

Quantum-Mechanical Description of Spin-1/2 Particles and Nuclei Channeled in Bent Crystals

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General quantum-mechanical description of relativistic particles and nuclei with spin $1/2$ channeled in bent crystals is performed with the use of the cylindrical coordinate system. The previously derived Dirac equation in this system is added by terms characterizing anomalous magnetic and electric dipole moments. A transformation to the Foldy-Wouthuysen representation, a derivation of the quantum-mechanical equations of motion for particles and their spins, and a determination of classical limit of these equations are fulfilled in the general case. A physical nature of main peculiarities of description of particles and nuclei in the cylindrical coordinate system is ascertained.

1 Introduction

The strict quantum-mechanical description of relativistic spin- $1/2$ particles and nuclei channeled in bent crystals is an important problem [1]. Taking spin effects into account is very important in such a description. It is known that, during planar channeling in bent crystals, the particle and nucleus spins are rotated by a rather large angle. This effect was first found in the papers of Baryshevsky [2, 3], where he also proposed its use to determine the magnetic moments of shortlived particles. The simple dependence between the rotation angles of particles and their spins in bent crystals was found by Lyuboshits [4]. In such crystals, the centrifugal force acting on particles or nuclei moving along bent trajectories is compensated by the Coulomb force, which leads to the appearance of a rather strong electric field rotating the spin. The effect of spin rotation was observed experimentally in [5, 6].

Although particle and nucleus channeling in many cases can be adequately described by methods of classical theory, a thorough quantum-mechanical analysis of the problem is also necessary. So, the discreteness of the energy spectrum is very often important for

relativistic positrons (electrons). To determine a given spectrum correctly, it is necessary to adequately take into account spin effects for relativistic particles. As is well known, in the Dirac equation, as in the Dirac-Pauli equation, which takes the anomalous magnetic moment (AMM) into account, Dirac matrices determine the spin projections on the axes of the Cartesian system of coordinates. It is convenient to use such a system of coordinates only for channeling in unbent crystals, and the choice of the cylindrical system of coordinates is natural for bent crystals (if the radius of curvature is approximately constant).

The author of [7] presented the quantum-mechanical description of spin $1/2$ particles (nuclei) for planar channeling in unbent and bent crystals. Special attention was focused on determination of the spin dynamics. In this paper, the author solved the Dirac equation (supplemented with terms describing the AMM) in the Foldy-Wouthhuysen (FW) representation, constructed the operator equation of spin motion, and calculated the average value of its precession frequency. The results obtained using the quantum-mechanical description agreed entirely with the corresponding classical results. The author of [7] used the Cartesian system of coordinates rather than the cylindrical one and took the presence of crystal bending into account formally by including an additional potential energy determining the correction for the centrifugal force into the Hamilton operator in the FW representation.

Naturally, such an approach is not rigorous, although it leads to reasonable results. In the present paper, the Dirac equation in the cylindrical system of coordinates derived in [8] is used as the initial one. We supplement it with terms describing the AMM and the electric dipole moment (EDM) and perform transformation into the FW representation by a method developed in [9]. We use the obtained Hamilton operator in this representation to derive general equations describing the motion of particles and nuclei and the spin evolution.

We let the respective Greek and Roman letters $\alpha, \mu, \nu, \dots = 0, 1, 2, 3$ and $i, j, k, \dots = 1, 2, 3$ denote the world and space indices in four-dimensional spacetime. Using the apparatus of the theory of general relativity, tetrad indices are denoted by the initial letters of the Roman alphabet $a, b, c, \dots = 0, 1, 2, 3$. The time and space tetrad indices are singled out by hats. The signature has the form $(+ - - -)$. We here use the system of units $\hbar = 1$, $c = 1$. In some cases, to make the presentation clearer, we include Planck's constant in the corresponding formulas. The notations $[\dots, \dots]$ $\{\dots, \dots\}$ determine the commutators and anticommutators, respectively.

2 Dirac-Pauli equations in the cylindrical system of coordinates

The standard DiracPauli equation (in the Cartesian system of coordinates) has the form

$$\left[\gamma^\mu \pi_\mu - m + \frac{\mu'}{2} \sigma^{\mu\nu} F_{\mu\nu} \right] \Psi = 0, \quad (1)$$

where γ^μ and $\sigma^{\mu\nu} = i(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)/2$ are the Dirac matrices, $F_{\mu\nu} = (\mathbf{E}, \mathbf{B})$ is the electromagnetic field tensor, μ' is the AMM, and $\pi_\mu = iD_\mu = i\hbar(\partial/\partial x^\mu) - eA_\mu$. Here \mathbf{E}, \mathbf{B} and A^μ are the electric strength, the magnetic induction, and the four-potential of the electromagnetic field. In [10], this equation was supplemented with a term describing the EDM d :

$$\left[\gamma^\mu \pi_\mu - m + \frac{\mu'}{2} \sigma^{\mu\nu} F_{\mu\nu} + \frac{d}{2} \sigma^{\mu\nu} G_{\mu\nu} \right] \Psi = 0, \quad (2)$$

where $G_{\mu\nu} = (-\mathbf{B}, \mathbf{E})$ is the tensor dual to $F_{\mu\nu}$.

