Journal of Algebra 459 (2016) 189–212

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Pure projective modules over chain domains with Krull dimension ☆

Pavel Příhoda [a,*], Gena Puninski [b]

[a] *Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, Praha 8, 18675, Czech Republic*
[b] *Belarusian State University, Faculty of Mechanics and Mathematics, av. Nezalezhnosti 4, Minsk 220030, Belarus*

A R T I C L E I N F O

Article history:
Received 6 August 2015
Available online 28 April 2016
Communicated by J.T. Stafford

MSC:
primary 16D70
16P70

Keywords:
Chain domain
Pure projective module
Uniserial module
Finitely presented module

A B S T R A C T

We will prove that over a chain domain with Krull dimension each pure projective module decomposes into a direct sum of finitely presented modules.

1. Introduction

A ring is said to be a chain ring if its right ideals are linearly ordered by inclusion, and the same holds true for its left ideals.

☆ The paper was written during two stays of the second author in ECI, Prague. Both authors were partially supported by GA ČR P201/12/G028.

* Corresponding author.
E-mail addresses: prihoda@karlin.mff.cuni.cz (P. Příhoda), punins@mail.ru (G. Puninski).

http://dx.doi.org/10.1016/j.jalgebra.2016.04.010

A clever trick by Drozd (see [12, Theorem 2.1]) shows that every matrix over a chain ring R can be diagonalized by elementary row and column transformations. In particular every finitely presented right module over R is isomorphic to a direct sum of cyclically presented modules R/rR, $r \in R$; furthermore (see [5, Theorem 9.19]) this decomposition is essentially unique. It follows from Warfield (see [18, 33.6]) that every pure projective module over a chain ring is isomorphic to a direct summand of a direct sum of uniserial modules R/rR. If each module R/rR has a local endomorphism ring, then the extended version of the Crawley–Jønsson theorem (see [5, Theorem 5.2]) shows that every pure projective module is isomorphic to a direct sum of finitely presented modules. For example this is the case when R is a commutative chain ring.

However for a general chain ring the situation is less orderly. For instance, Puninski [13] constructed a non-finitely generated uniserial pure projective module over a nearly simple chain domain; and also (see [14]) found a pure projective module without an indecomposable decomposition over an exceptional chain ring.

In this paper we will completely characterize chain domains R such that every pure projective R-module is isomorphic to a direct sum of finitely presented modules. Namely this is the case if and only if R contains no idempotent 2-sided ideal RrR, where $0 \neq r \in R$ is not a unit. For instance each chain domain with Krull dimension possesses this property.

To prove something nice we have to avoid complicated things, so the bulk of the paper is spent analyzing various anomalies of direct sum decompositions of serial modules. The main tool in this analysis is the so-called dimension theory for serial modules, recently developed by Facchini and Příhoda [6]. It states (see Proposition 4.1 for an exact statement) that pure projective modules over a chain ring R are classified by tuples of dimensions: one for each module R/rR with a local endomorphism ring, and two for each module R/rR whose endomorphism ring is not local.

Any countably generated pure projective module can be constructed as a colimit of a special directed system (called a Mittag–Leffler system) of morphisms between finitely presented modules. In this paper we will suggest a very abstract version of this construction that covers all known examples of strange pure projective modules over chain rings. By evaluating it for various dimensions we will identify the main obstacles to the existence of a perfect (hence trivial) decomposition theory of pure projectives.

Because these anomalies are of great interest we will demonstrate non-regular behavior of pure projective modules by various examples. As a main running example we will use the chain domain (constructed by Dubrovin – see [3], or better [4]) which is associated with a special embedding of the group ring of the trefoil group into a skew field.

We are indebted to the referee for a very careful reading of the manuscript, which greatly improved the presentation.

2. Chain rings

A module M is said to be *uniserial* if its lattice of submodules is a chain, and *serial* if M is a direct sum of uniserial modules. A ring R is said to be *chain* if R is uniserial as a right and left module over itself. Thus for any $a, b \in R$ either $aR \subseteq bR$ or $bR \subseteq aR$ (or both) holds true, i.e. there are $r, s \in R$ such that $a = br$ or $b = as$. The left ideals Ra and Rb are also comparable by inclusion but, as we will see in examples below, this inclusion may work in the opposite direction.

In this paper we will concentrate on chain rings without zero divisors, i.e. *chain domains*. Thus, with few exceptions, R **will denote a chain domain**. For instance R is a local ring whose Jacobson radical J consists of non-units, hence $U = R \setminus J$ is the set of invertible elements of R. If $0 \neq a, b \in R$ then the equality $aR = bR$ holds true iff $a = bu$ for a unit u.

All results of this section are folklore, so we will prove only short (or missing) implications.

Commutative chain domains are commonly called *commutative valuation domains*. Over such a domain every module R/rR has a local endomorphism ring. We will see below that this is no longer true in the noncommutative setting.

We will often deal with morphisms between indecomposable finitely presented modules R/rR and R/sR. Usually we will consider a 'generic' case $0 \neq r, s \in J$, i.e. when both modules are nonzero and non-projective. Note that for $r \neq 0$ we have $\mathrm{Hom}(R/rR, R) = 0$, since R is a domain.

If $0 \neq r, s \in J$ then any morphism $f : R/rR \to R/sR$ is given by left multiplication by $a \in R$ such that $ar = sb$ for some $b \in R$; furthermore b is uniquely determined by a, and a is uniquely determined modulo sR. We will write this morphism as $a \times -$.

Note that $\ker f$ is always a cyclic submodule of R/rR. Namely we may assume that f is nonzero, hence $s = aj$ for some $j \in J$ yields $\ker f = jR/rR$.

The following lemma is almost trivial.

Lemma 2.1. *f is a monomorphism if and only if b is a unit, and f is an epimorphism if and only if a is a unit.*

We will draw a few corollaries from this lemma.

Corollary 2.2. *Let $0 \neq r \in J$ and $M = R/rR$. Left multiplication by an element $a \in R$ defines a monomorphic endomorphism f of M if and only if $arR = rR$.*

Proof. $\Rightarrow$. Follows from Lemma 2.1.

$\Leftarrow$. From $ar \notin rR$ it would follow that $arR \supset rR$, a contradiction, hence $ar = rb$ for some $b \in R$. If $b \in J$ then we obtain $arR = rbR \subset rR$, a contradiction again. Thus $b \in U$, and therefore f is a monomorphism. $\square$

The following corollary is crucial for this paper.

Corollary 2.3. *(See [12, Proposition 2.21].) Let $0 \neq r, s$ be elements of a chain ring R. Then the following are equivalent.*

1) $R/rR \cong R/sR$;
2) $RrR = RsR$;
3) $r = usv$ *for units* $u, v \in R$.

Another will also be used frequently.

Corollary 2.4.

1) *If* $R/rR \to R/sR \to R/rR$ *are monomorphisms then* $R/rR \cong R/sR$.
2) *If* $R/rR \to R/sR \to R/rR$ *are epimorphisms then* $R/rR \cong R/sR$.

Proof. 1) By Lemma 2.1 we have $ar = su$ and $bs = rv$ for some $a, b \in R$ and $u, v \in U$. But this clearly implies $RrR = RsR$, hence $R/rR \cong R/sR$ by Corollary 2.3. □

Because R/rR is a uniserial module, it follows from [5, Theorem 9.1] that its endomorphism ring $S = \mathrm{End}(R/rR)$ contains at most two maximal (right, left or 2-sided) ideals: K consisting of non-epimorphisms and L consisting of non-monomorphisms. For instance, S is local if and only if K and L are comparable by inclusion.

Now it is easy to give a criterion for when S is local.

Proposition 2.5. *(See [12, Lemma 1.26, Theorem 2.13].) Let $0 \neq r \in J$. Then the following are equivalent.*

1) *The endomorphism ring of the module* $M = R/rR$ *is local.*
2) $r \neq srt$ *for any* $s, t \in J$.
3) *The right ideal* rR *is 2-sided or the left ideal* Rr *is 2-sided.*
4) $(1 + J)r \subseteq rU$ *and* $r(1 + J) \subseteq Ur$.

Chain domains satisfying these conditions for every $0 \neq r \in J$ are called *semi-invariant* or *semi-duo* in [1, Section 7.3]. An example of a chain domain which is semi-duo but not duo (i.e. some 1-sided ideals are not 2-sided) can be found in [1, Section 7.5].

Proof. 1) ⇒ 2). From $r = srt$ where $s, t \in J$ it follows that $(1 + s)^{-1}sr = r(t + 1)^{-1}$. Hence left multiplication by $(1 + s)^{-1}s$ defines a monomorphism f which is not epi. Similarly the equality $(1 + s)^{-1}r = rt(1 + t)^{-1}$ gives the epimorphism $g = (1 + s)^{-1} \times -$ which is not mono.

2) ⇒ 3). If neither rR nor Rr are 2-sided, there are $u, v \in R$ and $j, k \in J$ such that $urj = r$ and $krv = r$. Then $r = (ku)r(jv)$ contradicts 2).

