# INVARIANT $f$-STRUCTURES IN GENERALIZED HERMITIAN GEOMETRY *

VITALY V. BALASHCHENKO †

*Faculty of Mathematics and Mechanics, Belarusian State University, Independence av. 4, Minsk 220030, BELARUS*

*E-mail: balashchenko@bsu.by, vitbal@tut.by*

We collect the recent results on invariant $f$-structures in generalized Hermitian geometry. Here the canonical $f$-structures on homogeneous $k$-symmetric spaces play a remarkable role. Specifically, these structures provide a wealth of invariant examples for the classes of nearly Kähler $f$-structures, Hermitian $f$-structures and some others. Finally, we consider all invariant $f$-structures on the complex flag manifold $SU(3)/T_{max}$ and describe them in the sense of generalized Hermitian geometry. In particular, we present first invariant examples of Killing $f$-structures.

## 1. Introduction

Invariant structures on homogeneous manifolds are traditionally among the most important objects in differential geometry, specifically, in Hermitian geometry. In particular, a special role is played by a significant class of invariant nearly Kähler structures based on the canonical almost complex structure on homogeneous 3-symmetric spaces (see [51], [58], [21], [34]). It should be mentioned that the canonical almost complex structure on such spaces became an effective tool and a remarkable example in some deep constructions of differential geometry and global analysis such as homogeneous structures, Einstein metrics, holomorphic and minimal submanifolds, real Killing spinors.

The concept of generalized Hermitian geometry created in the 1980s (see, for example, [35], [38]) is a natural consequence of the development of Her-

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mitian geometry and the theory of almost contact structures. One of its central objects is the metric $f$-structures of the classical type ($f^3 + f = 0$), which include the class of almost Hermitian structures. Many important classes of metric $f$-structures such as Kähler, Killing, nearly Kähler, Hermitian $f$-structures and some others were introduced and intensively investigated in various aspects (see [35], [36], [38], [39] etc.). Specifically, Killing and nearly Kähler $f$-structures became natural generalizations of classical nearly Kähler structures in Hermitian geometry. However, this theory had not provided new invariant examples of its own up to the recent period.

There has recently been a qualitative change in the situation, related to the complete solution of the problem of describing canonical structures of classical type on regular $\Phi$-spaces [17]. A rich collection of canonical $f$-structures has been discovered (including almost complex structures) leading to the presentation of wide classes of invariant examples in generalized Hermitian geometry (see [5]-[8], [18] and others). In particular, nearly Kähler $f$-structures were provided with a remarkable class of their own invariant examples (see [7], [8]). This has ensured a continuation of the classical results of J. A. Wolf, A. Gray, V. F. Kirichenko and others. As to Killing $f$-structures, it is an essential problem to find proper non-trivial invariant examples. Moreover, the possibilities for constructing such examples are fairly limited (see [5]).

The main goals of this paper are

(i) to give a very brief survey on invariant structures in Hermitian and generalized Hermitian geometry and

(ii) to characterize all invariant $f$-structures on the flag manifold $SU(3)/T_{max}$ in the sense of generalized Hermitian geometry, in particular, to present first invariant examples of Killing $f$-structures.

Sections 2-4 are of short survey character. In Section 2, we mention some basic notions and results on homogeneous $\Phi$-spaces and canonical affinor structures of classical types. In particular, the exact formulae for canonical $f$-structures on 4- and 5-symmetric spaces are included. In Section 3, we recall the main classes of almost Hermitian structures following the Gray-Hervella division (see [27]). Besides, we select particular results related to invariant almost Hermitian structures. Further, in Section 4, we describe main classes of metric $f$-structures in generalized Hermitian geometry. Here we also refer to the recent results on invariant nearly Kähler, $G_1f$-, Hermitian, Killing $f$-structures as well as invariant $f$-structures admitting $(1,2)$-symplectic metrics. In this consideration, the canonical $f$-structures on homogeneous 4- and 5-symmetric spaces are especially important.

