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The algebra of canonical affinor structures on homogeneous k-symmetric spaces[1]

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Abstract. The commutative algebra of all canonical affinor structures on homogeneous k-symmetric spaces is completely described. It gives a classification of these spaces with respect to the algebra.

Keywords. Homogeneous Φ-space, canonical affinor structure, k-symmetric space.

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1. Introduction

Invariant structures on homogeneous spaces reveal an important information about the geometry of these spaces. In this sense homogeneous spaces defined by Lie group automorphisms Φ (briefly, homogeneous Φ-spaces) possess invariant structures of special interest. More exactly, any homogeneous regular Φ-space admits in a natural way the commutative algebra $\mathcal{A}(\theta)$ of all canonical affinor structures, see [3]. It is well known ([13]) that the class of regular Φ-spaces includes a widespread class of homogeneous Φ-spaces of any finite order k (homogeneous k-symmetric spaces, [10]). Specifically, for any homogeneous symmetric space (the case $k = 2$) the algebra $\mathcal{A}(\theta)$ is isomorphic to $\mathbb{R}$.

The main goal of the paper is to describe completely the algebra $\mathcal{A}(\theta)$ for arbitrary homogeneous k-symmetric spaces. This description gives the opportunity to classify homogeneous k-symmetric spaces with respect to $\mathcal{A}(\theta)$.

The paper is organized as follows.

In Section 2, we collect some basic notions and results about homogeneous regular Φ-spaces and canonical affinor structures. In particular, the full algebraic description of all canonical structures of classical type (almost complex, almost prod-

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

uct, f-structures, etc.) is formulated as well as a geometric idea of the structures is presented.

In Section 3, the algebraic structure of the algebra $\mathcal{A}(\theta)$ for any homogeneous k-symmetric space is completely characterized. This structure is entirely determined by the spectrum of the operator θ. As an example, for all homogeneous 3-symmetric spaces $\mathcal{A}(\theta)$ is isomorphic to $\mathbb{C}$, where the imaginary unit is just the classical canonical almost complex structure (see [6, 14, 16]).

Finally, in Section 4, we consider several particular examples the algebra $\mathcal{A}(\theta)$ of which is $\mathbb{C}$ or $\mathbb{R} \oplus \mathbb{C}$. They are the spheres S^2, S^5, S^6 in their non-symmetric representations, the 6-dimensional generalized Heisenberg group (two representations), the group of hyperbolic motions of the plane $\mathbb{R}^2$.

2. Canonical structures on regular Φ-spaces

Here we briefly formulate some basic definitions and results related to regular Φ-spaces and canonical affinor structures on them. More detailed information can be found in [3, 4, 10, 13, 14, 16].

Let G be a connected Lie group, Φ its (analytic) automorphism. Denote by G^Φ the subgroup of all fixed points of Φ and G^Φ_o the identity component of G^Φ. Suppose a closed subgroup H of G satisfies the condition

$$G^\Phi_o \subset H \subset G^\Phi.$$

Then G/H is called a *homogeneous Φ-space*.

Homogeneous Φ-spaces include homogeneous symmetric spaces ($\Phi^2 = \mathrm{id}$) and, more general, *homogeneous Φ-spaces of order k* ($\Phi^k = \mathrm{id}$) or, in other terminology, *homogeneous k-symmetric spaces* (see [10]). Note that there exist homogeneous Φ-spaces that are not reductive. That is why so-called regular Φ-spaces first introduced by N.A. Stepanov ([13]) are of fundamental importance.

Let G/H be a homogeneous Φ-space, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - \mathrm{id}$ and the Fitting decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to A, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote 0- and 1-component of the decomposition respectively. It is clear that $\mathfrak{h} = \mathrm{Ker}\, A$ and $\mathfrak{h} \subset \mathfrak{g}_0$.

