpha - T/\tau_0). \qquad (31)$$

From the distribution functions $g(T)$ plotted in Figure 4 follows that with growing premodulation, the autophasing time and its RMSD thereof decrease.

Using the distribution function $g(T)$, we shall compute the relative RMSD δ_T:

$$\delta_T = \frac{\sqrt{| < T^2 > - < T >^2 |}}{< T >}$$
$$= \frac{\sqrt{\pi^2/6 - e^{-2N_e\alpha}F_{-1}^2(N_e\alpha) + e^{-N_e\alpha}G_{-1}(N_e\alpha)}}{e^{N_e\alpha}(\gamma_e + \ln N_e) + F_{-1}(N_e\alpha)},$$
$$F_x(y) = \frac{\partial L_x(y)}{\partial x},$$
$$G_x(y) = \frac{\partial^2 L_x(y)}{\partial x^2}. \qquad (32)$$

Here, $\gamma_e = 0.577$ is the Euler constant and $L_x(y)$ is the Laguerre polynomial. If $N_e\alpha \ll 1$, then (32) transforms to the form

$$\delta_T = \frac{\pi}{\sqrt{6}(\gamma_e + \ln N_e)}, \qquad (33)$$

in the opposite case ($N_e\alpha \gg 1$),

$$\delta_T = \frac{\sqrt{2}}{\sqrt{N_e\alpha}\ln(1/\alpha)}. \qquad (34)$$

The Fig. 5 plots δ_T versus $\lg N_e$, respectively. It is obvious from 5 that when the number of particles exceeds a certain value depending on the degree of premodulation, the logarithmic N_e dependence of the relative RMSD of the autophasing time δ_T ($\delta_T \sim 1/\ln N_e$) goes to a square-root dependence ($\delta_T \sim 1/\sqrt{N_e}$). As follows from (32),

the value of $\delta_T \sqrt{N_e} = 7.67$ at $\delta_\phi = 0.05$ and $0 < \lg \theta < 2$ agree with the simulated ones (Fig. 3) within 20% accuracy.

Let us note a very important circumstance. The duration of the cooperative radiation pulse from non-isochronous electrons-oscillators is $\sim \tau_0$ [45]. The RMSD of the autophasing time in the absence of premodulation has the same order of magnitude:

$$\Delta T = \sqrt{|<T^2> - <T>^2|} \approx \pi \tau_0 / \sqrt{6} \qquad (35)$$

at $N_e \alpha \ll 1$. However, phase premodulation of particles leads to an appreciable decrease in the autophasing time spread:

$$\Delta T \approx \tau_0 \sqrt{2/N_e \alpha} \qquad (36)$$

at $N_e \alpha \gg 1$.

In coherent summation of cooperative-radiation pulses, the oscillation phases of all single-pass generators must differ by $\Delta \phi \ll \pi$, thus posing the following limitation on the average statistical spread of autophasing time $\Delta t = T_U \Delta T$:

$$\Omega T_U \Delta T \ll \pi, \qquad (37)$$

giving, after the substitution of T_U (19) and ΔT (36)

$$\sqrt{N_e} \delta_\phi \gg \frac{4\sqrt{2}}{\pi} \frac{\theta}{\mathrm{Re}(-1 + \sqrt{1 + 4i\theta})} \frac{c^2}{v_{\perp 0}^2}. \qquad (38)$$

Let $v/c \sim 0.4$ and $\theta = 100$, then we have the following estimate of the required degree of premodulation: $\sqrt{N_e} \delta_\phi \gg 86$. At $N_e = 10^9$, the parameter of premodulation δ_ϕ must be greater than $3 \cdot 10^{-3}$.

V. ENERGY SPREAD

To take account of the electron energy spread $E_k(0) \approx a_k^2 / N_e$, we assume the initial amplitudes $a_k(0)$ to be Gaussian random variables whose mean equals unity and the relative RMSD $\delta_a \approx \delta_E / 2$ (δ_E is the relative RMSD deviation of particle energy).

It should be noted the Penman-McNeil algorithm isn't applicable in the presence of energy spread. Therefore instead of the number $N = 36$ of simulated electrons we took the number $N = 288 \gg 1$ of particles with the initial phases uniformly distributed in the interval $[0; 2\pi)$.