Finally, in Section 5, we examine in detail all invariant $f$-structures on the complex flag manifold $SU(3)/T_{max}$ with respect to all invariant Riemannian metrics. We discuss belonging these structures to the main classes of metric $f$-structures above mentioned. In particular, invariant non-trivial Killing $f$-structures with the corresponding Riemannian metrics are first presented.

Note that more detailed version of this paper is available in [12].

## 2. Homogeneous Φ-spaces and canonical affinor structures

Here we briefly formulate some basic definitions and results related to regular Φ-spaces and canonical affinor structures on them. More detailed information can be found in [17], [11], [58], [41], [20], [50], [51].

Let $G$ be a connected Lie group, $\Phi$ its (analytic) automorphism, $G^\Phi$ the subgroup of all fixed points of $\Phi$, and $G_o^\Phi$ the identity component of $G^\Phi$. Suppose a closed subgroup $H$ of $G$ satisfies the condition $G_o^\Phi \subset H \subset G^\Phi$. Then $G/H$ is called a *homogeneous* Φ*-space*.

Homogeneous Φ-spaces include homogeneous symmetric spaces ($\Phi^2 = id$) and, more general, *homogeneous* Φ*-spaces of order* $k$ ($\Phi^k = id$) or, in the other terminology, *homogeneous* $k$*-symmetric spaces* (see [41]).

For any homogeneous Φ-space $G/H$ one can define the mapping

$$S_o = D\colon\ G/H \to G/H,\ xH \to \Phi(x)H.$$

It is known [50] that $S_o$ is an analytic diffeomorphism of $G/H$. $S_o$ is usually called a "symmetry" of $G/H$ at the point $o = H$. It is evident that in view of homogeneity the "symmetry" $S_p$ can be defined at any point $p \in G/H$.

Note that there exist homogeneous Φ-spaces that are not reductive. That is why so-called regular Φ-spaces first introduced by N. A.Stepanov [50] are of fundamental importance.

Let $G/H$ be a homogeneous Φ-space, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for $G$ and $H$, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - id$ and the Fitting decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to $A$, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote 0- and 1-component of the decomposition respectively. It is clear that $\mathfrak{h} = Ker\, A$, $\mathfrak{h} \subset \mathfrak{g}_0$. Recall that a homogeneous Φ-space $G/H$ is called a *regular* Φ*-space* if $\mathfrak{h} = \mathfrak{g}_0$ [50]. Note that other equivalent defining conditions can be found in [17], [11].

We formulate two basic facts [50]:

*Any homogeneous* Φ*-space of order* $k$ ($\Phi^k = id$) *is a regular* Φ*-space.*

*Any regular $\Phi$-space is reductive. More exactly, the Fitting decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \;\; \mathfrak{m} = A\mathfrak{g} \tag{1}$$

*is a reductive one.*

Decomposition (1) is the *canonical reductive decomposition* corresponding to a regular $\Phi$-space $G/H$, and $\mathfrak{m}$ is the *canonical reductive complement.*

Decomposition (1) is obviously $\varphi$-invariant. Denote by $\theta$ the restriction of $\varphi$ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$. We note that $\theta$ commutes with any element of the linear isotropy group $Ad(H)$ (see [50]). It also should be noted (see [50]) that $(dS_o)_o = \theta$.

An *affinor structure* on a manifold is known to be a tensor field of type $(1, 1)$ or, equivalently, a field of endomorphisms acting on its tangent bundle. Suppose $F$ is an invariant affinor structure on a homogeneous manifold $G/H$. Then $F$ is completely determined by its value $F_o$ at the point $o$, where $F_o$ is invariant with respect to $Ad(H)$. For simplicity, we will denote by the same manner both any invariant structure on $G/H$ and its value at $o$ throughout the rest of the paper.

Recall [16],[17] that an invariant affinor structure $F$ on a regular $\Phi$-space $G/H$ is called *canonical* if its value at the point $o = H$ is a polynomial in $\theta$.

Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on a regular $\Phi$-space $G/H$. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on $G/H$. It is evident that the algebra $\mathcal{A}(\theta)$ for any symmetric $\Phi$-space ($\Phi^2 = id$) is isomorphic to $\mathbb{R}$. As to arbitrary regular $\Phi$-space $(G/H, \Phi)$, the algebraic structure of its commutative algebra $\mathcal{A}(\theta)$ has been recently completely described (see [10]). It should be also mentioned that all canonical structures are, in addition, invariant with respect to the "symmetries" $\{S_p\}$ of $G/H$.