Definition 1 ([3, 4, 13]). *A homogeneous Φ-space G/H is called a regular Φ-space if one of the following equivalent conditions is satisfied*:

1. $\mathfrak{h} = \mathfrak{g}_0$;
2. $\mathfrak{g} = \mathfrak{h} \oplus A\mathfrak{g}$;
3. *The restriction of the operator A to $A\mathfrak{g}$ is non-singular*;
4. $A^2X = 0 \Longrightarrow AX = 0$ *for all $X \in \mathfrak{g}$*;
5. *The matrix of the automorphism φ can be represented in the form*

$$\begin{pmatrix} E & 0 \\ 0 & B \end{pmatrix},$$

where the matrix B does not admit the eigenvalue 1.

We recall two basic facts:

Theorem 1 ([13]).
– *Any homogeneous* Φ*-space of order* k ($\Phi^k = \mathrm{id}$) *is a regular* Φ*-space.*
– *Any regular* Φ*-space is reductive. More exactly, the Fitting decomposition*

(1) $$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} = A\mathfrak{g}$$

is a reductive one.

Decomposition (1) is the *canonical reductive decomposition* corresponding to a regular Φ-space G/H, and $\mathfrak{m}$ is the *canonical reductive complement.*

Decomposition (1) is obviously φ-invariant. Denote by θ the restriction of φ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$. It is important to note that the operator θ commutes with every element of the linear isotropy group $\mathrm{Ad}(H)$ (see [13]).

An *affinor structure* on a manifold is known to be a tensor field of type $(1{,}1)$. Suppose F is an invariant affinor structure on a homogeneous manifold G/H. Then F is completely determined by its value F_o at the point o, where F_o is invariant with respect to $\mathrm{Ad}(H)$. For simplicity, we will denote by the same manner both any invariant structure on G/H and its value at o throughout the rest of the paper.

Definition 2 ([3]). *An invariant affinor structure* F *on a regular* Φ*-space* G/H *is called* canonical *if its value at the point* $o = H$ *is a polynomial in* θ*.*

Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on a regular Φ-space G/H. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. Moreover,

$$\dim \mathcal{A}(\theta) = \deg \nu \le \dim G/H,$$

where ν is the minimal polynomial of the operator θ. Note that the algebra $\mathcal{A}(\theta)$ for any symmetric Φ-space ($\Phi^2 = \mathrm{id}$) consists of scalar structures only, i.e., it is isomorphic to $\mathbb{R}$.

It should be mentioned that all canonical structures are invariant with respect to the “symmetries” of G/H, which are generated by the automorphism Φ (see [13]).

The most remarkable example of canonical structures is the canonical almost complex structure $J = (1/\sqrt{3})(\theta - \theta^2)$ on a homogeneous 3-symmetric space (see [6, 14, 16]). It turns out that it is not an exception. In other words, the algebra $\mathcal{A}(\theta)$ contains many affinor structures of classical type.

In the sequel we will concentrate on the following affinor structures of classical type: *almost complex structures* J ($J^2 = -1$); *almost product structures* P ($P^2 = 1$); f*-structures* ($f^3 + f = 0$), see [17]; f-structures of hyperbolic type or, briefly, h*-structures* ($h^3 - h = 0$), see [9]. Clearly, f-structures and h-structures are generalizations of structures J and P respectively.

All the canonical structures of classical type on regular Φ-spaces have already been completely described, see [1, 3]. In particular, for homogeneous k-symmetric spaces, precise computational formulae were indicated. For future reference we select here some results.

Denote by $\tilde{s}$ (respectively, s) the number of all irreducible factors (respectively, all irreducible quadratic factors) over $\mathbb{R}$ of a minimal polynomial ν.