3) $\Rightarrow$ 1) Suppose that rR is a 2-sided ideal. We prove that every epimorphic endomorphism f of M is mono. Since f is epi, by Lemma 2.1 it is given by left multiplication by a unit $u \in R$. To show that f is mono, by the same Lemma, we need to check that $urR = rR$. Since rR is a 2-sided ideal, otherwise we obtain $ur = rj$ for some $j \in J$, hence $r = u^{-1}rj \in rJ$, a contradiction.

If Rr is a 2-sided ideal, similar arguments show that every monomorphic endomorphism of M is epi. Thus $\mathrm{End}(M)$ is local whenever 3) holds.

The equivalence of 3) and 4) follows from [12, Lemma 1.26]. $\square$

Let f and g be as in the above proof. Then the image of f equals sR/rR and the kernel of g is generated by $(1 + s)r + rR$, hence equals srR/rR. In particular $\ker g \subset \operatorname{im} f$, hence $gf \neq 0$. This kind of comparison of kernels of epimorphisms with images of monomorphisms will be important in what follows.

Now we will give an example of a chain domain where locality fails.

2.1. Example

(See [1, Section 7.6].)

Let $\mathbb{Z}[i]$ be the ring of Gaussian integers whose field of quotients is $\mathbb{Q}[i]$. Let $V = \mathbb{Z}_{(2-i)}$ be the localization of $\mathbb{Z}[i]$ with respect to the maximal ideal $(2 - i)\mathbb{Z}[i]$, hence V is a commutative valuation domain. Let σ denote the automorphism of $\mathbb{Q}[i]$ given by complex conjugation, in particular σ sends $2 - i \in J(V)$ to the unit $2 + i$.

Let R denote the ring of skew formal power series $\alpha_0 + \alpha_1 x + \dots$, where $\alpha_0 \in V$ and $\alpha_j \in \mathbb{Q}[i]$ for $j \geq 1$, and the multiplication is given by the rule $x\alpha = \sigma(\alpha)x$. In particular $J = (2 - i)R = R(2 - i)$ is the largest proper ideal of R, x^{2n} is in the center of R, but x^{2n+1} does not commute with $2 - i$.

It is not difficult to check that the principal right ideals of R form the following chain:

$$R \supset (2 - i)R \supset (2 - i)^2 R \supset \dots \supset x(2 - i)^{-1}R \supset xR \supset x(2 - i)R \supset \dots ,$$

and we have a similar chain of principal left ideals, hence R is a chain domain. Let $M = R/xR$. By Lemma 2.5 the equality $x = \frac{2-i}{2+i} \cdot x \cdot \frac{2-i}{2+i}$ shows that the ring $\mathrm{End}(M)$ is not local. Namely $(2 - i)x = x(2 + i)$ gives the monomorphism $(2 - i) \times -$ of M which is not epi, and the epimorphism $(2 + i) \times -$ which is not mono. Or notice that $(2 + i)^{-1}x \notin xR$ and $x(2 + i)^{-1} \notin Rx$.

There is a general definition of Krull dimension for noncommutative rings (see [8, Chapter 6]). By Müller's result (see [12, Proposition 1.30]) for chain domains it boils down to the following. By induction on ordinals we define the iterated powers, $J(\lambda)$, of the radical. Namely set $J(0) = J$ and $J(\lambda+1) = \cap_{n=1}^{\infty} J(\lambda)^n$. Finally define $J(\mu) = \cap_{\lambda<\mu} J(\lambda)$ for limit ordinals μ. Then the least λ such that $J(\lambda) = 0$ equals the *Krull dimension* of R (if no such λ exists then R has no Krull dimension).

For instance, in the above example we have $J(0) = (2-i)R$, $J(1) = RxR$ and $J(2) = 0$, therefore $\mathrm{Kdim}(R) = 2$.

If $M = R/rR$ then M_e denotes the sum of kernels of all epimorphic endomorphisms of M. If $M_e \neq 0$ then this module cannot be cyclic, so in particular $M_e \subset M$. For, if it were cyclic, then $M_e = \ker f$ for an epimorphism f which is not mono. But then $\ker f \subset \ker f^2 \subseteq M_e$, a contradiction.

In fact this module can be calculated precisely.

Fact 2.6. (See [10, Lemma 4.3].) $(R/rR)_e = RrR/rR$.

If R is as in Example 2.1 and $M = R/xR$ then $M_e = RxR/xR$ is nonzero, but for $N = R/x^2R$ we have $N_e = 0$, because the ideal x^2R is 2-sided.

Recall that an ideal P is said to be *prime* if $aRb \subseteq P$ implies $a \in P$ or $b \in P$; and *completely prime* if $ab \in P$ yields $a \in P$ or $b \in P$, thus each completely prime ideal is prime. Below we will see chain domains with prime ideals which are not completely prime.

We will also use the following remark.

Fact 2.7. (See [1, Theorem 1.15].) Every idempotent ideal of a chain domain is completely prime. Furthermore for any proper ideal I of a chain domain, the ideal $I(1)$ is completely prime.

3. Pure projective modules

Recall that over a chain domain R a module M is *pure projective* if it is a direct summand of a direct sum of uniserial modules R/rR, $r \in R$, i.e. if M is in Add of the category of finitely presented modules. Our aim is to prove that over certain chain domains pure projective modules decompose into direct sums of finitely presented modules. Thus we are looking for obstructions for this to happen. One has been known for a long time.

Proposition 3.1. *([9, Theorem 1.1] and [16, Theorem 5.8]) Suppose that $0 \neq r \in J$. Then the following are equivalent.*

1) *There exists a uniserial, hence indecomposable, non-finitely generated pure projective module V which is a direct summand of $(R/rR)^{(\omega)}$, i.e. $V \in \mathrm{Add}(R/rR)$.*
2) *There is an epimorphic endomorphism g of R/rR which is not mono and a monomorphic endomorphism f of R/rR which is not epi such that $gf = 0$.*
3) *The ideal RrR is idempotent.*

Observe that from the stronger condition $r = sr^2t$ for some $s, t \in J$ we can produce a pair of morphisms f, g as in 2) with $\ker g = \mathrm{im}\, f$. Namely from $(1+sr)^{-1}sr \cdot r = r(1+t)^{-1}$ it follows that $(1 + sr)^{-1}sr \times -$ defines the monomorphism of R/rR whose image equals srR/rR. Similarly $(1+s)^{-1}r = r \cdot rt(1+rt)^{-1}$ shows that $(1+s)^{-1} \times -$ is the epimorphism of R/rR whose kernel is generated by $(1 + s)r$, hence equals srR/rR.

In fact if 1) holds true we can always find $r' \in R$ such that $Rr'R = RrR$ and $r' \in Jr'^2J$. Namely, by [16, Theorem 5.8], it implies that $r = ab$ for some $a, b \in R$ such that $RaR = RbR = RrR$. By writing $a = urv$ and $b = u'rv'$ for units $u, u, v, v' \in R$ (see Fact 2.3), such an r' is easily found.

Note that a chain domain with Krull dimension has no (non-trivial) idempotent ideals. Moreover, no commutative valuation domain (with or without Krull dimension) contains a non-trivial idempotent ideal of the form RrR. Thus we need a more advanced example to satisfy the conditions of Proposition 3.1.

3.1. Example

Recall that a chain domain is said to be *nearly simple* if $J = J^2$ is the only nonzero 2-sided ideal of R. For the following construction of a nearly simple chain domain we refer to [1, p. 51].

Let $G = \{at + b \mid a, b \in \mathbb{Q}, a > 0\}$ be the group of affine linear transformations of the plane, where the multiplication is given by composition. For instance $(2t+1)\cdot(t/3+1) = 2(t/3 + 1) + 1 = 2t/3 + 3$.

Recall that a multiplicative subset T of a domain S is said to be a *right Ore set* if, for every $r \in T$ and $0 \neq s \in S$, there are nonzero $a \in S$ and $b \in T$ such that $ra = sb$. We call this common value a *right common denominator* of r and s.

We follow [1, p. 52] in the proof of the following statement.

Remark 3.2. FG is an Ore domain for any field F.

Proof. Clearly $H = \{t + b \mid b \in \mathbb{Q}\}$ is a subgroup of G isomorphic to the additive group of $\mathbb{Q}$; and $U = \{at \mid a \in \mathbb{Q}, a > 0\}$ is a subgroup of G isomorphic to the multiplicative group of positive rationals. Furthermore U acts on H by conjugation: $(at)^{-1}(t + b)at = a^{-1}t \cdot (at + b) = t + a^{-1}b$. Thus G is the semidirect product of H and U, in particular G is metabelian.