In principle, it is possible to pass to cylindrical coordinates, not changing the definition of the Dirac matrices and using the relations

$$\gamma^\rho = \boldsymbol{\gamma} \cdot \mathbf{e}_\rho = \gamma^x \cos \phi + \gamma^y \sin \phi, \quad \gamma^\phi = \boldsymbol{\gamma} \cdot \mathbf{e}_\phi = -\gamma^x \sin \phi + \gamma^y \cos \phi.$$

Naturally, such a way is not convenient, and the subsequent transformations are accompanied by very cumbersome calculations. Of course, all required calculations can be carried out in the Cartesian system of coordinates. However, to specify an external field, if the symmetry of the problem is taken into account, it is more convenient to use the cylindrical coordinates specifically. A convenient form of the Dirac equation in the cylindrical system of coordinates was found in [8], the authors of which used the fundamental Pauli theorem [11] determining the relationship between different sets of Dirac matrices satisfying the required commutation and anticommutation relations. The Dirac equation derived in [8] (for $\mu' = 0$) has a very simple form and formally coincides with the initial equation:

$$(\gamma^\mu \pi_\mu - m) \Psi = 0. \quad (3)$$

The matrices γ^μ are ordinary Dirac matrices. However, here, the indices 1, 2, and 3 correspond to the cylindrical coordinates ρ, ϕ , and z , and

$$(\pi_1, \pi_2, \pi_3) = \left(i\hbar \frac{\partial}{\partial \rho} - eA_\rho, i\hbar \frac{1}{\rho} \frac{\partial}{\partial \phi} - eA_\phi, i\hbar \frac{\partial}{\partial z} - eA_z \right). \quad (4)$$

The contravariant vector potential \mathbf{A} , whose components are $-A_i$ is usually used.

It follows from Eq. (3) that the operator π_i contains the nabla operator in the cylindrical system of coordinates. The Dirac equation in the spherical system of coordinates has similar properties [8]. The result obtained in [8] is completely natural and shows that the transformation into cylindrical and spherical coordinates which retains the form of the Dirac matrices γ^μ does not violate the covariance of the Dirac equation.

We use this fact to include terms proportional to the AMM and the EDM into the equation. Such inclusion will be substantiated additionally in the next section by comparison with the results obtained within the framework of the theory of general relativity.

The covariance of Eq. (3) is not violated if its generalization is written in a form (2), where the matrices $\sigma^{\mu\nu}$ have the usual form and the indices μ and ν correspond to the cylindrical system of coordinates. The components of the tensors $F_{\mu\nu}$ and $G_{\mu\nu}$ are also determined in this system of coordinates.

It is convenient to multiply the obtained equation by the matrix γ^0 and represent it into the Hamiltonian form

$$i\hbar \frac{\partial \Psi}{\partial t} = \mathcal{H} \Psi, \quad \mathcal{H} = \boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \beta m + e\Phi + \mu'(-\boldsymbol{\Pi} \cdot \mathbf{B} + i\boldsymbol{\gamma} \cdot \mathbf{E}) - d(\boldsymbol{\Pi} \cdot \mathbf{E} + i\boldsymbol{\gamma} \cdot \mathbf{B}), \quad (5)$$

$$\boldsymbol{\pi} = -i\hbar \nabla - e\mathbf{A},$$

where $\Phi \equiv A_0$ is the scalar potential and $\boldsymbol{\pi} = -(\pi_\rho, \pi_\phi, \pi_z)$ is the kinetic momentum operator in the cylindrical system of coordinates. Unlike the Cartesian system of coordinates, the different components of the operators $\boldsymbol{\pi}$ and ∇ do not commute with each other in the general case.

Equation (5) is initial for the next transition to the FW representation. Nevertheless, the obtained result requires additional substantiation because the terms describing the AMM and EDM were included in (3) obtained as a result of the transformation, rather than in the standard Dirac equation. Such substantiation can be obtained within the framework of gravitation theory. Using gravitation theory makes it possible to relatively simply understand the physical meaning of the difference between the dynamics of particles in the Cartesian and cylindrical systems of coordinates.

3 Deriving the Dirac-Pauli Hamiltonian for particles in the cylindrical system of coordinates using the methods of gravity

The correctness of the above generalization of Eq. (3), which makes it possible to take into account the possible presence of the AMM and the EDM of the particle, can be confirmed by analyzing the covariant Dirac equation in gravitation theory. This equation describes the electromagnetic interaction of the Dirac particle in Riemannian spacetime and has the form [12, 13]

$$(i\hbar\gamma^a D_a - m)\psi = 0, \quad a = 0, 1, 2, 3. \quad (6)$$

In Eq. (6) we have

$$D_a = e_a^\mu \left(\partial_\mu + \frac{ie}{\hbar} A_\mu \right) + \frac{i}{4} \sigma^{bc} \Gamma_{bca}, \quad (7)$$

where $\Gamma_{abc} = -\Gamma_{bac}$ are the Ricci rotation coefficients [14]) having the following form:

$$\Gamma_{abc} = \frac{1}{2} (C_{abc} - C_{bca} - C_{cab}), \quad C_{abc} = e_a^\mu e_b^\nu (\partial_\mu e_{c\nu} - \partial_\nu e_{c\mu}). \quad (8)$$

Here e_a^μ are tetrad coefficients determining the tetrad components of the covariant derivative ($D_a = e_a^\mu D_\mu$) and other four-vectors.