Analyzing the results of numerical experiments [47], we can see that the energy spread leads to a sharp drop in the radiated power (Fig. 6). This is well-illustrated by the Fig. 7, where the growing influence of the energy spread with higher θ is seen clearly: the energy spread leads to a stronger suppression of radiation at large θ, and peak radiated power and radiated energy both decrease. At $\delta_a = 0.02$ ($\delta_E = 0.04$), the maximum attainable radiated power reduces by more than a factor of two.
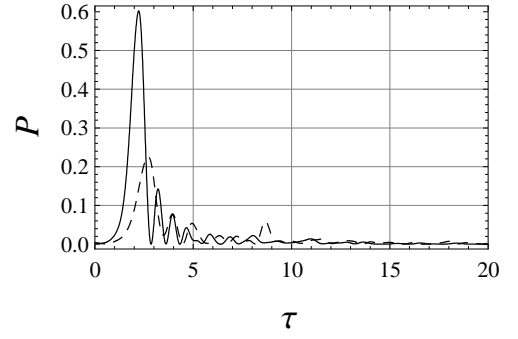

FIG. 6: Radiated power as a function of time at $\theta = 10$ and different δ_a [solid curve — $\delta_a = 0.0$, dashed curve — $\delta_a = 0.04$].
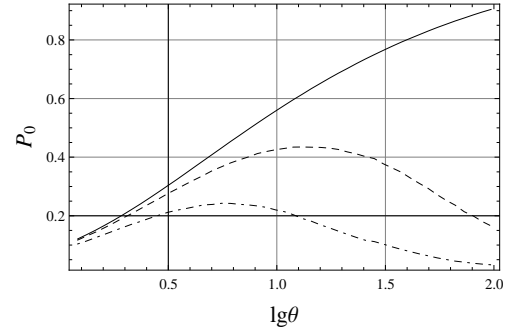

FIG. 7: Peak radiated power versus the nonisochronism parameter θ at different δ_a [black curve — $\delta_a = 0.00$, dashed curve — $\delta_a = 0.02$, and dot-dashed curve — $\delta_a = 0.04$].

VI. CONCLUSION

In this paper we have studied the statistical properties of cooperative radiation from an ensemble of non-isochronous electrons-oscillators interacting with one another via the radiation field. It has been shown that for the number of electrons $N_e \sim 10^9$–10^{12}, typical of modern acceleration facilities, the relative RMSD of the autophasing time from its mean is $\delta_T \approx 1.1/\ln N_e \sim 0.04$–$0.05$. The fluctuations of the peak radiated power appear to be negligibly small $\delta_P < 10^{-4}$.

The autophasing time distribution function depending on the number of particles N_e and the degree of premodulation has been obtained in the absence of energy spread. We used this function to show that when the number of particles exceeds a certain value depending on the degree of premodulation, the logarithmic dependence of the relative RMSD of the autophasing time on the number of electrons-oscillators ($\delta_T \sim 1/\ln N_e$) goes to a square-root dependence ($\delta_T \sim 1/\sqrt{N_e}$).

It has also been demonstrated that a slight energy spread ($\sim 4\%$) results in a twofold drop of the maximum attainable power of cooperative radiation.

The analysis made here indicates that shot noise, and

electron velocity spread pose considerable constraints on the output characteristics of single-pass CRMs and limits the possibility of coherent summation of cooperative radiation pulses from several sources.

Appendix A: Relativistic electrons-oscillators

Let in laboratory reference there be a relativistic electron beam whose velocity is directed at a small angle to the magnetic-field vector. The particles in the beam move in a spiral under Lorentz force $\vec{v} \times \vec{H}/c$. Let us denote the particle velocity components perpendicular and parallel to vector $\vec{H}$ by $\vec{u}_\perp$ and $\vec{u}_z$, respectively. Then the velocity component $\vec{v}_\perp$ perpendicular to the magnetic field $\vec{H}$ in the reference frame attached to the beam is given by [50]:

$$\vec{v}_\perp = \frac{\vec{u}_\perp}{\sqrt{1 - \frac{u_z^2}{c^2}}}. \tag{A1}$$

Let us assume that $v_\perp \ll c$, which leads to the limitations on the acceptable value of $u_\perp$:

$$\frac{u_\perp}{c} \ll \sqrt{1 - u_z^2/c^2} \approx \frac{1}{\gamma}. \tag{A2}$$

In this section we shall focus on the distribution of the cooperative radiation from relativistic electrons-oscillators in the laboratory frame if the condition (A2) is fulfilled. In a moving frame of reference, the radiated power per unit solid angle is given by [50, 51]

$$\frac{dP}{d\Omega} = \frac{3P}{16\pi}(1 + \cos^2\theta), \tag{A3}$$

where P is the total power of electrons-oscillators. The question naturally arises of how the radiation intensity (A3) can be transformed into the lab reference frame.