Since H is a linearly ordered abelian group, $S := FH$ is a commutative domain, hence Ore. For any finitely generated subgroup U_0 of U the subring SU_0 of FG is a crossed product $S * U_0$. Furthermore since U_0 is abelian and free, SU_0 is obtained from S as an iterated skew Laurent polynomial ring (cf. [8, Proposition 1.5.11]), and hence is an Ore domain. The proof of this fact in [7, Theorem 10.28, Exercise 14 on page 318] gives a (not very fast) algorithm for finding common denominators of elements of FG (see an example below). □

Fix an irrational real ε close to but larger than 1, for instance $\varepsilon = 1 + 2^{-10.5}$ will do. Let P consist of the $f \in G$ such that $f(\varepsilon) \geq \varepsilon$, in particular the identity $e = t$ belongs to P but $t/2 + 1/2$ does not. Then $P \cup P^{-1} = G$ and $P \cap P^{-1} = \{e\}$, i.e. P is a *right pure cone* in G (see [2] for more on right cones).

In particular G can be *left (linearly) ordered* by setting $g \le h$ if $g^{-1}h \in P$, i.e. $g(\varepsilon) \le h(\varepsilon)$. For instance $t+1 < 2t$ in G, because $(t+1)(1) = 2t(1) = 2$ but the slope of $2t$ is larger, hence $(t+1)(\varepsilon) < 2t(\varepsilon)$.

Note that this ordering respects left multiplication: $g \le h$ implies $ug \le uh$ for any $u \in G$. However $\le$ does not respect right multiplication, for instance $(t+1)\cdot(t/3+1/3) = t/3+4/3 > 2t\cdot(t/3+1/3) = 2t/3+2/3$, because $(t/3+4/3)(\varepsilon) \approx 5/3 > (2t/3+2/3)(\varepsilon) \approx 4/3$.

Another way to say this is the following. Define the *right ordering* on G by setting $g \le_r h$ if $hg^{-1} \in P$ i.e. if $h^{-1} \le g^{-1}$. This linear ordering respects right multiplication by elements of G and differs from $\le$.

Let P^+ denote the subsemigroup of P consisting of $f \in P$ such that $f(\varepsilon) > \varepsilon$, for instance $e \notin P^+$. It is easily checked that FP^+ is a 2-sided ideal in the semigroup ring FP.

Lemma 3.3. *$T = FP \setminus FP^+$ is a right (and left) Ore set.*

Proof. For simplicity we will write 1 instead of e. Note that each element in T is a scalar multiple of $r = 1 + \sum_i \alpha_i g_i$, $g_i \in P^+$. Thus when proving that T is right Ore it suffices to consider only such r and arbitrary $0 \ne s = \sum_j \beta_j h_j$, $h_j \in P$. Since FG is Ore, there are nonzero $a, b \in FG$ such that $ra = sb$. By factoring out the least (in right ordering) terms we can write $a = (1+\sum_k \gamma_k u_k)\gamma u$, where $u_k \in P^+$, $u \in G$ and $b = (1+\sum_l \delta_l v_l)\delta v$, where $v_l \in P^+$, $v \in G$.

By multiplying out $ra = sb$ we see that u is the least term in ra and the least term in sb is of the form pv for some $p \in P$. Therefore $uv^{-1} \in P$ and multiplying $ra = sb$ by $\delta^{-1}v^{-1}$ on the right we obtain $r(1+\sum_k \gamma_k u_k)\gamma\delta^{-1}uv^{-1} = s(1+\sum_l \delta_l v_l)$, as desired. $\square$

Despite this procedure being algorithmic, it is usually quite lengthy to execute the required calculations in practice. For instance suppose that $r = 1 + (t/3 + 4/3)$ and $s = 1 + (t/3 + 1)$. Then one possibility for a right common denominator of r and s is the following:

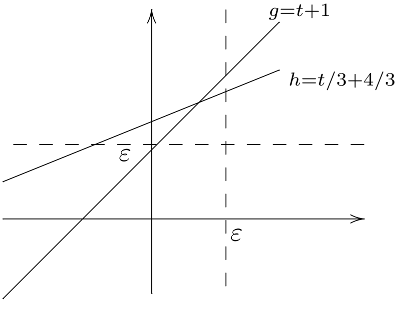
$$1 + (t/3 + 1) + (t/9 + 5/3) - (t/9 + 16/9) - (t/3 + 7/3) - (t + 3)\,.$$

Namely one should take $a = 1 + (t/3 + 1) - (t/3 + 4/3) - (t + 3)$, $b = 1 + (t/3 + 2) - (t/3 + 7/3) - (t + 3)$ and multiply.

Thus we can localize FP with respect to T and obtain a nearly simple chain domain $R = FP_T$. Every element of R can be written as a right fraction $\sum_i \alpha_i g_i \cdot (1+\sum_j \beta_j h_j)^{-1}$ where $g_i \in P$ and $h_i \in P^+$; or as a similar left fraction.

It follows that each principal right ideal of R is of the form gR, $g \in P$; and $g \in hR$ if and only if $g \ge h$. Similarly each principal left ideal of R is of the form Rg, $g \in P$; and $g \in Rh$ if and only if $g \ge_r h$. For instance, let $g = t + 1$ and $h = t/3 + 4/3$, both are in P. Then $t + 1 = (t/3 + 4/3)(3t - 1)$ implies $gR \subset hR$. On the other hand, $(t/3 + 1)(t + 1) = t/3 + 4/3$ yields $Rg \supset Rh$.

It follows that the inclusion ordering on right ideals gR is given by comparing the values $g(\varepsilon)$; and the inclusion ordering on left ideals Rh is determined by the points x such that $h(x) = \varepsilon$, i.e. by intercepts of h with the line $y = \varepsilon$.



Now we are in a position to produce the desired example.

Let $r = t + 1$ and $M = R/rR$. Then the equality $t + 1 = (t/3 + 1) \cdot (t + 1)^2 \cdot (3t - 2)$ shows that $r = sr^2h$ for $s = t/3 + 1$ and $h = 3t - 2$, all elements being in P^+. Thus $f = [1 + (t/3 + 4/3)]^{-1}(t/3 + 4/3) \times -$ and $g = [1 + (t/3 + 1)]^{-1} \times -$ is the pair of morphisms from Proposition 3.1 with $\ker g = \operatorname{im} f = (t/3 + 4/3)R/(t + 1)R$.

Recall that J is the only nonzero 2-sided ideal of R. Furthermore it is easily seen that J is a simple radical ring, even *1-simple*, i.e. for each $a \in J$ there are $b, c \in J$ such that $a = bac$.

4. Dimension theory

In this section we will briefly recall, adapting to our situation, the so-called dimension theory of direct summands of serial modules which was developed in [6]. The module R/rR is said to be of *type 1* if its endomorphism ring S is local and of *type 2* otherwise (see Proposition 2.5).

Let $\mathcal{PP}$ denote the category of pure projective (right) modules over R, i.e. the category of direct summands of direct sums of finitely presented R-modules. For each module R/rR of type 1 define the ideal $\mathcal{J}$ of $\mathcal{PP}$, *associated* to $J(S)$, in the following way. A morphism $f : M \to N$ of pure projective modules is in $\mathcal{J}$ if, for any morphisms $g : R/rR \to M$ and $h : N \to R/rR$, their composition hfg belongs to $J(S)$.

$$\begin{tikzcd}
R/rR \arrow[r, dashed, "J(S)"] \arrow[d, "g"'] & R/rR \\
M \arrow[r, "f"'] & N \arrow[u, "h"']
\end{tikzcd}$$

It follows from [6, Proposition 3.2] that the corresponding factor category $\mathcal{PP}/\mathcal{J}$ is equivalent to the category of (right) vector spaces over the skew field S/J. If M is pure projective then we assign to M the dimension of the image of M in this factor category, written $\dim_{R/rR} M$.

Suppose that R/rR has type 2, therefore its endomorphism ring S has two incomparable maximal ideals: K consisting of non-epimorphisms and L consisting of non-monomorphisms. As above one can factor the category of pure projectives by the ideal associated to K obtaining the category of vector spaces over the skew field S/K. The corresponding dimension of M is called an *e-dimension* (epi-dimension), written $\mathrm{edim}_{R/rR} M$. For instance, $\mathrm{edim}(M) \geq n$ if and only if there exist morphisms $f : (R/rR)^n \to M$ and $g : M \to (R/rR)^n$ whose composition is epi. Furthermore for a countably generated pure projective module M, from $\mathrm{edim}_{R/rR} M \geq n$ for every n it follows that $\mathrm{edim}_{R/rR} M = \omega$.

Similarly using the ideal L one defines an *m-dimension* (mono-dimension), written $\mathrm{mdim}_{R/rR} M$, for every pure projective module M. Here is the main result of [6] applied to our setting. In the case when M, N are countably generated (which will be sufficient for us) it follows directly from [6, Lemma 2.7].

Proposition 4.1. *(see [6, Theorem 7.4]) Let M and N be pure projective modules over a chain domain R. Then $M \cong N$ if and only if*

1) $\dim_{R/rR} M = \dim_{R/rR} N$ *for every module R/rR of type 1; and*
2) $\mathrm{mdim}_{R/rR} M = \mathrm{mdim}_{R/rR} N$ *and* $\mathrm{edim}_{R/rR} M = \mathrm{edim}_{R/rR} N$ *for every module R/rR of type* 2.