General equation (6) can be used for cylindrical and other curvilinear coordinates. The transition from x, y, z to ρ, ϕ, z in the expression for the squared infinitesimal interval ds^2 gives the following form for the metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (9)$$

Metric tensor (9) describes a flat spacetime.

In this case, the nonzero Ricci rotation coefficients are

$$\Gamma_{212} = -\Gamma_{122} = \frac{1}{\rho}. \quad (10)$$

The covariant Dirac equation obtained by substituting (10) into (6) and (7) does not coincide with Eq. (3). However, we note that to find the Hermitian Hamiltonian, it is necessary to

perform the nonunitary transformation of the wave function of the initial covariant Dirac equation (6) which has the form [15]

$$\Psi = \left(\sqrt{-g}e_0^0\right)^{1/2} \psi = \sqrt{\rho}\psi, \quad (11)$$

where g is the determinant of the metric tensor. To obtain the Hermitian Hamiltonian from Eq. (3), such a transformation is not required.

The natural generalization of Eq. (6) making it possible to describe the AMM and the EDM and retaining the equation covariance has the form

$$\left(i\hbar\gamma^a D_a - m + \frac{\mu'}{2}\sigma^{ab}F_{ab} + \frac{d}{2}\sigma^{ab}G_{ab}\right)\Psi = 0. \quad (12)$$

Here, $F_{ab} = e_a^\mu e_b^\nu F_{\mu\nu}$, $G_{ab} = e_a^\mu e_b^\nu G_{\mu\nu}$, and $\sigma^{ab} = i(\gamma^a\gamma^b - \gamma^b\gamma^a)/2$.

In the general case, Eq. (6) was transformed into the Hamilton form in [15], and the authors of [16] transformed the obtained Hamiltonian into the FW representation also in the general case. For metric (9), using the method for finding the Hamilton form of Eq. (12) proposed in [15] leads to an expression for the Hermitian Hamiltonian which coincides with (5). Thus, the correctness of Eq. (5) is completely confirmed strictly by the methods of quantum mechanics of Dirac particles in gravitational fields.

Gravitation theory also enables us to determine the distinguishing features of particle dynamics in the cylindrical system of coordinates. The particle velocity in this system has the form $\mathbf{v} = v_\rho \mathbf{e}_\rho + v_\phi \mathbf{e}_\phi + v_z \mathbf{e}_z$. Naturally, its constancy ($\mathbf{v} = \text{const}$) means that a force, keeping the particle in a circular orbit, acts on it. This force can be determined using the formalism of gravitoelectromagnetic fields which makes it possible to describe relativistic particles in arbitrarily strong gravitational fields. This formalism was proposed by Pomeranskii and Khriplovich [17] and underwent further development in [15, 16, 18, 19]. It is convenient to introduce gravitoelectromagnetic fields to determine the equations of motion described in local Lorentz (tetrad) reference systems:

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\Omega} \times \mathbf{s}, \quad \boldsymbol{\Omega} = \frac{1}{u^0} \left(-\mathbf{B} + \frac{\hat{\mathbf{u}} \times \boldsymbol{\mathcal{E}}}{u^{\hat{0}} + 1} \right), \quad (13)$$

$$\frac{d\hat{\mathbf{u}}}{dt} = \frac{u^{\hat{0}}}{u^0} \left(\boldsymbol{\mathcal{E}} + \frac{\hat{\mathbf{u}} \times \mathbf{B}}{u^{\hat{0}}} \right), \quad \frac{du^{\hat{0}}}{dt} = \frac{\boldsymbol{\mathcal{E}} \cdot \hat{\mathbf{u}}}{u^0}. \quad (14)$$

These equations are analogous to equations for a particle with a Dirac magnetic moment ($g = 2$) in an electromagnetic field. The spin \mathbf{s} , the gravitoelectric field $\boldsymbol{\mathcal{E}}$, and the gravitomagnetic field \mathbf{B} are determined precisely in these local Lorentz reference systems; i.e., their

components are of the tetrad type. The general equation for the fields has the form

$$\mathcal{E}_{\hat{i}} = \Gamma_{0ic} u^c, \quad \mathcal{B}_{\hat{i}} = -\frac{1}{2} e_{ikl} \Gamma_{klc} u^c, \quad (15)$$

where e_{ikl} is the antisymmetric tensor with spatial components.