Let us denote the radiated power per unit solid angle in the lab frame by $dP_u/d\Omega_u$. Before seeking $dP_u/d\Omega_u$, let us note that the quantity

$$X_1 = \frac{dP}{d\Omega}\frac{d\Omega dt}{\hbar\omega} \tag{A4}$$

that equals the number of quanta emitted in solid angle $d\Omega$ during the time dt is the Lorentz invariant, and X_2 is the Lorentz invariant, too [50]

$$X_2 = \frac{d^3k}{\hbar\omega} = \frac{k^2 dk d\Omega}{\hbar\omega}. \tag{A5}$$

The division of X_1 by X_2 gives another invariant

$$X_3 = \frac{dt}{k^2 dk}\frac{dP}{d\Omega}. \tag{A6}$$

By equating the invariants in both frames of reference we get

$$\frac{dP_u}{d\Omega_u} = \frac{k_u^2 dk_u dt}{k^2 dk dt_u}\frac{dP}{d\Omega}. \tag{A7}$$

Because the absolute values of the wave vectors in moving (k) and fixed (k_u) frames of reference are related as [50]

$$k = k_u \frac{1 - \frac{u_z \cos(\theta_u)}{c}}{\sqrt{1 - \frac{u_z^2}{c}}}, \tag{A8}$$

where θ_u is the angle between vectors $\vec{H}$ and $\vec{k}_u$, whereas

$$\frac{dt}{dt_u} = \frac{1}{\sqrt{1 - u_z^2/c^2}} \approx \gamma, \tag{A9}$$

we can find

$$\frac{dP_u}{d\Omega_u} = \left(1 + \left(\frac{\cos\theta_u - u_z/c}{1 - u_z \cos\theta_u/c}\right)^2\right) \times \frac{(1 - u_z^2/c^2)^2}{(1 - u_z \cos\theta_u/c)^3}\frac{3P}{16\pi}. \tag{A10}$$

According to [50, 51], the angular distribution of the power $dP_{det}/d\Omega_u$ detected by the resting detector in the lab frame is related to $dP_u/d\Omega_u$ as

$$\frac{dP_{det}}{d\Omega_u} = \frac{1}{1 - u_z \cos\theta_u/c}\frac{dP_u}{d\Omega_u}. \tag{A11}$$

Integration of (A11) with due account of (A10) finally gives $(\gamma \gg 1)$

$$P_{det} = \frac{5 + 2u_z^2}{5 - 5u_z^2}P \approx \frac{7\gamma^2 P}{5}. \tag{A12}$$

Appendix B: Effects of Coulomb and induction fields

Let us study the applicability boundaries of (10). The equation set (10) was derived in the assumption that the beam size R_0 is much less than the radiation wavelength

$$R_0 \ll \lambda = \frac{mc^2}{eH}. \tag{B1}$$

We also neglected the Coulomb repulsion and induction fields. The latter assumption is often taken in solving specific problems [45], so we shall qualitatively outline its legitimacy.

It has been shown in [44, 45] that the possibility of phasing electrons-oscillators without the external field arises because the particles oscillate nonisochronously, i.e., the particle oscillation frequency depends on energy (see (7)). As a result, a nonlinear term having the same order of magnitude as $\Omega v^3/c^2$ appears in (10). The Coulomb and induction fields have no effect of autophasing if the acceleration $\sim \frac{Ne^2}{mR_0^2}$ induced to the electron by the Coulomb forces (induction fields are by a factor of c/v less than the Coulomb ones) is much smaller than $\Omega v^3/c^2 = \frac{eHv^3}{c^3}$. Let us write this condition in the form:

$$R_0 \gg R_{min} = \sqrt{\frac{N_e e c^3}{Hv^3}}. \tag{B2}$$

Let $H = 6$ kGs, $v/c = 0.4$, and $N_e = 10^9$. Then $\lambda = 0.3$ cm, $R_{min} = 0.04$ cm, and the particle orbit radius $R_{orbit} = v/\Omega = mcv/eH = 0.1$ cm is between R_{min} and λ. That is why if in this specific case the beam size is the same as the particle orbit, we can satisfy both (B1) and (B2) requirements.