Thus every pure projective module over a chain domain is uniquely determined by its *dimension vector*, which is a cardinal-valued vector of dimensions defined above (usually we write epi and mono dimensions relative to a given module R/rR of type 2 in pairs).

If $M = R/rR$ is of type 1, it follows that its relative dimension $\dim_{R/rR} M$ equals 1, and all the remaining dimensions are zero. Similarly if $M = R/rR$ is of type 2 then we have $\mathrm{mdim}_{R/rR} M = \mathrm{edim}_{R/rR} M = 1$, and all the remaining dimensions are zero. If one is lucky enough to construct a sufficient supply of pure projective modules, it is possible to complete a classification of such modules by matching this list against possible dimension vectors.

By Corollary 2.3 one could parameterize the set of isomorphism types of modules R/rR by the set of ideals which can be generated by one element. For instance let R be the chain domain from Example 3.1. It has only one (indecomposable finitely presented) module R of type 1, with dimension $(1, [0, 0])$; and one module R/rR, $0 \neq r \in J$ of type 2, with dimension $(0, [1, 1])$. Furthermore there exists (see Proposition 3.1) a (unique!) module $V \in \mathrm{Add}(R/rR)$ with dimension $(0, [1, 0])$, where 1 is the m-dimension. In this way one could complete a classification of pure projective modules over nearly simple chain domains (see [13]).

It may be advantageous to explain the dimension theory from the point of view of idempotent matrices. Suppose that a pure projective module M is realized as a direct summand of a direct sum $\oplus_{i\in I} R/r_iR$ of finitely presented modules. Let us assume that $r_i = r_j$ whenever R/r_iR and R/r_jR are isomorphic. The corresponding projection to M is given by a column finite $I \times I$ idempotent matrix acting on the column R/r_iR, $i \in I$ of height I by multiplication on the left. The (i,j) entry of this matrix is in $\mathrm{Hom}(R/r_jR, R/r_iR)$. For $k \in I$ let $I_k = \{i \in I \mid r_i = r_k\}$.

To measure the m-dimension with respect to R/r_kR consider the $I_k \times I_k$-submatrix. This is a column finite matrix over the ring $S_k = \mathrm{End}(R/r_kR)$. After modding out the ideal L_k consisting of non-monos we obtain a column finite matrix over S_k/L_k. The m-dimension of M equals the column rank of this matrix.

We will use the following straightforward consequence of Proposition 4.1.

Corollary 4.2. *Let N be a pure projective module over a chain domain R. Then N is a direct sum of finitely presented modules if and only if* $\mathrm{mdim}_M N = \mathrm{edim}_M N$ *for every $M = R/rR$ of type 2.*

Proof. If $N = \oplus_{i\in I} R/s_iR$ and $M = R/rR$ is of type 2 then both dimensions $\mathrm{mdim}_M(N)$ and $\mathrm{edim}_M(N)$ are equal to the cardinality of the set $\{i \in I \mid Rs_iR = RrR\}$.

Conversely suppose that N is a direct summand of $\oplus_{i\in I} R/s_iR$ such that $\mathrm{mdim}_M N = \mathrm{edim}_M N$ for every $M = R/rR$ of type 2. Let T be a set of representatives of isoclasses of modules $R/s_iR, i \in I$, i.e. for every $i \in I$ there exists a unique $M \in T$ such that $M \simeq R/s_iR$. For any $M \in T$ let d_M be a dimension of N related to M (by our assumption $\mathrm{mdim}_M N$ and $\mathrm{edim}_M N$ coincide if M is of type 2). Let $N' = \oplus_{M\in T} M^{(d_M)}$.

Comparing dimension, by Proposition 4.1, we conclude that $N \cong N'$. □

We conclude this section with a useful observation saying that a finitely presented module is a direct summand of a pure projective module unless it is obstructed by the dimension.

Lemma 4.3. *Suppose that $M = R/rR$ is of type 2 and let N be a pure projective module such that* $\mathrm{mdim}_M N > 0$ *and* $\mathrm{edim}_M N > 0$*. Then M is a direct summand of N.*

Proof. Let $\alpha \in \mathrm{End}(M)$ be a non-epic monomorphism and let $\beta \in \mathrm{End}(M)$ be a non-monic epimorphism. Since $\mathrm{mdim}_M N > 0$ there are morphisms $f_1 : M \to N$, $g_1 : N \to M$ such that $g_1 f_1$ is mono. Similarly, $\mathrm{edim}_M N > 0$ implies that there exist morphisms $f_2 : M \to N$ and $g_2 : N \to M$ such that $g_2 f_2$ is epi.

Let $h_1 = f_1\alpha + f_2\beta : M \to N$ and $h_2 = \alpha g_1 + \beta g_2 : N \to M$. Then

$$h_2 h_1 = \alpha g_1 f_1 \alpha + \alpha g_1 f_2 \beta + \beta g_2 f_1 \alpha + \beta g_2 f_2 \beta\,.$$

The only monomorphism in this sum is $\alpha g_1 f_1 \alpha$, hence $h_2 h_1$ is mono. Furthermore the only epimorphism is $\beta g_2 f_2 \beta$, hence $h_2 h_1$ is epi. It follows that $h_2 h_1$ is an isomorphism of M, in particular $N = h_1(M) \oplus \ker h_2$, where $h_1(M) \cong M$. □

5. The Mittag–Leffler construction

Recall (see [5, Theorem 2.47]) that every pure projective module M is a direct sum of countably generated modules, therefore in most considerations we may assume that M is countably generated. There is a uniform way to construct such modules from finitely presented ones.

Recall that a directed system of morphisms $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots$ is said to be *Mittag–Leffler*, ML for short, if there are morphisms $g_n : M_{n+2} \to M_{n+1}$ such that $g_n f_{n+1} f_n = f_n$:

$$\begin{tikzcd} M_1 \arrow[r, "f_1"'] & M_2 \arrow[r, "f_2"'] & M_3 \arrow[r, "f_3"'] \arrow[l, bend right, "g_1"'] & M_4 \arrow[r, "f_4"'] \arrow[l, bend right, "g_2"'] & \dots \arrow[l, bend right, "g_3"'] \end{tikzcd}$$

The following fact found some uses in both algebra (see [17]) and model theory (see [16]).

Fact 5.1. The direct limit of a Mittag–Leffler directed system of finitely presented modules is a countably generated pure projective module. Furthermore each countably generated pure projective module can be obtained as a direct limit of a Mittag–Leffler directed system of finitely presented modules.

It follows from the proof of this result that one can choose the f_i to be monomorphisms, but this restriction is rather limiting in applications. To prove the Fact, one shows that the canonical exact sequence

$$0 \to \bigoplus_{i\in I} M_i \xrightarrow{f} \bigoplus_{i\in I} M_i \to M = \varinjlim M_i \to 0\,,$$

where $f(m_i) = m_i - f_i(m_i), m_i \in M_i$, splits, therefore $M \bigoplus \bigoplus_{i\in I} M_i \cong \bigoplus_{i\in I} M_i$, in particular $M \in \mathrm{Add}(\bigoplus_{i\in I} M_i)$.

The following construction is the main technical tool in this paper, and it works over any ring. To some extent it is a basic free version of the model theoretic construction of [14].

Suppose that M, N are R-modules and there are endomorphisms f_k, f_{k+1}, α_k of M and morphisms $g_k : M \to N$, $\beta_k : N \to M$ such that $f_k - \alpha_k f_{k+1} f_k = \beta_k g_k$:

$$\begin{tikzcd}
M \arrow[rr, "f_k"] \arrow[dr, "g_k"'] & & M \arrow[r, "f_{k+1}"] & M \arrow[l, bend left, "\alpha_k"] \\
& N \arrow[ur, "\beta_k"'] & &
\end{tikzcd}$$

The meaning is that $f_k - \alpha_k f_{k+1} f_k$ factors through N, hence the sequence of morphisms $M \xrightarrow{f_1} M \xrightarrow{f_2} \dots$ becomes Mittag–Leffler when we mod out the ideal of $\operatorname{Mod} R$ generated by 1_N. Now we will produce a concrete ML-sequence.

For every k set $F_k = M \oplus N^{k-1}$, for instance $F_1 = M$, $F_2 = M \oplus N$ and $F_3 = M \oplus N^2$. We define morphisms $\varphi_k : F_k \to F_{k+1}$ by the following formula:

$$\varphi_k = \begin{pmatrix} f_k & 0_{1\times k-1} \\ g_k & 0_{1\times k-1} \\ 0_{k-1\times 1} & I_{k-1} \end{pmatrix}$$

if $k \geq 2$, and $\varphi_1 = \left(\begin{smallmatrix} f_1 \\ g_1 \end{smallmatrix}\right)$. Thus φ_k is obtained by mapping the first component M into $M \oplus N$ via (f_k, g_k), and shifting the copy of N^{k-1} by one to the right: $\varphi_k(m, n_1, \dots, n_{k-1}) = (f_k(m), g_k(m), n_1, \dots, n_{k-1})$.