Theorem 2 ([1, 3]). *Let G/H be a regular Φ-space.*

1. *The algebra $\mathcal{A}(\theta)$ contains precisely $2^{\tilde{s}}$ structures P.*
2. *G/H admits a canonical structure J if and only if $s = \tilde{s}$. In this case $\mathcal{A}(\theta)$ contains 2^s different structures J.*
3. *G/H admits a canonical f-structure if and only if $s \neq 0$. In this case $\mathcal{A}(\theta)$ contains $3^s - 1$ different f-structures. Suppose $s = \tilde{s}$. Then 2^s f-structures are almost complex and the remaining $3^s - 2^s - 1$ have non-trivial kernels.*
4. *The algebra $\mathcal{A}(\theta)$ contains $3^{\tilde{s}}$ different h-structures. All these structures form a (commutative) semigroup in $\mathcal{A}(\theta)$ and include a subgroup of order $2^{\tilde{s}}$ of canonical structures P.*

Further, let G/H be a homogeneous k-symmetric space. Then $\tilde{s} = s + 1$ if $-1 \in$ spec θ, and $\tilde{s} = s$ in the opposite case. We indicate explicit formulae enabling us to compute all canonical f-structures and h-structures. We shall also use the notation

$$u = \begin{cases} n & \text{if } k = 2n+1, \\ n-1 & \text{if } k = 2n. \end{cases}$$

Theorem 3 ([1, 3]). *Let G/H be a homogeneous Φ-space of order k.*

1. *All non-trivial canonical f-structures on G/H can be given by the operators*

$$f = \frac{2}{k} \sum_{m=1}^{u} \left(\sum_{j=1}^{u} \zeta_j \sin\left(\frac{2\pi mj}{k}\right)\right)(\theta^m - \theta^{k-m}),$$

where $\zeta_j \in \{-1; 0; 1\}$, $j = 1, 2, \ldots, u$, and not all coefficients ζ_j are zero. Moreover, the polynomials f define canonical structures J if and only if all $\zeta_j \in \{-1; 1\}$.

2. *All canonical h-structures on G/H can be given by the polynomials*

$$h = \sum_{m=0}^{k-1} a_m \theta^m,$$

where:

(*a*) *if $k = 2n + 1$, then*

$$a_m = a_{k-m} = \frac{2}{k} \sum_{j=1}^{u} \xi_j \cos\left(\frac{2\pi mj}{k}\right);$$

(*b*) *if $k = 2n$, then*

$$a_m = a_{k-m} = \frac{1}{k}\left(2 \sum_{j=1}^{u} \xi_j \cos\left(\frac{2\pi mj}{k}\right) + (-1)^m \xi_n\right).$$

Here the numbers ξ_j, $j = 1, 2, \ldots, u$, take their values from the set $\{-1; 0; 1\}$ and the polynomials h define canonical structures P if and only if all $\xi_j \in \{-1; 1\}$.

We now particularize the results for homogeneous Φ-spaces of orders 3 and 4 only.

Corollary 1 ([1, 3]). *Let G/H be a homogeneous Φ-space of order* 3*. There are* (*up to the sign*) *only the following canonical structures of classical type on G/H*:

$$J = \frac{1}{\sqrt{3}}(\theta - \theta^2), \qquad P = 1.$$

We note that the existence of the structure J and its properties are well known (see [6, 14, 16]).

Corollary 2 ([1, 3]). *On a homogeneous Φ-space of order* 4 *there are* (*up to the sign*) *the following canonical classical structures*:

$$P = \theta^2, \qquad f = \tfrac{1}{2}(\theta - \theta^3), \qquad h_1 = \tfrac{1}{2}(1 - \theta^2), \qquad h_2 = \tfrac{1}{2}(1 + \theta^2).$$

The operators h_1 and h_2 form a pair of complementary projectors: $h_1 + h_2 = 1$, $h_1^2 = h_1$, $h_2^2 = h_2$*. Moreover, the following conditions are equivalent*:

1. $-1 \notin \operatorname{spec}\theta$;
2. *the structure P is trivial* ($P = -1$);
3. *the f-structure is an almost complex structure*;
4. *the structure h_1 is trivial* ($h_1 = 1$);
5. *the structures h_2 is null.*

It is important to note that the procedure of describing all canonical structures on homogeneous Φ-spaces of finite order is constructive (see [3], § 4 and § 5). We briefly present it for future reference.