Appendix C: Near-field of charged particles

Let us consider a short beam of charged particles moving with nonrelativistic velocities. The beam particles produce the electromagnetic field that can be described using the scalar ϕ and vector $\vec{A}$ retarded potentials [50]:

$$\phi(\vec{r}, t) = \int \frac{1}{R} \rho(\vec{r}_1, t - R/c) dV_1,$$
$$\vec{A}(\vec{r}, t) = \int \frac{1}{R} \vec{j}(\vec{r}_1, t - R/c) dV_1, \qquad (C1)$$

where $R = |\vec{r} - \vec{r}_1|$ is the distance from the volume element dV_1 to the point of observation $\vec{r}$, where we seek the potential difference, ρ and $\vec{j}$ are the current and charge densities, respectively. The electric $\vec{E}$ and magnetic $\vec{H}$ fields relate to the potentials ϕ and $\vec{A}$ as

$$\vec{E} = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla \cdot \phi,$$
$$\vec{H} = \nabla \times \vec{A}. \qquad (C2)$$

Using these relation, we can readily find the force acting on each charge from all beam particles

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{H}/c). \qquad (C3)$$

Because the velocities of all charges are small compared to the speed of light, their distribution hardly can change a lot during the time $\sim R_0/c$ (R_0 is the size of the beam). That is why, to find forces acting on the charges, we can expand ρ and $\vec{j}$ into power series in R/c. For the scalar potential, we find accurate up to the third order in terms of c^{-1}:

$$\phi(\vec{r}, t) = \int \frac{\rho}{R} dV - \frac{1}{c}\frac{\partial}{\partial t}\int \rho dV$$
$$+ \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV - \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int R^2\rho dV, \qquad (C4)$$

But $\int \rho dV$ is the constant total charge of the system; Then the second term in this expression equals zero, and so

$$\phi(\vec{r}, t) = \int \frac{\rho}{R} dV + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho dV$$
$$- \frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int R^2\rho dV. \qquad (C5)$$

We can do the same with $\vec{A}$. But c^{-1} appearing in the expression for the vector potential is multiplied by c^{-1} when $\vec{A}$ is substituted into the expression for the force. Because we seek the potentials within the accuracy up to the third order in terms of c^{-1}, in the expansion of $\vec{A}$ suffice it to find the first two terms, i.e.,

$$\vec{A}(\vec{r}, t) = \frac{1}{c}\int \frac{\vec{j}}{R} dV - \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{j} dV, \qquad (C6)$$

Let us perform gauge transformation of the potentials:

$$\phi_{new} = \phi - \frac{1}{c}\frac{\partial f}{\partial t},$$
$$\vec{A}_{new} = \vec{A} + \nabla f, \qquad (C7)$$

choosing the function f such that the scalar potential ϕ_{new} becomes

$$\phi_{new} = \int \frac{\rho}{R} dV : \qquad (C8)$$

$$f = \frac{1}{2c}\frac{\partial}{\partial t}\int R\rho dV - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int R^2\rho dV. \qquad (C9)$$

Then a new vector potential is

$$\vec{A}_{new} = \frac{1}{2c}\nabla\left(\frac{\partial}{\partial t}\int R\rho dV\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\int \vec{j} dV$$
$$- \frac{1}{3c^2}\frac{\partial^2}{\partial t^2}\int \vec{R}\rho dV. \qquad (C10)$$

Passing from integration to summation over individual charges, we get [50]

$$\vec{A}_{new} = \vec{A}_1 + \vec{A}_2,$$
$$\vec{A}_1 = \sum_j e \frac{\vec{v}_j + (\vec{v}_j \vec{n})\vec{n}}{2cR_j},$$
$$\vec{A}_2 = -\sum_j \frac{2}{3c^2} e \dot{\vec{v}}_j \qquad (C11)$$

and

$$\phi_{new} = \sum_j \frac{e}{R_j}, \qquad (C12)$$

where $\vec{n}_j = \vec{R}_j/R_j$, $\vec{R}_j = |\vec{r} - \vec{r}_j|$.

The first term $\vec{A}_1$ in (C11) determines the induction field of the system of charges and the scalar potential $\vec{A}$ determines the electric field of the spatial charge. Both fields are non-uniform and depend on the averaged charge distribution in the electron beam [45]. The second term $\vec{A}_2$ in (C11) is uniform and arises through radiation; the related electric field is given by

$$\vec{E}_{rad} = \sum_j \frac{2}{3c^2} e \ddot{\vec{v}}_j. \qquad (C13)$$

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