We will show that this defines a directed system (F_k, φ_k) which is Mittag–Leffler. For this consider morphisms $\psi_k : F_{k+2} \to F_{k+1}$ defined by the following formula:

$$\psi_k = \begin{pmatrix} \alpha_k & 0 & \beta_k & 0_{1\times k-1} \\ 0 & 0 & 1 & 0_{1\times k-1} \\ 0_{k-1\times 1} & 0_{k-1\times 1} & 0_{k-1\times 1} & I_{k-1} \end{pmatrix}$$

for $k \geq 2$, and $\psi_1 = \left(\begin{smallmatrix} \alpha_1 & 0 & \beta_1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

It is easily checked that $\psi_k \varphi_{k+1} \varphi_k = \varphi_k$ for every k, for instance

$$\psi_2\varphi_3\varphi_2 = \begin{pmatrix} \alpha_2 & 0 & \beta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_3 & 0 & 0 \\ g_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_2 & 0 \\ g_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha_2 f_3 & \beta_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_2 & 0 \\ g_2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_2 f_3 f_2 + \beta_2 g_2 & 0 \\ g_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f_2 & 0 \\ g_2 & 0 \\ 0 & 1 \end{pmatrix} = \varphi_2 \,.$$

If M, N are finitely presented then $W = \varinjlim F_k$ is a countably generated pure projective module such that $W \bigoplus \bigoplus_k F_k \cong \bigoplus_k F_k = M^{(\omega)} \bigoplus N^{(\omega)}$, in particular $W \in \operatorname{Add}(M \oplus N)$.

Furthermore by the construction N^k is a direct summand of W for any k, in particular W is decomposable. If R is a chain ring, it follows from the dimension theory that $N^{(\omega)}$ is a direct summand of W.

We will derive two useful facts.

Lemma 5.2. *Suppose that in the above construction R is a chain ring, $M = R/rR$ has type 2, $N = R/sR$ is not isomorphic to M, and each f_n is a monomorphism which is not epi. Then* $\operatorname{mdim}_M W = 1$ *and* $\operatorname{edim}_M W = 0$.

Similarly if each f_n is epi and not mono, then $\operatorname{mdim}_M W = 0$ *and* $\operatorname{edim}_M W = 1$.

Proof. We will prove only the first part. Recall that $S = \operatorname{End}(M)$ contains a maximal ideal L consisting of non-monomorphisms such that $D = S/L$ is a skew field. We look at the additive functor G (see [6]) from the category of pure projective R-modules into the category of D-modules, for instance G preserves arbitrary direct sums. It is easily derived that G preserves direct limits of Mittag–Leffler sequences.

Apply this functor to the ML-sequence (F_i, f_i) defining W. Since N is not isomorphic to M, it is annihilated when applying this functor. Thus the resulting directed system has the form $D \xrightarrow{G(f_1)} D \xrightarrow{G(f_2)} \ldots$ with nonzero maps $G(f_i)$. It follows that the direct limit of this sequence is isomorphic to D, therefore $\operatorname{mdim}_M(W) = 1$.

For a similar functor H to the category of S/K-modules, where K is the ideal of S consisting of non-epimorphisms, the resulting maps $H(f_i)$ are zero, therefore the direct limit is a zero object. It follows that $\operatorname{edim}_M(W) = 0$. □

6. Epi-dimension

Now we will give the first application of our main construction. Recall (see after Example 2.1) that M_e denotes the sum of kernels of all epic endomorphisms of a module M.

Proposition 6.1. *Let $M = R/rR$ be a module of type 2. Then the following are equivalent.*

1) *There exists a countably generated pure projective module W such that* $\operatorname{mdim}_M W = 0$ *and* $\operatorname{edim}_M W = 1$.
2) *There exists a countably generated pure projective module W such that* $\operatorname{mdim}_M W < \operatorname{edim}_M W$.
3) *There exists a module $N = R/sR$ not isomorphic to M and morphisms $g : M \to N$, $\beta : N \to M$ such that $\beta g(M_e) \neq 0$.*
4) *There exists a module $N = R/sR$ not isomorphic to M, endomorphisms f and α of M, where f is epi not mono, and morphisms $g : M \to N$, $\beta : N \to M$ such that $\alpha(f^2 + \beta g) = f$.*
5) *There exists a module $N = R/sR$ not isomorphic to M, epimorphisms $f_1, f_2 : M \to M$, where f_1 is not mono, and morphisms $g' : M \to N$ and $\beta' : N \to M$ such that $f_2 f_1^2 - f_1 = \beta' g'$.*

Proof. 1) ⇒ 2) is trivial and 2) ⇒ 1) follows from Lemma 4.3.

1) ⇒ 3). Realize W as a direct summand of a direct sum of finitely presented modules $M_k = R/r_k R$, $k = 1, \ldots$ hence as a column finite idempotent $\omega \times \omega$ matrix which we also denote by W. The entries of this matrix are morphisms f_{ij} from M_j to M_i.

Since $\mathrm{edim}_M W > 0$ there are $i, j \in \omega$ such that $M_i, M_j \cong M$ and f_{ij} is epi (and not mono, since $\mathrm{mdim}_M W = 0$). Fix such a j. Because the matrix W is column finite, there are only finitely many nonzero morphisms f_{lj}, $l \in \omega$. Choose i such that $M_i \cong M$ and f_{ij} has the smallest kernel among morphisms f_{lj} with $M_l \cong M$. Because some f_{lj} is epi, we conclude that $0 \subset \ker f_{ij} \subseteq (M_j)_e$.

Since W is idempotent we have $f_{ij} = \sum_k f_{ik} f_{kj}$, where almost all summands are zero. If $M_k \cong M$ and $f_{kj} \neq 0$, then the kernel of $f_{ik} f_{kj}$ is strictly larger than the kernel of f_{ij}. Namely, it follows from $M_k \cong M$ and $\mathrm{mdim}_M W = 0$ that $\ker f_{ik} \neq 0$, hence $\ker f_{ik} f_{kj} \supset \ker f_{kj} \supseteq \ker f_{ij}$. Thus there is a k such that $\ker f_{ik} f_{kj} \subseteq \ker f_{ij}$ and M_k is not isomorphic to M.

Since $M_i \cong M_j \cong M$ and $\ker f_{ij} \subset (M_j)_e$ we conclude that $(M_j)_e$ is not contained in the kernel of $f_{ik} f_{kj}$. Fix isomorphisms $h_j : M \to M_j$ and $h_i : M_i \to M$.

$$\begin{tikzcd}
M \arrow[r, "h_j"] & M_j \arrow[rr, "f_{ij}"] \arrow[dr, "f_{kj}"'] & & M_i \arrow[r, "h_i"] & M \\
& & M_k \arrow[ur, "f_{ik}"'] & &
\end{tikzcd}$$

Clearly we can take $N = M_k$ and $g = f_{kj} h_j : M \to M_k$, $\beta = h_i f_{ik} : M_k \to M$.

3) $\Rightarrow$ 4). Because $\beta g(M_e) \neq 0$ it follows that $\ker \beta g \subset M_e$. Since $\ker \beta g$ is a cyclic submodule of M, $\ker \beta g \subseteq \ker f$ for a certain epimorphism f of M with a nonzero kernel.

From $\ker \beta g \subset \ker f^2$ we conclude that the kernel of $f^2 + \beta g$ equals $\ker \beta g$. Because f is epi and βg is not (by Corollary 2.4, since M is not isomorphic to N) it follows that $f^2 + \beta g$ is epi. Clearly there is a unique morphism α of M such that $\alpha(f^2 + \beta g) = f$.

4) $\Rightarrow$ 5) Take $f_1 = f$, $f_2 = \alpha$ and $g' = g$, $\beta' = -\alpha\beta$.

5) $\Rightarrow$ 1) follows from Lemma 5.2 if we set $f_k = f_{k+1} = f_1$, $\alpha_k = f_2$, $\beta_k = \beta'$ and $g_k = g'$. □

A module $M = R/rR$ satisfying the conditions of Proposition 6.1 is said to have *type 2e*. We will see later that there are no such modules over chain domains with Krull dimension, therefore Example 2.1 does not work. If R is as in Example 3.1 then M is the only candidate for a module of type $2e$. Furthermore $\mathrm{Hom}(M, R) = 0$ for the only remaining (indecomposable finitely presented) module R. Thus applying item 5) of the Proposition we obtain $f_2 f_1^2 = f_1$, which is impossible (comparing the kernels) because f_1 is not mono.

So we have to work harder to produce an example. The following one is due to Dubrovin [3] (see [2] for explanations).

6.1. Example

Recall that the trefoil group G is given by generators u, v and relation $u^3 = v^2$. We will use another presentation of G, choosing as generators $x = u^{-1}vu^{-1}$ and $y = u$. It is easily checked that the relation becomes $xy^2x = y$.