The tetrad and world directions coincide in terms of the problem under consideration, and the fields are determined by the expressions

$$\mathcal{E} = 0, \quad \mathcal{B}_{\hat{\rho}} = \mathcal{B}_{\hat{\phi}} = 0, \quad \mathcal{B}_{\hat{z}} = \frac{u^{\hat{\phi}}}{\rho}. \quad (16)$$

Because $e_{\hat{\phi}}^{\hat{\phi}} = 1/\rho$, we have $u^{\hat{\phi}} = u^{\hat{\phi}}/\rho$.

Equations (14) and (16) show that the force determined by the gravitomagnetic field acting in the cylindrical system of coordinates is an analogue of the Lorentz force. Its appearance is a consequence of the fact that, if the azimuthal angle of the particle changes by $d\phi$, the horizontal axes of the cylindrical and Cartesian systems of coordinates rotate by the same angle with respect to each other; i.e., the cylindrical system of coordinates rotates with an instantaneous angular velocity $-d\phi/dt = -v_{\phi}/\rho$ with respect to the Cartesian one [20].

For the detailed quantum-mechanical description of spin 1/2 particles and nuclei, it is convenient to use the FW representation, the transition to which is performed in the following section.

4 Dirac-Pauli Hamiltonian in the Foldy-Wouthuysen representation for particles in the cylindrical system of coordinates

The method of FW transformation for relativistic particles in arbitrary external fields developed in [9, 21, 22, 23], is based on the following representation of the initial Hamiltonian:

$$\mathcal{H}_D = \beta m + \mathcal{E} + \mathcal{O}, \quad \beta \mathcal{E} = \mathcal{E} \beta, \quad \beta \mathcal{O} = -\mathcal{O} \beta, \quad (17)$$

where $\beta \equiv \gamma^0$. In this equation, the initial Hamiltonian is divided into even (\mathcal{E}) and odd (\mathcal{O}) terms, which are diagonal and off-diagonal in two spinors, respectively. In Eq. (17), we have

$$\mathcal{E} = e\Phi - \mu' \mathbf{\Pi} \cdot \mathbf{B} - d\mathbf{\Pi} \cdot \mathbf{E}, \quad \mathcal{O} = \boldsymbol{\alpha} \cdot \boldsymbol{\pi} + i\mu' \boldsymbol{\gamma} \cdot \mathbf{E} - id\boldsymbol{\gamma} \cdot \mathbf{B}. \quad (18)$$

The FW transformation proved itself to be the best method for finding the quasiclassical approximation and the classical limit of relativistic quantum mechanics in the case of the single-particle approach. In this representation, the Hamiltonian is diagonal in two spinors (block-diagonal), the probability interpretation of the wave function is restored, and the operators have the same form as in nonrelativistic quantum mechanics. The transition to the FW representation is widely used for all fundamental interactions.

If the terms that are bilinear in external fields \mathbf{E} and \mathbf{B} are disregarded, the result of the transformation can be written in the form

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E} - \frac{1}{8} \left\{ \frac{1}{\epsilon(\epsilon + m)}, [\mathcal{O}, [\mathcal{O}, \mathcal{F}]] \right\}, \quad \epsilon = \sqrt{m^2 + \mathcal{O}^2}. \quad (19)$$

This Hamiltonian in the FW representation contains exactly determined terms of the first and second orders with respect to \hbar/S_0 , where S_0 is the value of the action dimensionality [23]. Terms of second and higher order with respect to \hbar/S_0 are also determined exactly in the case where they appear as a result of calculating the given Hamiltonian. In particular, this is related to the Darwin (contact) interaction.

For the problem under consideration, the commutator of the operators π_i and π_j is equal to the sum of the two terms:

$$[\pi_i, \pi_j] = -\hbar^2 [\nabla_i, \nabla_j] - ie\hbar (\nabla_i A_j - \nabla_j A_i) = -\hbar^2 [\nabla_i, \nabla_j] + ie\hbar e_{ijk} B^k.$$

The first term is equal to zero if Cartesian coordinates are used. For cylindrical coordinates, the commutator

$$[\nabla_\rho, \nabla_\phi] = -\frac{1}{\rho^2} \frac{\partial}{\partial \phi} = \frac{p_\phi}{i\hbar\rho}$$

has a nonzero value.

As a result, the operator \mathcal{O}^2 is defined by the following exact expression:

$$\mathcal{O}^2 = \boldsymbol{\pi}^2 - e\hbar \boldsymbol{\Sigma} \cdot \mathbf{B} - \hbar \Sigma_z \frac{p_\phi}{\rho} + \beta (\boldsymbol{\Sigma} \cdot [\boldsymbol{\pi} \times \mathbf{G}] - \boldsymbol{\Sigma} \cdot [\mathbf{G} \times \boldsymbol{\pi}] - \hbar \nabla \cdot \mathbf{G}) + \mathbf{G}^2, \quad (20)$$

where $\mathbf{G} = \mu' \mathbf{E} - d\mathbf{B}$.