Suppose the spectrum $\operatorname{spec}\theta$ contains s pairs of conjugate k-th roots of unity (apart from -1, which can also be an eigenvalue). Consider the corresponding θ-invariant decomposition of the canonical reductive supplement $\mathfrak{m}$ from formula (1):

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s, \tag{2}$$

where $\mathfrak{m}_0$ is the subspace for the eigenvalue -1 (if $-1 \in \operatorname{spec}\theta$) and $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ the subspaces for s pairs of roots. Then any canonical f-structure can be represented in the form

$$f = (0, \zeta_1 J_1, \ldots, \zeta_s J_s), \tag{3}$$

where $J_1, \ldots, J_s$ are the specially defined complex structures on $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ respectively, $\zeta_j \in \{-1; 0; 1\}$, $j = 1, 2, \ldots, s$. As to any canonical h-structure, it can be represented in the form

$$h = (\xi_0 I_0, \xi_1 I_1, \ldots, \xi_s I_s), \tag{4}$$

where $I_0, I_1, \ldots, I_s$ are the identical operators on $\mathfrak{m}_0$, $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ respectively, $\xi_j \in \{-1, 0, 1\}$, $j = 0, 1, 2, \ldots, s$.

It should be noted that in particular case of homogeneous Φ-spaces of any odd order $k = 2n + 1$ the method of constructing invariant almost complex structures was described in [10]. It is easy to see that all these structures are canonical in the above sense.

3. The algebraic structure of $\mathcal{A}(\theta)$ for homogeneous k-symmetric spaces

Here we explicitly characterize the construction of the algebra $\mathcal{A}(\theta)$ for arbitrary homogeneous k-symmetric spaces.

Theorem 4. *Let G/H be a homogeneous Φ-space of order k, s the number of pairs of conjugate k-th roots of unity, which are included into the spectrum* spec θ.

1. *If $-1 \in \operatorname{spec}\theta$, then the algebra $\mathcal{A}(\theta)$ is isomorphic to*

$$\mathbb{R} \oplus \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{s}.$$

2. *If $-1 \notin \operatorname{spec}\theta$, then*

$$\mathcal{A}(\theta) \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{s}.$$

Proof. By the procedure of describing canonical f-structures and h-structures we construct the isomorphism required (see Section 2).

1) Suppose $-1 \in \operatorname{spec}\theta$. Consider the corresponding decomposition of the canonical reductive supplement $\mathfrak{m}$:

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_s.$$

Using (3) and (4), we define the canonical h-structures and f-structures on G/H by setting

$$\begin{aligned} h_0 &= (I_0, 0, \ldots, 0),\\ h_1 &= (0, I_1, \ldots, 0),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ h_s &= (0, 0, \ldots, I_s),\\ f_1 &= (0, J_1, \ldots, 0),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ f_s &= (0, 0, \ldots, J_s), \end{aligned} \tag{5}$$

where $I_0, I_1, \ldots, I_s$ are the identical operators on the subspaces $\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_s$ respectively and $J_1, \ldots, J_s$ are special complex structures on $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ respectively. By Theorem 3, there exist real polynomials $h_0(x), h_1(x), \ldots, h_s(x)$ and $f_1(x), \ldots, f_s(x)$ such that

$$\begin{aligned} h_i(\theta) &= h_i, \quad i = 0, 1, \ldots, s,\\ f_j(\theta) &= f_j, \quad j = 1, \ldots, s. \end{aligned}$$

It is evident that operators $\{h_i, f_j\}$ in (5) are linearly independent. Since in the case $\dim \mathcal{A}(\theta) = \deg \nu = 2s + 1$, we obtain that the collection $h_0, h_1, \ldots, h_s$,

$f_1, \ldots, f_s$ is exactly a basis in $\mathcal{A}(\theta)$. It means that for any $F \in \mathcal{A}(\theta)$ we have

$$F = a_0 h_0 + \sum_{m=1}^{s} (a_m h_m + b_m f_m),$$

where $a_0, a_m, b_m \in \mathbb{R}, \quad m = 1, \ldots, s$.