Dubrovin proved that the semigroup P generated by e, x and y is a right pure cone in G, so we can define the left linear ordering on G by setting $g \leq h$ if $g^{-1}h \in P$. For instance, $x < y$ because $x^{-1}y = y^2x \in P$. Again, this ordering respects left but not right multiplication by elements of G. For example multiplying $1 < x$ by y on the right we obtain $y > xy$, because $y^{-1} \cdot xy = y^{-1}x \cdot y = x^{-1}y^{-2} \cdot y = x^{-1}y^{-1} \in P^{-1}$.

The corresponding right linear ordering, defined by $g \leq_r h$ if $hg^{-1} \in P$, respects right multiplication but differs from the left ordering: $x < y$ but $x >_r y$ because $y^{-1} < x^{-1}$.

For any field F, Dubrovin [3] constructed an embedding of the group ring FG into a skew field D and associated with this embedding a chain domain R, a non-classical localization of the semigroup ring FP. Every principal right ideal of R is of the form gR for some $g \in P$, and every principal left ideal is of the form Rh, $h \in P$. From $x < y$ it follows that $xR \supset yR$, but $x >_r y$ yields the opposite inclusion $Rx \subset Ry$ of principal left ideals.

Recall that y^3 generates the center of G. Namely $x^{-1}y^3x = x^{-1}y \cdot y^2x = y^2x \cdot y^2x = y^3$. It follows that the 2-sided ideals of R form the following chain:

$$R \supset J = xR \supset x^2R \supset \ldots \supset RyR = Ry^2R \supset y^3R \supset Ry^4R \supset y^6R \supset \ldots .$$

This ring has only three completely prime ideals J, RyR and 0; furthermore y^3R is a prime ideal which is not completely prime.

Now we are ready for the next example. By Proposition 3.1 the equality $xy^2x = y$ implies that the module $M = R/yR$ possesses a monomorphism f which is not epi, and an epimorphism g which is not mono such that $\ker g = \operatorname{im} f$, in particular M has type 2.

Lemma 6.2. *$M = R/yR$ has type 2e.*

Proof. Take $N = R/y^3R$ in Proposition 6.1. Since y^3 is a central element, this module is of type 1. We will check item 3) of the Proposition.

Clearly the left multiplication by $g = y^2$ defines the morphism from R/yR to R/y^3R. Similarly $x^2y^3 = y^3x^2 \in yR$ shows that $\beta = x^2 \times -$ is the morphism from R/y^3R to R/yR.

$$\begin{tikzcd}
R/yR \arrow[rr, "x^2y^2\times -"] \arrow[dr, "y^2\times -"'] & & R/yR \\
& R/y^3R \arrow[ur, "x^2\times -"'] &
\end{tikzcd}$$

Thus the composite endomorphism βg of M is given by left multiplication by x^2y^2.

Note that $xy < y$, hence the class $m = \overline{xy}$ of this element is nonzero in R/yR. Furthermore $xy \in RyR$ implies $m \in M_e$ (by Corollary 2.6). Also $\beta g(m) = x^2y^2xy = x \cdot xy^2x \cdot y = xy^2 < y$, therefore $\beta g(m) \neq 0$. □

Thus by Proposition 6.1 there exists a countably generated pure projective module $W \in \mathrm{Add}(R/yR \oplus R/y^3R)$ whose dimension vector is $([0,1], \omega)$, where 1 is the epi-dimension. Comparing dimensions it is easily seen that this module has no indecomposable decomposition: each direct sum decomposition of W has the form $W = W \oplus (R/y^3R)^{(\lambda)}$, $\lambda \leq \omega$.

Recall that a chain ring R' is said to be *exceptional* if it is prime, contains zero divisors, and its Jacobson radical J' is the only nonzero ideal of R'. One could factorize R by y^3R and localize with respect to the completely prime ideal RyR obtaining a coherent exceptional chain ring R'.

The image W' of W has similar properties over R'. This is exactly the module that was constructed in [14] using model theory. Note that pure projective modules over exceptional chain rings (coherent or not) can be classified following Příhoda's approach (see [11] and [15]).

7. Mono-dimension

We would like to dualize Proposition 6.1 (by interchanging epis and monos) but this is far from straightforward. First we need some preliminaries.

Recall that every endomorphism of the module $M = R/rR$ is given by left multiplication by an element $s \in R$ such that $sr \in rR$, and s is uniquely determined modulo rR. We will show that under mild restrictions the left ideal Rs is uniquely determined.

Lemma 7.1. *Suppose that $f = a \times -$ and $g = b \times -$ define the same endomorphism of $M = R/rR$ and $a \notin RrR$, hence $b \notin RrR$. Then $Ra = Rb$, therefore the left ideal Ra is uniquely determined by f.*

For instance this is the case when RrR is not an idempotent ideal and f is mono.

Proof. Otherwise we may assume that $b = ja$ for some $j \in J$. From $f = g$ we conclude that $a - ja = rt$ for some $t \in R$, and therefore $a = (1-j)^{-1}rt \in RrR$, a contradiction.

If $a \times -$ is a monomorphism, then $ar = ru$ for a unit u and therefore $r = aru^{-1}$. Then the assumption $a \in RrR$ would lead to $r \in (RrR)^2$, a contradiction. □

Recall that $S = \mathrm{End}(M)$ denotes the endomorphism ring of a module M. If $f_1, f_2 \in S$ and $\ker f_1 \subset \ker f_2$ then obviously $f_1 \notin Sf_2$; otherwise it is quite difficult to decide whether $f_1 \in Sf_2$ or not. In some cases we can conclude.

Lemma 7.2. *Suppose that the monomorphic endomorphisms f_1, f_2 of $M = R/rR$ are given by left multiplication by t_1, t_2. Then $t_1 \in Rt_2$ implies $f_1 \in Sf_2$.*

Furthermore if $t_1, t_2 \notin RrR$ (for instance, if RrR is not idempotent) then the converse also holds true.

Proof. Let $t_1 = st_2$ for some $s \in R$. By Corollary 2.2 we have $t_1rR = rR = t_2rR$, hence $srR = st_2rR = t_1rR = rR$. By the same corollary the map $s \times -$ defines an endomorphism g of M which is mono, and clearly $f_1 = gf_2$.

For the converse suppose that $f_1 = hf_2$, where h is given by multiplication by s. By Lemma 7.1 we obtain $Rt_1 = Rst_2$, therefore $t_1 \in Rt_2$. □

Now we formulate the main result of this section. This is almost, but not completely, dual to Proposition 6.1. Namely there are no analogs of items 3) and 4) of that Proposition. Perhaps this reflects an elusive non-symmetry between monomorphisms and epimorphisms of uniserial modules.

Proposition 7.3. *Let $M = R/rR$ be a module of type 2. Then the following are equivalent.*

1) *There exists a countably generated pure projective module W such that $\operatorname{mdim}_M W = 1$ and $\operatorname{edim}_M W = 0$.*
2) *There exists a countably generated pure projective module W such that $\operatorname{mdim}_M W > \operatorname{edim}_M W$.*
3) *There exists a module $N = R/sR$ not isomorphic to M, monomorphisms $f_1, f_2 : M \to M$, where f_1 is not epi, and morphisms $g : M \to N$, $\beta : N \to M$ such that $f_2f_1^2 - f_1 = \beta g$.*

Proof. 1) ⇒ 2) is obvious and 2) ⇒ 1) follows from Lemma 4.3.

1) ⇒ 3). Suppose first that RrR is an idempotent ideal. Using Proposition 3.1 choose a monomorphism $f : M \to M$ which is not epi and an epimorphism $g : M \to M$ which is not mono such that $gf = 0$. It follows that $f + g$ is invertible, hence the equality $(g + f)f = f^2$ implies $f = (g + f)^{-1}f^2$. So we can set $f_1 = f$, $f_2 = (g + f)^{-1}$ and $\beta = g = 0$, hence $N = 0$.

Therefore we may assume that the ideal RrR is not idempotent. Realize W as a direct summand of a direct sum of finitely presented modules $M_k = R/r_kR$, $k = 1, \dots,$ hence as a column finite idempotent $\omega \times \omega$ matrix W. The (i, j) entry of this matrix is a morphism $f_{ij} = s_{ij} \times -$ from M_j to M_i. To simplify our arguments we assume that $r_i = r_j$ whenever $M_i \simeq M_j$.

Since $\operatorname{mdim}_M W > 0$ there are i, j such that $M_i, M_j \cong M$ and f_{ij} is a monomorphism (which is not epi, since $\operatorname{edim}_M W = 0$). Fix such i, j. By identifying things we will assume that $M_i = M_j = M$ and $r_i = r_j = r$. Because RrR is not idempotent it is easily seen that $s_{ij} \notin RrR$. In particular, by Lemma 7.1, the left ideal Rs_{ij} is uniquely determined by f_{ij}.