When calculating the general expression for the Hamiltonian in the FW representation, it is possible to disregard terms of second and higher degrees with respect to external fields and terms of third and higher degrees with respect to the Planck constant. With this accuracy, the Hamiltonian is determined by the equations

$$\mathcal{H}_{FW} = \mathcal{H}_{FW}^{(0)} + \mathcal{H}_{FW}^{(MDM)} + \mathcal{H}_{FW}^{(EDM)}, \quad (21)$$

$$\mathcal{H}_{FW}^{(0)} = \beta\epsilon' + e\Phi - \frac{\hbar}{4}\Pi_z \left\{ \frac{1}{\epsilon'}, \frac{p_\phi}{\rho} \right\}, \quad \epsilon' = \sqrt{m^2 + \boldsymbol{\pi}^2}, \quad (22)$$

$$\begin{aligned} \mathcal{H}_{FW}^{(MDM)} = & \frac{1}{4} \left\{ \left(\frac{\mu_0 m}{\epsilon' + m} + \mu' \right) \frac{1}{\epsilon'}, \left(\boldsymbol{\Sigma} \cdot [\boldsymbol{\pi} \times \mathbf{E}] - \boldsymbol{\Sigma} \cdot [\mathbf{E} \times \boldsymbol{\pi}] - \hbar \nabla \cdot \mathbf{E} \right) \right\} \\ & - \frac{1}{2} \left\{ \left(\frac{\mu_0 m}{\epsilon'} + \mu' \right), \boldsymbol{\Pi} \cdot \mathbf{B} \right\} \\ & + \beta \frac{\mu'}{4} \left\{ \frac{1}{\epsilon'(\epsilon' + m)}, \left[(\mathbf{B} \cdot \boldsymbol{\pi})(\boldsymbol{\Sigma} \cdot \boldsymbol{\pi}) + (\boldsymbol{\Sigma} \cdot \boldsymbol{\pi})(\boldsymbol{\pi} \cdot \mathbf{B}) + 2\pi\hbar(\boldsymbol{\pi} \cdot \mathbf{j} + \mathbf{j} \cdot \boldsymbol{\pi}) \right] \right\}, \end{aligned} \quad (23)$$

$$\begin{aligned} \mathcal{H}_{FW}^{(EDM)} = & -d\boldsymbol{\Pi} \cdot \mathbf{E} + \frac{d}{4} \left\{ \frac{1}{\epsilon'(\epsilon' + m)}, \left[(\mathbf{E} \cdot \boldsymbol{\pi})(\boldsymbol{\Pi} \cdot \boldsymbol{\pi}) + (\boldsymbol{\Pi} \cdot \boldsymbol{\pi})(\boldsymbol{\pi} \cdot \mathbf{E}) \right] \right\} \\ & - \frac{d}{4} \left\{ \frac{1}{\epsilon'}, \left(\boldsymbol{\Sigma} \cdot [\boldsymbol{\pi} \times \mathbf{B}] - \boldsymbol{\Sigma} \cdot [\mathbf{B} \times \boldsymbol{\pi}] \right) \right\}, \end{aligned} \quad (24)$$

where $\mu_0 = e\hbar/(2m)$ is the Dirac magnetic moment and \mathbf{j} is the density of the external current satisfying the Maxwell equation

$$\mathbf{j} = \frac{1}{4\pi} \left(\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} \right).$$

The quantities $\mathcal{H}_{FW}^{(MDM)}$ and $\mathcal{H}_{FW}^{(EDM)}$ define the contributions of the magnetic and electric dipole moments (the MDM and the EDM), respectively. In accordance with the Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

the EDM does not contribute to contact interactions with external charges and currents.

Equations (21)–(24) give a general solution of the quantum-mechanical description of the electromagnetic interaction of the Dirac particle in the cylindrical system of coordinates. Comparison with the results in [10] shows that the operators $\mathcal{H}_{FW}^{(MDM)}$ and $\mathcal{H}_{FW}^{(EDM)}$ have the same form as in the Cartesian system of coordinates, and a new spin-dependent term appears in $\mathcal{H}_{FW}^{(0)}$.

5 General equations of particle dynamics

Deriving quantum-mechanical equations of particle and nucleus dynamics in the cylindrical system of coordinates has important specific features related to the noncommutation of the operators ∇_ρ and ∇_ϕ . The form of the dynamic equations is universal:

$$\frac{d\boldsymbol{\pi}}{dt} = \frac{i}{\hbar} [\mathcal{H}_{FW}, \boldsymbol{\pi}] + \frac{\partial \boldsymbol{\pi}}{\partial t} = \frac{i}{\hbar} [\mathcal{H}_{FW}, \boldsymbol{\pi}] - e \frac{\partial \mathbf{A}}{\partial t}, \quad (25)$$

$$\frac{d\boldsymbol{\Pi}}{dt} = \frac{i}{\hbar} [\mathcal{H}_{FW}, \boldsymbol{\Pi}] = \boldsymbol{\Omega} \times \boldsymbol{\Pi}, \quad (26)$$

where $\mathbf{\Omega}$ is the operator of the angular velocity of the spin precession. However, the above noncommutation leads to the appearance of new terms in dynamic equations compared to the corresponding equations in the Cartesian system of coordinates.