The most remarkable example of canonical structures is the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on a homogeneous 3-symmetric space (see [51], [58], [21]). It turns out that it is not an exception. In other words, the algebra $\mathcal{A}(\theta)$ contains many affinor structures of classical types.

We will concentrate on the following affinor structures of classical types:

*almost complex structures* $J$ ($J^2 = -1$);

*almost product structures* $P$ ($P^2 = 1$);

*$f$-structures* ($f^3 + f = 0$) [59]; $f$-structures of hyperbolic type or, briefly, *$h$-structures* ($h^3 - h = 0$), [35].

Clearly, $f$-structures and $h$-structures are generalizations of structures $J$ and $P$ respectively.
All the canonical structures of classical type on regular $\Phi$-spaces were completely described [16],[17],[9]. In particular, for homogeneous $k$-symmetric spaces, precise computational formulae were indicated. For future reference we select here the results for canonical $f$-structures (including structures $J$) on homogeneous $\Phi$-spaces of orders 3, 4, and 5 only:

$$
\begin{aligned}
&k=3: \ J=\frac{1}{\sqrt{3}}(\theta-\theta^2) \qquad k=4: \ f=\frac{1}{2}(\theta-\theta^3);\\
&k=5: \ J_1=\alpha(\theta-\theta^4)-\beta(\theta^2-\theta^3); \quad J_2=\beta(\theta-\theta^4)+\alpha(\theta^2-\theta^3);\\
&\qquad\quad f_1=\gamma(\theta-\theta^4)+\delta(\theta^2-\theta^3); \quad f_2=\delta(\theta-\theta^4)-\gamma(\theta^2-\theta^3);
\end{aligned}
$$

where $\alpha=\frac{\sqrt{5+2\sqrt{5}}}{5}$; $\beta=\frac{\sqrt{5-2\sqrt{5}}}{5}$; $\gamma=\frac{\sqrt{10+2\sqrt{5}}}{10}$; $\delta=\frac{\sqrt{10-2\sqrt{5}}}{10}$.
We note that the existence of the structure $J$ and its properties are well known (see [51],[58],[21],[34]). Besides, general properties of the canonical structure $f$ on homogeneous 4-symmetric spaces were investigated in [14].

## 3. Almost Hermitian structures

Let $M$ be a smooth manifold, $\mathfrak{X}(M)$ the Lie algebra of all smooth vector fields on $M$, $d$ the exterior differentiation operator. An *almost Hermitian structure* on $M$ (briefly, *AH-structure*) is a pair $(g,J)$, where $g=\langle\cdot,\cdot\rangle$ is a (pseudo)Riemannian metric on $M$, $J$ an almost complex structure such that $\langle JX,JY\rangle=\langle X,Y\rangle$ for any $X,Y\in\mathfrak{X}(M)$. It follows immediately that the tensor field $\Omega(X,Y)=\langle X,JY\rangle$ is skew-symmetric, i.e. $(M,\Omega)$ is an almost symplectic manifold. $\Omega$ is usually called a *fundamental form* (the *Kähler form*) of an $AH$-structure $(g,J)$.

Further, denote by $\nabla$ the Levi-Civita connection of the metric $g$ on $M$. We recall below some main classes of $AH$-structures together with their defining properties (see, for example, [27]):

| | | |
|---|---|---|
| **K** | *Kähler structure*: | $\nabla J=0$, |
| **H** | *Hermitian structure*: | $\nabla_X(J)Y-\nabla_{JX}(J)JY=0$, |
| $\mathbf{G}_1$ | *AH-structure of class $G_1$*, or *$G_1$-structure*: | $\nabla_X(J)X-\nabla_{JX}(J)JX=0$, |
| **QK** | *quasi-Kähler structure*: | $\nabla_X(J)Y+\nabla_{JX}(J)JY=0$, |
| **AK** | *almost Kähler structure*: | $d\,\Omega=0$, |
| **NK** | *nearly Kähler structure,* or *NK-structure*: | $\nabla_X(J)X=0$. |