Because the matrix W is column finite there are only finitely many nonzero morphisms f_{lj} with this fixed j. Choose t such that $M_t \cong M$ and the ideal Rs_{tj} is largest among the ideals Rs_{lj} with $M_l \cong M$. In particular, as above, the left ideal Rs_{tj} is also uniquely determined by f_{tj}.

From the equality $f_{tj} = \sum_k f_{tk} f_{kj}$ we obtain $Rs_{tj} = R \sum_k s_{tk} s_{kj}$. If $M_k \cong M$ then $\mathrm{edim}_M W = 0$ implies $s_{tk} \in J$. So $Rs_{tk}s_{kj} \subset Rs_{kj} \subseteq Rs_{tj}$ by our choice of t. Therefore there exists a k such that $M_k \ncong M$ and $s_{tj} \in Rs_{tk}s_{kj}$. Furthermore $s_{ij} \in Rs_{tj}$ yields $s_{ij} \in Rs_{tk}s_{kj}$.

Let $f_1 = f_{ij} \in \mathrm{End}(R/rR)$. Since f_{kj} is not mono it follows that $f_1^2 + f_{tk}f_{kj}$ is a monomorphism given by $(s_{ij}^2 + s_{tk}s_{kj}) \times -$. Obviously $Rs_{ij} \subseteq Rs_{tk}s_{kj} = R(s_{ij}^2 + s_{tk}s_{kj})$ therefore (by Lemma 7.2) there exists $f_2 \in \mathrm{End}(R/rR)$ such that $f_1 = f_2(f_1^2 + f_{tk}f_{kj})$.

It remains to take $N = M_k$, $\beta = -f_2 f_{tk}$ and $g = f_{kj}$.

3) $\Rightarrow$ 1). Follows from Lemma 5.2. $\square$

We will say that a module $M = R/rR$ satisfying the conditions of Proposition 7.3 has *type 2m*.

Let R be a nearly simple chain domain from Example 3.1 and $M = R/(t+1)R$. We have constructed in this example a pair of endomorphisms f, g of M such that f is mono not epi, g is epi not mono and $\ker g = \operatorname{im} f$. It follows from the proof of the proposition that the directed system $M \xrightarrow{f} M \xrightarrow{f} \dots$ is Mittag–Leffler and its direct limit V is a pure projective (uniserial countably generated) module in $\mathrm{Add}(M)$ of relative dimension $[1, 0]$, where 1 is the m-dimension.

In this case $N = 0$ in the item 3) of the Proposition.

8. Factoring morphisms

Informally we will consider elements of R as being located on a vertical line. We will put an element s below r if $RsR \subset RrR$. According to this convention we will consider morphisms $R/rR \to R/sR$ as going down and morphisms $R/sR \to R/rR$ as going up.

The following technical lemma on factorization of descending and ascending morphisms has a straightforward proof.

Lemma 8.1. *Let R be a chain domain and $0 \neq r \in J$ is such that rR is not a 2-sided ideal. Let $P = RrR$ and let Q denote the largest ideal which is contained in rR, hence $Q \subset rR \subset P$. Suppose that f is an endomorphism of $M = R/rR$.*

1) *f factors through R/Q if and only if it factors through R/sR for some $s \in Q$.*
2) *f factors through R/P if and only if it factors through R/sR for some $s \in R$ such that $P \subseteq sR$.*

In what follows we will often consider the situation as in the Lemma. Define $L = \{t \in R \mid tr \in Q\}$. Clearly $0 \in L, 1 \notin L$ and L respects left multiplication by elements of R, therefore L is a left ideal. Furthermore if P is not an idempotent ideal then $rR \subseteq L$, hence $P \subseteq L$.

To show that L is not always a right ideal let us consider Example 6.1. Take $r = y$ therefore $P = RyR$ and Q is generated by the central element y^3. Clearly $y^2 \in L$ but $y^2x \notin L$, i.e. $s = y^2xy \notin y^3R$. Namely when multiplying s by x on the left we obtain $xs = xy^2xy = y^2$, which is not in y^3R.

Below we will often need to verify the condition $LP = P$. In the above example we have $y^2 \in L$ and $xy \in P$, therefore $y^2xy \in LP$. It is easily checked that this element generates P as a 2-sided ideal, hence $LP = P$.

Now we will add one more equivalent condition to Proposition 6.1.

Proposition 8.2. *Suppose that $M = R/rR$ has type 2, in particular rR is not a left ideal and P, Q and L are defined as in Lemma 8.1. Then the following are equivalent.*

1) *There exists a countably generated pure projective module W with* $\mathrm{mdim}_M W = 0$ *and* $\mathrm{edim}_M W > 0$.
2) *$LP = P$.*

Proof. Clearly we have strict inclusions $Q \subset rR \subset P$.

1) ⇒ 2). By item 3) of Proposition 6.1 there exists a module $N = R/sR$ not isomorphic to M and morphisms $g : M \to N$, $\beta : N \to M$ such that $\beta g(M_e) \neq 0$. Recall that, by Fact 2.6, we have $M_e = RrR/rR = P/rR$.

If $s \notin P$ then P is a proper subset of sR. Suppose that g is given by left multiplication by $w \in R$. Then $g(M_e) = (wP + sR)/sR = sR/sR$ is zero in R/sR, a contradiction. Thus $s \in P$.

Because M is not isomorphic to N it follows that $RsR \subset RrR$, hence $s \in Q$ (because there are no 2-sided ideals between Q and P). Let $g : M \to N$ be given by left multiplication by x, hence $xr = st$ for some $t \in R$. From $s \in Q$ we conclude that $x \in L$.

If $xP \subseteq Q$ then, for any $y \in R$, we obtain $yxP \subseteq Q \subseteq rR$, hence $\beta g(M_e) = 0$, a contradiction. Otherwise $RxP \subseteq LP$ is a 2-sided ideal properly containing Q hence equal to P. Thus $LP = P$.

2) ⇒ 1). Since rR is not a 2-sided ideal, there exists $x \in P \setminus rR$. Because $P = LP$ we can write $x = yz$ for some $y \in L$ and $z \in P$, in particular $yr \in Q$ and yP is not contained in rR.

Since $yr \in Q$ we conclude that $N = R/yrR$ is not isomorphic to M and also that yrR is a proper subset of rR. Let $N = R/yrR$ and let $g : M \to N$ be given by left multiplication by y. Further let $\beta : N \to M$ be the natural projection. Since yP is not contained in rR, we conclude that $\beta g(M_e) \neq 0$, as desired. □

9. Main results

In this section we will prove the following main result of the paper.

Theorem 9.1. *Let R be a chain domain. Then the following are equivalent.*

1) *For every $0 \neq r \in J$, RrR is not an idempotent ideal.*
2) *Every pure projective R-module is a direct sum of finitely presented modules.*

Notice from the dimension theory that the decomposition $M = \oplus_{i \in I} R/r_iR$ as in 2) is essentially unique: if $M = \oplus_{j \in J} R/s_jR$ is another such decomposition then there is a bijection $f : I \to J$ such that $R/r_iR \cong R/s_{f(i)}R$, i.e. $Rr_iR = Rs_{f(i)}R$: combine [5, Theorem 9.12] and Corollary 2.4 to see this.

Because there are no non-trivial idempotent ideals in chain domains with Krull dimension we immediately obtain

Corollary 9.2. *If a chain domain R has Krull dimension then every pure projective R-module is isomorphic to the module $\oplus_{i \in I} R/r_iR$, $r_i \in R$.*

Note that Theorem 9.1 applies to commutative valuation domains without Krull dimension, but Corollary 9.2 does not.

We proceed with the proof of the Theorem. If RrR is a (nonzero) idempotent ideal, it follows from Proposition 3.1 that R possesses an infinitely generated uniserial pure projective module, from which 2) $\Rightarrow$ 1) follows.

Thus it remains to prove 1) $\Rightarrow$ 2). **From now on we assume that R is a chain domain without idempotent ideals RsR, $0 \neq s \in J$.**

Suppose by a way of contradiction that there exists a pure projective R-module W which is not a direct sum of finitely presented modules. Because every pure projective is a direct sum of countably generated modules we may assume that W is countably generated.

Applying Corollary 4.2 and Lemma 4.3 it follows that there exists a finitely presented module $M = R/rR$ of type 2 such that either a) $\operatorname{mdim}_M W > 0$ and $\operatorname{edim}_M W = 0$ or b) $\operatorname{mdim}_M W = 0$ and $\operatorname{edim}_M W > 0$.

First we eliminate a).

Lemma 9.3. *Suppose that there exists a countably generated pure projective module W such that $\operatorname{mdim}_M W > 0$ and $\operatorname{edim}_M W = 0$. Then there is $0 \neq x \in J$ such that the ideal RxR is idempotent.*

Proof. By item 3) of Proposition 7.3 there is a module $N = R/sR$ not isomorphic to M, monomorphisms $f_1, f_2 : M \to M$ such that f_1 is not epi, and morphisms $g : M \to N$ and $h : N \to M$ such that $f_2 f_1^2 - f_1 = \beta g$.