As is known, the total force acting on the particle with spin has a spin-dependent component. This component is usually called the Stern-Gerlach force, and, for a moving particle, it depends on the magnetic and electric fields. Although the Stern-Gerlach force leads to the important effect of division of the beam into two beams with different polarizations, as a rule, it is small for charged particles as compared with the Lorentz force $e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. The author of [7] showed that, in the case of planar channeling, the influence of spin-dependent terms in the Hamiltonian in the FW representation on the particle trajectory is small and was not observed experimentally. Therefore, we do not take the spin-dependent force into account when determining the dynamics of the kinetic momentum of a particle.

In the approximation used, the equation of motion for kinetic momentum (25) takes the form

$$\begin{aligned} \frac{d\boldsymbol{\pi}}{dt} &= e\mathbf{E} + \beta \frac{e}{4} \left\{ \frac{1}{\epsilon'}, (\boldsymbol{\pi} \times \mathbf{B} - \mathbf{B} \times \boldsymbol{\pi}) \right\} + \mathcal{F}, \\ \mathcal{F} &= \left(\frac{\beta}{2} \left\{ \frac{1}{\epsilon'}, \frac{\pi_\phi p_\phi}{\rho} \right\}, -\frac{\beta}{4} \left\{ \frac{1}{\epsilon'}, \left\{ \pi_\rho, \frac{p_\phi}{\rho} \right\} \right\}, 0 \right). \end{aligned} \quad (27)$$

The first two terms in the obtained equation give the operator expression for the Lorentz force, and the third term describes the additional force acting in the cylindrical system of coordinates. It can be written in a more compact vector form. We introduce the operator $\mathbf{O} = O\mathbf{e}_z = (p_\phi/\rho)\mathbf{e}_z$, characterizing the orbital motion of the particle about the z axis. In this case,

$$\mathcal{F} = \frac{\beta}{4} \left\{ \frac{1}{\epsilon'}, (\boldsymbol{\pi} \times \mathbf{O} - \mathbf{O} \times \boldsymbol{\pi}) \right\}. \quad (28)$$

The author of [22] showed that, if the conditions of the quasiclassical approximation (the de Broglie wavelength is less than the characteristic size of the region of external field nonuniformity) are satisfied, using the FW representation it becomes possible to reduce determination of the classical limit of equations of relativistic quantum mechanics to replacing operators in the Hamiltonian and the quantum-mechanical equations of motion with the corresponding classical quantities. In this case, it is possible to disregard the noncommutation of the operators in the quantum-mechanical expressions. In the case under consideration, $\mathbf{O} \rightarrow \epsilon'\boldsymbol{\omega} = m\gamma\boldsymbol{\omega}$, where $\boldsymbol{\omega} = \omega\mathbf{e}_z = (v_\phi/\rho)\mathbf{e}_z$ is the instantaneous angular velocity of the orbital particle motion about the z axis and γ is the Lorentz factor. The classical limit of

the expression for the additional force has the form

$$\mathcal{F} \rightarrow -\boldsymbol{\omega} \times \boldsymbol{\pi}. \quad (29)$$

It is easy to see that, for particle motion along a circular arc, this equation reproduces the centrifugal force $\pi_\phi v_\phi / \rho$, and the force \mathcal{F} changes the instantaneous angular velocity of the orbital particle motion by the quantity $-\boldsymbol{\omega}$.

Expression (29) is completely analogous to the corresponding expression for the force acting in a rotating reference system (Eq. (3.37) in Ref. [16]).¹ The analogy between the cylindrical system of coordinates and the rotating reference system was demonstrated previously in [20] within the framework of the classical approach.

However, it is important that the rotating reference system can be used only within the framework of the one-particle description, while the cylindrical system of coordinates can also be used to describe a beam of particles or nuclei with various energies. The presence of such a possibility is an important advantage of the cylindrical system of coordinates.

The force \mathcal{F} is similar to forces appearing in noninertial reference systems. However, the latter are real forces, which, in particular, can be measured by a dynamometer, while the force \mathcal{F} is fictitious. Its presence does not affect the dynamometer readings, although it affects the particle and nucleus motion in the cylindrical system of coordinates. An interesting effect of the mutual influence of particle motion in the directions \mathbf{e}_ρ and \mathbf{e}_ϕ follows from Eqs. (27) and (29). In particular, oscillator motion in the radial direction causes the appearance of an oscillating term in the expression for p_ϕ and vice versa. This effect has kinematic nature. It is due to the rotation of the axes \mathbf{e}_ρ and \mathbf{e}_ϕ (about the Cartesian system of coordinates) as the azimuth ϕ changes, and it disappears at $p_\phi = 0$ as follows from the definition of \mathbf{O} .