Now we define the following mapping:

$$\mathcal{A}(\theta) \longrightarrow \mathbb{R} \oplus \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{s},$$

$$\begin{aligned} F &= (a_0 I_0, a_1 I_1 + b_1 J_1, \ldots, a_s I_s + b_s J_s) \\ &\to (a_0, a_1 + b_1 i, \ldots, a_s + b_s i). \end{aligned}$$

It is not difficult to verify that the correspondence is an isomorphism of the commutative algebras under consideration. This proves the required result.

2) Suppose $-1 \notin \operatorname{spec} \theta$. Then we obtain the decomposition

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s,$$

i.e., $\mathfrak{m}_0 = \{0\}$. In the case $\dim \mathcal{A}(\theta) = \deg \nu = 2s$. Here we put

$$\begin{aligned} h_1 &= (I_1, 0, \ldots, 0), \\ &\cdots\cdots\cdots\cdots \\ h_s &= (0, 0, \ldots, I_s), \\ f_1 &= (J_1, 0, \ldots, 0), \\ &\cdots\cdots\cdots\cdots \\ f_s &= (0, 0, \ldots, J_s). \end{aligned} \tag{6}$$

All the other arguments can be realized in the same manner. This completes the proof. □

Definition 3. The canonical structures (5), respectively (6), on a homogeneous k-symmetric space G/H corresponding to the case $-1 \in \operatorname{spec} \theta$, respectively $-1 \notin \operatorname{spec} \theta$, are called *canonical generators* of the algebra $\mathcal{A}(\theta)$.

Now we dwell on particular cases of Theorem 4.

Corollary 3. *For any homogeneous* 3*-symmetric space G/H its algebra $\mathcal{A}(\theta)$ is isomorphic to $\mathbb{C}$. Moreover, the canonical almost complex structure*

$$J = \frac{1}{\sqrt{3}} (\theta - \theta^2)$$

(*see Corollary* 1) *plays a role of the imaginary unit in $\mathcal{A}(\theta)$.*

Proof. Indeed, in this case $\operatorname{spec} \theta = \{\varepsilon, \bar{\varepsilon}\}$, where ε is a primitive third root of unity. It follows from Theorem 4 (2) that $P = 1$ and J are canonical generators in $\mathcal{A}(\theta) \cong \mathbb{C}$. □

We recall that reductive homogeneous space G/H with the corresponding reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is called *locally symmetric* if $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ (see [14]).

Corollary 4. *For any homogeneous* 4*-symmetric space G/H there are the following two possibilities*:

1. *If* spec $\theta = \{i; -1; -i\}$, *then $\mathcal{A} \cong \mathbb{R} \oplus \mathbb{C}$. Besides, the canonical structures h_1, h_2, f (see Corollary* 2) *are canonical generators in $\mathcal{A}(\theta)$.*

2. *If* spec $\theta = \{i, -i\}$, *then $\mathcal{A}(\theta) \cong \mathbb{C}$. Moreover, G/H is a locally symmetric homogeneous space. In particular, $f = J = \theta$ is the integrable canonical almost complex structure on G/H.*

Proof. (1) This statement directly follows from Theorem 4 (1).

(2) The subspace $\mathfrak{m}_0 = \operatorname{Ker} f$ is trivial in the case. Then, using [2], we obtain: $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. It immediately implies that the canonical structure $f = J = \theta$ is integrable. □

Remark 1. It is clear that there are no fundamental obstructions to considering homogeneous Φ-spaces of higher orders k. We also emphasize that the key point for the algebraic structure of $\mathcal{A}(\theta)$ is the spectrum spec θ but not an order k. As a result, Theorem 4 gives a possibility to classify all homogeneous k-symmetric spaces with respect to the algebra $\mathcal{A}(\theta)$. Obviously, this classification contains a denumerable set of equivalent classes.