It is well known (see, for example, [27]) that

$$\mathbf{K} \subset \mathbf{H} \subset \mathbf{G_1};\ \ \mathbf{K} \subset \mathbf{NK} \subset \mathbf{G_1};\ \ \mathbf{NK} = \mathbf{G_1} \cap \mathbf{QK};\ \ \mathbf{K} = \mathbf{H} \cap \mathbf{QK}.$$

As usual, we will denote by $N$ the Nijenhuis tensor of an almost complex structure $J$, that is,

$$N(X,Y) = \frac{1}{4}\left([JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]\right)$$

for any $X, Y \in \mathfrak{X}(M)$. Then the condition $N = 0$ is equivalent to the integrability of $J$. Moreover, an almost Hermitian structure $(g, J)$ belongs to the class $\mathbf{H}$ if and only if $N = 0$ (see, for example, [27]).

As was already mentioned, the role of homogeneous almost Hermitian manifolds is particularly important "because they are the model spaces to which all other almost Hermitian manifolds can be compared" (see [22]). A wealth of examples for the most classes above noted, both of general and specific character, can be found in [58], [21], [22], [34] and others. In particular, after the detailed investigation of the 6-dimensional homogeneous nearly Kähler manifolds V. F. Kirichenko proved [34] that naturally reductive strictly nearly Kähler manifolds $SO(5)/U(2)$ and $SU(3)/T_{max}$ are not isometric even locally to the 6-dimensional sphere $S^6$. These examples gave a negative answer to the conjecture of S. Sawaki and Y. Yamanoue (see [52]) claimed that any 6-dimensional strictly $NK$-manifold was a space of constant curvature. It should be noted that the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on homogeneous 3-symmetric spaces plays a key role in these and other examples of homogeneous $AH$-manifolds.

Let $g$ be an invariant (pseudo-)Riemannian metric on a homogeneous space $G/H$. Suppose $G/H$ is a reductive homogeneous space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the reductive decomposition of the Lie algebra $\mathfrak{g}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$. Then the invariant metric $g$ is completely defined by its value at the point $o$. For convenience we denote by the same manner both any invariant metric $g$ on $G/H$ and its value at $o$.

Recall that $(G/H, g)$ is *naturally reductive* with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ [40] if $g([X,Y]_{\mathfrak{m}}, Z) = g(X, [Y,Z]_{\mathfrak{m}})$ for all $X, Y, Z \in \mathfrak{m}$. Here the subscript $\mathfrak{m}$ denotes the projection of $\mathfrak{g}$ onto $\mathfrak{m}$ with respect to the reductive decomposition.

We select here some known results closely related to the main subject of our future consideration.

**Theorem 3.1** *[1] Any invariant almost Hermitian structure on a naturally reductive space $(G/H, g)$ belongs to the class $\boldsymbol{G}_1$.*

**Theorem 3.2** *[58], [21] A homogeneous* 3*-symmetric space* $G/H$ *with the canonical almost complex structure* $J$ *and an invariant compatible metric* $g$ *is a quasi-Kähler manifold. Moreover,* $(G/H, J, g)$ *belongs to the class* ***NK*** *if and only if* $g$ *is naturally reductive.*

**Theorem 3.3** *[44], [23], [37] A* 6*-dimensional strictly nearly Kähler manifold is Einstein.*

Finally, we dwell on some recent results obtained in [48] for flag manifolds. Let $G$ be a complex semi-simple Lie group, $\mathfrak{g}$ its Lie algebra. Consider the corresponding maximal flag manifold $\mathbb{F} = G/P$, where $P$ is a Borel (minimal parabolic) subgroup of $G$. For any maximal compact subgroup $U$ of $G$ it is possible to write $\mathbb{F} = U/T$, where $T \subset U$ is a maximal torus. Studying $U$-invariant almost Hermitian structures on $\mathbb{F}$ the following result was proved:

**Theorem 3.4** *[48] Let* $G$ *be a complex simple Lie group. Any invariant nearly Kähler structure on* $\mathbb{F}$ *is Kähler if* $\mathfrak{g}$ *is not* $A_2$*. In the case* $A_2$ *there exists one equivalence class of invariant almost complex structures admitting a unique (up to homothety) nearly Kähler metric.*

Further, in accordance with 16 classes of almost Hermitian structures (see [27]), it was shown in [48] that in the invariant setting on $\mathbb{F}$ these 16 classes collapse down to four classes of invariant almost Hermitian structures with three possibilities for the invariant almost complex structures. More exactly, the following results were summarized [48]:

There are the following classes of invariant almost Hermitian structures on $\mathbb{F}$:

1) *Kähler*: $\{0\}$; $W_1$ (nearly Kähler); $W_2$ (almost Kähler); $W_3$; $W_4$; $W_3 \oplus W_4$ (integrable); $W_2 \oplus W_4$; $W_1 \oplus W_4$; $W_2 \oplus W_3$; $W_2 \oplus W_3 \oplus W_4$.
2) $(1,2)$-*symplectic* (*quasi-Kähler*): $W_1 \oplus W_2$; $W_1 \oplus W_2 \oplus W_4$.
3) *Invariant*: $W_1 \oplus W_2 \oplus W_3$ (co-symplectic); $W_1 \oplus W_3$; $W_1 \oplus W_3 \oplus W_4$. (The last two for specific metrics and every invariant almost complex structure.)

## 4. Metric *f*-structures and homogeneous manifolds

An *f-structure* on a manifold $M$ is known to be a field of endomorphisms $f$ acting on its tangent bundle and satisfying the condition $f^3 + f = 0$ (see [59]). The number $r = dim\ Im\ f$ is constant at any point of $M$ and called a *rank* of the $f$-structure. Besides, the number $dim\ Ker\ f = dim\ M - r$ is usually said to be a *deficiency* of the $f$-structure and denoted by $def\ f$.

Recall that an $f$-structure on a (pseudo)Riemannian manifold $(M, g = \langle\cdot,\cdot\rangle)$ is called a *metric $f$-structure*, if $\langle fX, Y\rangle + \langle X, fY\rangle = 0,\ X, Y \in \mathfrak{X}(M)$ (see [35]). In the case the triple $(M, g, f)$ is called a *metric $f$-manifold*. It is clear that the tensor field $\Omega(X,Y) = \langle X, fY\rangle$ is skew-symmetric, i.e. $\Omega$ is a 2-form on $M$. $\Omega$ is called a *fundamental form* of a metric $f$-structure [38], [35]. It is easy to see that the particular cases $def\ f = 0$ and $def\ f = 1$ of metric $f$-structures lead to almost Hermitian structures and almost contact metric structures respectively.

Let $M$ be a metric $f$-manifold. Then $\mathfrak{X}(M) = \mathcal{L} \oplus \mathcal{M}$, where $\mathcal{L} = Im\ f$ and $\mathcal{M} = Ker\ f$ are mutually orthogonal distributions, which are usually called the *first* and the *second fundamental distributions* of the $f$-structure respectively. Obviously, the endomorphisms $l = -f^2$ and $m = id + f^2$ are mutually complementary projections on the distributions $\mathcal{L}$ and $\mathcal{M}$ respectively. We note that in the case when the restriction of $g$ to $\mathcal{L}$ is non-degenerate the restriction $(F, g)$ of a metric $f$-structure to $\mathcal{L}$ is an almost Hermitian structure, i.e. $F^2 = -id,\ \langle FX, FY\rangle = \langle X, Y\rangle,\ X, Y \in \mathcal{L}$.
A fundamental role in the geometry of metric $f$-manifolds is played by the *composition tensor* $T$, which was explicitly evaluated in [38]:

$$T(X,Y) = \frac{1}{4} f(\nabla_{fX}(f)fY - \nabla_{f^2X}(f)f^2Y), \tag{2}$$

where $\nabla$ is the Levi-Civita connection of a (pseudo)Riemannian manifold $(M, g)$, $\quad X, Y \in \mathfrak{X}(M)$.Using this tensor $T$, the algebraic structure of a so-called *adjoint $Q$-algebra* in $\mathfrak{X}(M)$ can be defined by the formula:
$X * Y = T(X,Y)$. It gives the opportunity to introduce some classes of metric $f$-structures in terms of natural properties of the adjoint $Q$-algebra (see [35], [38]). We enumerate below the main classes of metric $f$-structures together with their defining properties:

| | | |
|---|---|---|
| **Kf** | *Kähler $f$–structure*: | $\nabla f = 0$, |
| **Hf** | *Hermitian $f$–structure*: | $T(X,Y) = 0$, i.e. $\mathfrak{X}(M)$ is an abelian $Q$-algebra, |
| **$\mathbf{G_1f}$** | $f$-structure of class $G_1$, or *$G_1f$-structure*: | $T(X,X) = 0$, i.e. $\mathfrak{X}(M)$ is an anticommut. $Q$-algebra, |
| **QKf** | *quasi-Kähler $f$–structure*: | $\nabla_X f + T_X f = 0$, |
| **Kill f** | *Killing $f$-structure*: | $\nabla_X(f)X = 0$, |
| **NKf** | *nearly Kähler $f$-structure,* or *$NKf$-structure*: | $\nabla_{fX}(f)fX = 0$. |

The classes $\mathbf{Kf}$, $\mathbf{Hf}$, $\mathbf{G_1f}$, $\mathbf{QKf}$ (in more general situation) were introduced in [35] (see also [49]). Killing $f$-manifolds $\mathbf{Kill\ f}$ were defined and studied in [24], [25]. The class $\mathbf{NKf}$ was determined in [7], [8].

The following relationships between the classes mentioned are evident:

$$\mathbf{Kf} = \mathbf{Hf} \cap \mathbf{QKf}; \quad \mathbf{Kf} \subset \mathbf{Hf} \subset \mathbf{G_1f}; \quad \mathbf{Kf} \subset \mathbf{Kill\ f} \subset \mathbf{NKf} \subset \mathbf{G_1f}.$$

It is important to note that in the special case $f = J$ we obtain the corresponding classes of almost Hermitian structures (see [27]). In particular, for $f = J$ the classes $\mathbf{Kill\ f}$ and $\mathbf{NKf}$ coincide with the well-known class $\mathbf{NK}$ of *nearly Kähler structures.*

**Remark 4.1** Killing $f$-manifolds are often defined by requiring the fundamental form $\Omega$ to be a Killing form, i.e. $d\Omega = \nabla\Omega$ (see [24], [39]). It is not hard to prove that the definition is equivalent to the above condition $\nabla_X(f)X = 0$.

Now we dwell on invariant metric $f$-structures on homogeneous spaces.

Any invariant metric $f$-structure on a reductive homogeneous space $G/H$ determines the orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $\mathfrak{m}_1 = Im f$, $\mathfrak{m}_2 = Ker\, f$.

As it was already noted (see Section 3), the main classes of almost Hermitian structures are provided with the remarkable set of invariant examples. It turns out that there is also a wealth of invariant examples for the basic classes of metric $f$-structures. These invariant metric $f$-structures can be realized on homogeneous $k$-symmetric spaces with canonical $f$-structures. We select here only several results in this direction. More detailed information can be found in [5]-[8], [18], [43].

**Theorem 4.1** *[6] Any invariant metric $f$-structure on a naturally reductive space $(G/H, g)$ is a $G_1 f$-structure.*

As a special case ($Ker\ f = 0$), it follows Theorem 3.1.

We stress the particular role of homogeneous 4- and 5-symmetric spaces.