As above let Q be the largest ideal of R which is contained in rR and let $P = RrR$, hence $Q \subset rR \subset P$. Furthermore we may assume that $P^2 \subseteq Q$. If not, then $Q \subset P^2 = (RrR)^2$ implies $r \in (RrR)^2$ by the definition of Q, hence $(RrR)^2 = RrR$.

By Lemma 8.1 we may assume that $N = R/Q$ or $N = R/P$ (note that in both cases N is still not isomorphic to M).

Suppose first that $N = R/Q$. Let f_1, f_2 be given by left multiplication by t_1, t_2 and $g = a \times -$, $\beta = b \times -$. From $f_2 f_1^2 - f_1 = \beta g$ we obtain

$$(t_2 t_1 - 1)t_1 - ba = rs \quad \text{for some} \quad s \in R\,.$$

Because f_1 is not epi it follows that $t_1 \in J$ and therefore $u = t_2 t_1 - 1$ is a unit. Since f_1 is mono, there is a unit u_1 such that $t_1 r = r u_1$. Since g sends R/rR to R/Q we get $ar \in Q$, hence $bar = q \in Q$.

Multiplying the above equality by r on the right we obtain

$$uru_1 - q = rsr\,.$$

Then $(RrR)^2 \subseteq Q$ yields that $r \in Q$, a contradiction.

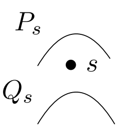
Now consider the case $N = R/P$ and let f_1, f_2, g and β be as above. Since βg factors through R/P we conclude that $M_e = P/rR \subseteq \ker \beta g$. Let f_1', f_2' be restrictions of f_1 and f_2 on M_e (since P is a 2-sided ideal these maps are well defined). It follows that $(f_2' f_1' - 1) f_1' = 0$, i.e. $(t_2 t_1 - 1)t_1 RrR \subseteq rR$. Furthermore because f_1 is mono, it follows that f_1' is mono.

If f_1' is epi then $t_1 RrR = RrR$. Because $t_2 t_1 - 1 = u$ is a unit we obtain $(t_2 t_1 - 1)t_1 RrR = uRrR = RrR$, hence $RrR \subseteq rR$.

Thus we may assume that f_1' is not epi (and mono), therefore $g' = f_2' f_1' - 1$ is epi. Since $g' f_1' = 0$, by [10, Lemma 2.9(ii)] there exists a uniserial R-module which is not quasi-small – see the definition and discussion of this notion in [10]. Then by Proposition 4.6 of this paper R contains an idempotent ideal of the form RxR, $0 \neq x \in J$. $\square$

Before considering b), we need a preliminary lemma. Recall (see Fact 2.7) that for every ideal I of a chain domain, the ideal $I(1) = \bigcap_{n=1}^{\infty} I^n$ is completely prime (though it could be zero).

Because we need to vary r in the proof below we will introduce the following notation. For each $0 \neq s \in J$ we set $P_s = RsR$, and let Q_s denote the largest ideal contained in sR. If sR is not a left ideal then clearly $s \in P_s \setminus Q_s$, hence $Q_s \subset sR \subset P_s$.



Lemma 9.4. *Suppose that $0 \neq s \in J$ is such that sR is not a left ideal. Then $ws \in Q_s$ implies $wP_s \subseteq sR$.*

Proof. Let X_s consist of those $j \in J$ such that $tsj = s$ for some $t \in R$. Since sR is not a left ideal it follows that X_s is nonempty and clearly that $P_s = \sum_{j \in X_s} sj^{-1}R$, where the inverse of j is taken in the quotient skew field of R.

For each $j \in X_s$ we set $Q(j) = (RjR)(1)$. Note that $tsj = s$ implies $t^n s j^n = s$ for each n, and therefore $s \in Q(j)$.

Let Q denote the intersection of all such ideals: $Q = \bigcap_{j \in X_s} Q(j)$, in particular $s \in Q$ and Q is a nonzero completely prime ideal (by Fact 2.7). Furthermore for each $j \in X_s$

we have $j \notin (RjR)^2$, therefore $j \notin Q(j)$ yields $j \notin Q$. Because Q is a completely prime ideal we derive $jQ = Q$ for each $j \in X_s$.

We will show that $Q_s = sQ$.

First let us check that sQ is a left (hence 2-sided) ideal. If $t \in R$ and $ts \in sR$ then $tsQ \subseteq sQ$, as desired. If $ts \notin sR$ then there exists $j \in X_s$ such that $tsj = s$ and therefore $tsQ = tsjQ = sQ$.

Now, since sQ is a 2-sided ideal and $s \notin sQ$ we conclude that $sQ \subseteq Q_s$. It remains to check the reverse inclusion $Q_s \subseteq sQ$.

Note that left multiplication by s provides an isomorphism of the chain of right ideals of R and the chain of submodules of sR, hence $Q_s = sI$ for a right ideal I. From $sQ \subseteq Q_s$ it follows that $Q \subseteq I$. Suppose that $Q \subset I$ hence there is a $j \in X_s$ such that $Q(j) \subset I$. For this j there exists n such that $(RjR)^n \subset I$, in particular $j^n R \subset I$ and therefore also $sj^n I \subset sI$. But then $s = t^n sj^n$ implies $sI = t^n sj^n I \subset t^n sI$, hence sI is not a left ideal, a contradiction.

Now we complete the proof of the lemma. Take $w \in R$ such that $ws \in Q_s = sQ$, say $ws = sq$ for some $q \in Q$. Further let $p \in P_s$ hence $p = sj^{-1}v$ for some $j \in X_s$ and $v \in R$. Since $j \notin Q(j)$ and $q \in Q(j)$ we conclude that $q \in Rj \cap jR$. Write $q = uj$ for some $u \in R$. Then

$$wp = wsj^{-1}v = sq \cdot j^{-1}v = suv \in sR\,,$$

as we wanted to show. $\square$

Now we are in a position to consider the remaining case b) and complete the proof of the Theorem 9.1. Recall that $M = R/rR$ has type 2 and b) claims the existence of $W \in \mathrm{Add}(M)$ such that $\mathrm{mdim}_M W = 0$ and $\mathrm{edim}_M W > 0$.

By Proposition 8.2 we obtain $LRrR = RrR$. By the definition of L we have $tr \in Q_r$ for each $t \in L$. But then Lemma 9.4 implies that $tRrR \subseteq rR$. It follows that $RrR = LRrR \subseteq rR$ and therefore rR is a left ideal, a contradiction.

References

[1] K. Bessenrodt, H.H. Brungs, G. Törner, Right Chain Rings. Part 1, Schriftenreihe des Fachbereich Mathematik, Universität Duisburg, 1990.

[2] H.H. Brungs, N.I. Dubrovin, A classification and examples of rank one chain domains, Trans. Amer. Math. Soc. 355 (2003) 2733–2753.

[3] N. Dubrovin, The rational closure of group rings of left ordered groups, Mat. Sb. 184 (1993) 3–48.

[4] N. Dubrovin, The Rational Closure of Group Rings in Left Ordered Groups, Schriftenreihe des Fachbereich Mathematik, Universität Duisburg, 1994.

[5] A. Facchini, Module Theory: Endomorphism Rings and Direct Sum Decompositions in Some Classes of Modules, Progress in Mathematics, vol. 167, Birkhäuser, 1998.

[6] A. Facchini, P. Příhoda, Functor categories and infinite direct sums, Int. Electron. J. Algebra 5 (2009) 135–168.

[7] T.Y. Lam, Lectures on Rings and Modules, Graduate Texts in Mathematics, vol. 189, Springer, 1998.

[8] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, revised edition, AMS Graduate Studies in Math., vol. 30, 2000.

[9] P. Příhoda, Add(U) of a uniserial module, Comment. Math. Univ. Carolin. 47 (2006) 391–398.
[10] P. Příhoda, On uniserial modules that are not quasi-small, J. Algebra 299 (2006) 329–343.
[11] P. Příhoda, Projective modules are determined by their radical factors, J. Pure Appl. Algebra 210 (2007) 827–835.
[12] G. Puninski, Serial Rings, Kluwer, 2001.
[13] G. Puninski, Some model theory over a nearly simple uniserial domain and decomposition of serial modules, J. Pure Appl. Algebra 163 (2001) 319–337.
[14] G. Puninski, Some model theory over an exceptional uniserial ring and decompositions of serial modules, J. Lond. Math. Soc. 64 (2001) 311–326.
[15] G. Puninski, Pure projective modules over exceptional chain noncoherent rings, J. Math. Sci. 187 (2012) 157–168.
[16] G. Puninski, P. Rothmaler, Pure projective modules, J. Lond. Math. Soc. 71 (2005) 304–320.
[17] J.M. Whitehead, Projective modules and their trace ideals, Comm. Algebra 8 (19) (1980) 1873–1901.
[18] R. Wisbauer, Foundations of Module and Ring Theory, Algebra, Logic and Application Series, vol. 3, Gordon and Breach, 1991.