4. Examples

4.1. The spheres S^2, S^5, S^6

It is well known that any standard sphere S^n is a Riemannian globally symmetric space. That is why its algebra of canonical affinor structures is isomorphic to $\mathbb{R}$. However among all the spheres only S^2, S^5, and S^6 can be realized as Riemannian homogeneous Φ-spaces with automorphisms Φ that are not involutions (see [10, 12]).

The sphere $S^2 \cong SO(3)/SO(2)$ can be represented as a homogeneous Φ-space of any order k ($k \geq 2$). If $k > 2$, then any invariant affinor structure on S^2 is canonical, i.e., $\mathcal{A}(\theta) = \mathcal{A}$ (see [3]). It is evident now that $\mathcal{A} = \mathcal{A}(\theta) \cong \mathbb{C}$. In particular, the standard complex structure on S^2 is determined by the canonical almost complex structure.

The sphere S^5 represented as a homogeneous space $SU(3)/SU(2)$ admits the structure of a homogeneous Φ-space of order 4 (see [10, 12]). The canonical f-structure for this representation was calculated in ([3]). It has deficiency 1 and defines an invariant almost contact structure on S^5. From Corollary 4 (1) it evidently follows that the algebra $\mathcal{A}(\theta)$ is isomorphic to $\mathbb{R} \oplus \mathbb{C}$. We note that the fact was obtained (in [11]) by the straightforward computation.

The sphere $S^6 \cong G_2/SU(3)$ is the most significant example in the theory of nearly Kähler manifolds (see, for instance, [5]). Its nearly Kähler structure is defined by the canonical almost complex structure of the corresponding homogeneous 3-symmetric space. It follows that $\mathcal{A}(\theta) \cong \mathbb{C}$ for the sphere S^6.

4.2. The 6-dimensional generalized Heisenberg group

We briefly formulate some notions and results related to the 6-dimensional generalized Heisenberg group (N, g). As to details, we refer to ([7, 8, 15]).

Let V and Z be two real vector spaces of dimension n and m ($m \geq 1$) both equipped with an inner product which we shall denote for both spaces by the same symbol $\langle\,,\rangle$. Further, let $j : Z \to \mathrm{End}(V)$ be a linear map such that

$$|j(a)x| = |x||a|, \quad j(a)^2 = -|a|^2 I, \quad x \in V,\ a \in Z.$$

Next we put $\mathfrak{n} := V \oplus Z$ together with the bracket defined by

$$[a + x, b + y] = [x, y] \in Z, \quad \big\langle [x, y], a\big\rangle = \big\langle j(a)x, y\big\rangle,$$

where $a, b \in Z$ and $x, y \in V$. It is a 2-step nilpotent Lie algebra with center Z.

The simply connected, connected Lie group N whose Lie algebra is $\mathfrak{n}$ is called a *generalized Heisenberg group*. Note that N has a left invariant metric g induced by the following inner product on $\mathfrak{n}$:

$$\langle a + x, b + y\rangle = \langle a, b\rangle + \langle x, y\rangle, \quad a, b \in Z,\ x, y \in V.$$

The 6-dimensional generalized Heisenberg group (N, g) is of especial interest (see [8, 15]). The brackets for the Lie algebra $\mathfrak{n} = L(x_1, x_2, x_3, x_4) \oplus L(a_1, a_2) = V \oplus Z$ were explicitly indicated, see [15], p. 111:

$$\begin{cases} [x_1, x_2] = a_1, \quad [x_1, x_3] = a_2, \\ [x_2, x_4] = -a_2, \quad [x_3, x_4] = a_1, \\ \text{all the other brackets being zero.} \end{cases}$$