Spin motion in the cylindrical system of coordinates has a simpler character than the dynamics of the kinetic momentum. The angular velocity operator of the spin precession, which is easily determined using Eq. (26), has the form

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}^{(0)} + \boldsymbol{\Omega}^{(MDM)} + \boldsymbol{\Omega}^{(EDM)}, \quad (30)$$

where

$$\boldsymbol{\Omega}^{(0)} = -\frac{\beta}{2} \left\{ \frac{1}{\epsilon'}, \mathbf{O} \right\}, \quad (31)$$

¹In Ref. [16], as in Eqs. (27) and (29), the force was defined as the time derivative of the covariant momentum operator. Coriolis factor 2 appears in the corresponding equations for the acceleration and the spatial component of the contravariant four-velocity [15, 24].

$$\begin{aligned} \Omega^{(MDM)} = \frac{1}{2\hbar} \left\{ \left(\frac{\mu_0 m}{\epsilon' + m} + \mu' \right) \frac{1}{\epsilon'}, (\boldsymbol{\pi} \times \mathbf{E} - \mathbf{E} \times \boldsymbol{\pi}) \right\} - \frac{\beta}{\hbar} \left\{ \left(\frac{\mu_0 m}{\epsilon'} + \mu' \right), \mathbf{B} \right\} \\ + \beta \frac{\mu'}{2\hbar} \left\{ \frac{1}{\epsilon'(\epsilon' + m)}, \left((\mathbf{B} \cdot \boldsymbol{\pi})\boldsymbol{\pi} + \boldsymbol{\pi}(\boldsymbol{\pi} \cdot \mathbf{B}) \right) \right\}, \end{aligned} \quad (32)$$

$$\begin{aligned} \Omega^{(EDM)} = -\beta \frac{2d}{\hbar} \mathbf{E} + \frac{d}{2\hbar} \left\{ \frac{1}{\epsilon'(\epsilon' + m)}, \left((\mathbf{E} \cdot \boldsymbol{\pi})\boldsymbol{\pi} + \boldsymbol{\pi}(\boldsymbol{\pi} \cdot \mathbf{E}) \right) \right\} \\ - \frac{d}{2\hbar} \left\{ \frac{1}{\epsilon'}, (\boldsymbol{\pi} \times \mathbf{B} - \mathbf{B} \times \boldsymbol{\pi}) \right\}. \end{aligned} \quad (33)$$

For a particle with a positive total energy in the FW representation, the lower spinor is zero. Therefore, passing to the classical limit eliminates the β matrices (and also the anticommutators).

It is easy to see that $\boldsymbol{\Omega}^{(0)} \rightarrow -\boldsymbol{\omega}$ in the classical limit. Thus, the passage from the Cartesian to the cylindrical coordinates changes the instantaneous angular velocities of orbital particle motion and the precession of its spin by the same quantity, so that the difference between them remains the same. The forms of the quantities $\boldsymbol{\Omega}^{(MDM)}$ and $\boldsymbol{\Omega}^{(EDM)}$ in the Cartesian [10] and cylindrical systems of coordinates coincide.

6 Motion of particle and nucleus spins at planar channeling in bent crystals

As was mentioned above, the quantum-mechanical description of spin 1/2 particles and nuclei during planar channeling in unbent and bent crystals was presented previously in [7], the author of which used the Cartesian system of coordinates, and the presence of crystal bending was taken into account by including an additional potential energy into the Hamilton operator in the FW representation. In our notation, this energy is

$$W = -\frac{p_\phi v_\phi x}{R}, \quad (34)$$

where R is the crystal bending radius and $x = 0$ corresponds to the middle of the distance between the crystal planes. During planar channeling in bent crystals, the interplanar distance d_p is negligibly small as compared with the crystal bending radius. Therefore, the approach used in [7] leads to the obtained correct results. The author of this paper gave detailed quantum-mechanical description of the effects occurring during particle and nucleus channeling in unbent and bent crystals. For this reason, we restrict ourselves to deriving well-known formulas [2, 4], describing the spin motion during planar channeling in bent crystals, starting from the general equations obtained above.

At planar channeling, the field of planes is characterized by the even potential $\Phi(x) = \Phi(-x)$. For nuclei that move in the channeling mode and have positive charges, this field can be approximated by the harmonic potential

$$\Phi(x) = \frac{ax^2}{2}, \quad a = \frac{8U_0}{d_p^2}, \quad (35)$$

where U_0 is the maximum value of the potential and d_p is the distance between the crystal planes. We assume that the crystal is bent so that the bending plane is perpendicular to the crystal planes and the curvature radius is R . In this case, the plane potential (35) becomes

$$\Phi(\rho) = \frac{a(\rho - R)^2}{2}. \quad (36)$$

We do not consider the magnetic field and neglect effects caused by possible electric dipole moments.

At channeling, the particle oscillates with respect to the equilibrium trajectory which is a circular arc. Therefore, the average force acting on the particle in the cylindrical system of coordinates is zero. As follows from Eqs. (27) and (29), in this case, in the classical limit, we have

$$e < \mathbf{E} > = < \boldsymbol{\omega} \times \boldsymbol{\pi} > .$$

Another form of this equation is

$$e < E_\rho > = -m|\boldsymbol{\omega}|\sqrt{\gamma^2 - 1}. \quad (37)$$

The fact that the kinetic energy of transverse motion is significantly smaller than that of longitudinal motion during channeling is taken into account in Eq. (37).

Substitution into Eqs. (30) and (32) gives the angular velocity of the spin precession

$$\boldsymbol{\Omega} = \frac{1}{\gamma} \left[\frac{g-2}{2} (\gamma^2 - 1) - 1 \right] \boldsymbol{\omega}. \quad (38)$$

When passing to the Cartesian system of coordinates, the quantity $-\boldsymbol{\omega}$ is added to the angular precession velocity, and we obtain the Lyuboshits formula [4] determining the relation between the angular velocities of the spin and momentum rotations:

$$\boldsymbol{\Omega}^{(Car)} = \frac{\gamma - 1}{\gamma} \left[\frac{g-2}{2} (\gamma + 1) + 1 \right] \boldsymbol{\omega}. \quad (39)$$

The formula determining the effect of particle and nucleus spin rotation during planar channeling in bent crystals was first derived in a paper by V.G. Baryshevsky [2]. The fast increase in the ratio Ω/ω with increasing Lorentz factor makes it possible to measure the magnetic moments of relativistic particles with short lifetimes [2].

7 Discussion and summary

It is natural to choose the cylindrical system of coordinates in the quantum-mechanical description of relativistic spin 1/2 particles and nuclei channeled in bent crystals. However, the problem of determining projections of the spin operator on the radial and azimuthal directions appears in this case. Using ordinary Dirac matrices for the given projections is its best solution, found in [8]. In this paper, the Dirac equation in cylindrical coordinates obtained in [8] is supplemented with terms describing the AMM and EDM (with strict substantiation of this procedure using methods of quantum mechanics of Dirac particles in gravitational fields). Although the form of the obtained equation does not differ from that of the corresponding equation in Cartesian coordinates, the difference between them is manifested after transformation into the FW representation. In this representation, the Hamiltonian in cylindrical coordinates differs from the corresponding Hamiltonian in Cartesian coordinates by the presence of an additional spin term. Although the spin-independent parts of two Hamiltonians determining the particle and nucleus motion coincide formally, the obtained equations of motion are significantly different. The equation of particle and nucleus motion in the cylindrical system of coordinates includes the centrifugal force and also determines the effect of the mutual influence of particle motion in the directions \mathbf{e}_ρ and \mathbf{e}_ϕ . This effect has a kinematic nature and, in particular, is manifested in the appearance of the oscillating force in the azimuthal direction during oscillatory motion in the radial direction and vice versa. Passing to the classical limit makes it possible to establish that the equations of motion for the kinetic momentum and spin in the two considered systems of coordinates agree completely with each other. As an example demonstrating the correctness of the description of physical phenomena using the equations obtained in this paper, we have derived a well-known relation between the angular velocities of the spin and momentum vector rotations during planar channeling in bent crystals.

Using methods developed in gravitation theory makes it possible not only to provide the correct description of particles and nuclei with the AMM and the EDM in cylindrical coordinates, but also to determine the physical nature of the main specific features of such a description. The analogy between the cylindrical system of coordinates and the rotating reference system demonstrated previously in [20] within the framework of the classical approach appears paradoxical at first glance. Although the cylindrical system of coordinates and the rotating reference system belong to planar spacetime manifolds, the structures of

the metric tensor in these systems are significantly different. The main physical properties of the rotating reference system are determined by off-diagonal components of the metric tensor g_{0i} , and the metric is stationary and nonstatic. As opposed to this, the metric of the cylindrical system of coordinates (9) is static and has a single nontrivial component. However, the calculation of gravitoelectromagnetic fields explains the indicated analogy exhaustively. In both cases, there is no gravitoelectric field, and the gravitomagnetic field \mathbf{B} is $\boldsymbol{\omega}u^{\hat{0}}$ in the rotating reference system and $(u^{\hat{\phi}}/\rho)\mathbf{e}_z$ in the cylindrical system of coordinates. If the condition $\boldsymbol{\omega} = \omega\mathbf{e}_z = [u^{\hat{\phi}}/(\rho u^{\hat{0}})]\mathbf{e}_z$ is satisfied, these quantities are equal to each other, and the dynamics of particles and their spins in the two systems becomes identical. This condition can always be satisfied for an individual particle, but it cannot be satisfied for an ensemble (beam) of particles with different momenta. The possibility of describing an ensemble of particles with an arbitrary distribution over momenta is an important advantage of the cylindrical system of coordinates.

Thus, in this paper, we presented the general quantum-mechanical description of relativistic spin 1/2 particles and nuclei channeled in bent crystals by using the cylindrical system of coordinates. The results of our study can be used to solve concrete problems of channeling theory.

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