It is known (see [15], p. 112) that (N, g) is a Riemannian homogeneous Φ-space of order 4. More exactly, the automorphism Φ is determined by means of the isometric automorphism φ of the Lie algebra $\mathfrak{n}$ such that $\varphi^4 = \mathrm{id}$. For convenience, we consider φ written in the form

$$\varphi : (x_1, x_2, x_3, x_4, a_1, a_2) \to (-x_4, -x_3, x_2, x_1, -a_1, -a_2).$$

By our notations we have $\theta = \varphi$. Directly calculating the canonical f-structure $f = \frac{1}{2}(\theta - \theta^3)$ on (N, g), we obtain

$$f : (x_1, x_2, x_3, x_4, a_1, a_2) \to (-x_4, -x_3, x_2, x_1, 0, 0).$$

It follows $f|_V = \varphi|_V$, $f|_Z = 0$, hence $\mathfrak{m}_1 = \mathrm{Im}\, f = V$, $\mathfrak{m}_2 = \mathrm{Ker}\, f = Z$. It means that f has deficiency 2. By Corollary 4 (1), we obtain: $\mathcal{A}(\theta) \cong \mathbb{R} \oplus \mathbb{C}$.

On the other hand, (N, g) is simultaneously a homogeneous $\widetilde{\Phi}$-space of order 3 (see [15], p. 111). Denote $\tilde{\theta} = \tilde{\varphi}$, where $\tilde{\varphi}$ is the corresponding isometric automorphism of $\mathfrak{n}$ such that $\tilde{\varphi}^3 = \mathrm{id}$. In the case we obviously obtain: $\mathcal{A}(\tilde{\theta}) \cong \mathbb{C}$. It can easily be checked that the canonical almost complex structure $J = (1/\sqrt{3})(\tilde{\theta} - \tilde{\theta}^2)$ is defined by the mapping

$$J : (x_1, x_2, x_3, x_4, a_1, a_2) \to (-x_4, -x_3, x_2, x_1, a_2, -a_1).$$

In particular, we have $J|_V = f|_V$.

Remark 2. The above example illustrates two distinct Φ-spaces having the same underlying structure of a homogeneous space. It follows that the same homogeneous space may correspond to various classes in the classification with respect to the algebra $\mathcal{A}(\theta)$, see Remark 1.

4.3. The group of hyperbolic motions of the plane $\mathbb{R}^2$

Finally, we consider the well-known example of a 3-dimensional Riemannian homogeneous Φ-space of order 4, see [10], p. 18.

Let

$$G = \left\{ \begin{pmatrix} e^{-c} & 0 & a \\ 0 & e^c & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

be the Lie group of hyperbolic motions of the plane $\mathbb{R}^2$. This is a solvable Lie group diffeomorphic to $\mathbb{R}^3$. The Riemannian metric g on G determined by the formula

$$ds^2 = e^{2c}\,da^2 + e^{-2c}\,db^2 + \lambda^2\,dc^2, \quad \lambda > 0$$

is invariant with respect to G for any λ indicated. The automorphism Φ of the Lie group G given by the formula

$$\Phi : \begin{pmatrix} e^{-c} & 0 & a \\ 0 & e^c & b \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{c} & 0 & -b \\ 0 & e^{-c} & a \\ 0 & 0 & 1 \end{pmatrix}$$

is an isometry of order 4 with the only fixed point. Hence $(G = \mathbb{R}^3(a, b, c), g)$ is a Riemannian Φ-space of order 4.

Directly calculating in the case the canonical f-structure $f = \frac{1}{2}(\theta - \theta^3)$, we obtain that the subspace $\mathfrak{m}_0 = \operatorname{Ker} f$ is 1-dimensional. Applying again Corollary 4 (1), we get

$$\mathcal{A}(\theta) \cong \mathbb{R} \oplus \mathbb{C}.$$

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