**Theorem 4.2** *[5]-[8] The canonical $f$-structure $f = \frac{1}{2}(\theta - \theta^3)$ on any naturally reductive 4-symmetric space $(G/H, g)$ is both a Hermitian $f$-structure and a nearly Kähler $f$-structure. Moreover, the following conditions are equivalent:*
*1) $f$ is a Kähler $f$-structure; 2) $f$ is a Killing $f$-structure; 3) $f$ is a quasi-Kähler $f$-structure; 4) $f$ is an integrable $f$-structure; 5) $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$; 6) $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$; 7) $G/H$ is a locally symmetric space: $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.*

**Theorem 4.3** *[5], [6], [8], [18] Let $(G/H, g)$ be a naturally reductive* 5*-symmetric space, $f_1$ and $f_2$, $J_1$ and $J_2$ the canonical structures on this space. Then $f_1$ and $f_2$ belong to both classes **Hf** and **NKf**. Moreover, the following conditions are equivalent:*
*1) $f_1$ is a Kähler $f$-structure; 2) $f_2$ is a Kähler $f$-structure; 3) $f_1$ is a Killing $f$-structure; 4) $f_2$ is a Killing $f$-structure; 5) $f_1$ is a quasi-Kähler $f$-structure; 6) $f_2$ is a quasi-Kähler $f$-structure; 7) $f_1$ is an integrable $f$-structure; 8) $f_2$ is an integrable $f$-structure; 9) $J_1$ and $J_2$ are $NK$-structures; 10) $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$ (here $\mathfrak{m}_1 = Im\ f_1 = Ker\ f_2, \mathfrak{m}_2 = Im\ f_2 = Ker\ f_1$); 11) $G/H$ is a locally symmetric space: $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.*

It should be mentioned that Riemannian homogeneous 4-symmetric spaces of classical compact Lie groups were classified and geometrically described in [29]. The similar problem for homogeneous 5-symmetric spaces was considered in [54]. By Theorem 4.2 and Theorem 4.3, it presents a collection of homogeneous $f$-manifolds in the classes **NKf** and **Hf**. Note that the canonical $f$-structures under consideration are generally non-integrable.

Besides, there are invariant $NKf$-structures and $Hf$-structures on homogeneous spaces $(G/H, g)$, where the metric $g$ is not naturally reductive. The example of such a kind can be realized on the 6-dimensional Heisenberg group $(N, g)$. As to details related to this group, we refer to [32], [33], [53].

**Theorem 4.4** *[6]-[8] The 6-dimensional generalized Heisenberg group $(N, g)$ with respect to the canonical $f$-structure $f = \frac{1}{2}(\theta - \theta^3)$ of a homogeneous $\Phi$-space of order* 4 *is both $Hf$- and $NKf$-manifold. This $f$-structure is neither Killing nor integrable on $(N, g)$.*

**Remark 4.2** Theorems 4.2 and 4.4, in particular, illustrate simultaneously the analogy and the difference between the canonical almost complex structure $J$ on homogeneous 3-symmetric spaces $(G/H, g, J)$ and the canonical $f$-structure on homogeneous 4-symmetric spaces $(G/H, g, f)$ (see Theorem 3.2).

Let us also remark that the 6-dimensional generalized Heisenberg group $(N, g)$ is an example of solvable type. In Section 5, we present $NKf$-structures with non-naturally reductive metrics of semi-simple type.

Finally, we briefly discuss the existence problem for invariant Killing $f$-structures. By Theorems 4.2 and 4.3, the canonical $f$-structures on naturally reductive 4- and 5-symmetric spaces are never strictly (i.e. non-Kähler) Killing $f$-structures. Moreover, we recall the following general result:

**Theorem 4.5** *[5] Let $(G/H, g, f)$ be a naturally reductive Killing $f$-manifold. Then the following relations hold:*

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_1 \oplus \mathfrak{h}, \quad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_2 \oplus \mathfrak{h}, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h}.$$

*In particular, both the fundamental distributions of the Killing $f$-structure generate invariant totally geodesic foliations on $G/H$.*

By the results in [24] and Theorem 4.5, it follows

**Corollary 4.1** *[5] There are no non-trivial (i.e. $def\ f > 0$) invariant Killing $f$-structures of the so-called fundamental type (see [24]) on naturally reductive homogeneous spaces $(G/H, g)$.*

This fact is a wide generalization of the similar result of A.Gritsans obtained for Riemannian globally symmetric spaces. Besides, it shows a substantial difference between invariant Killing $f$-structures and invariant $NK$-structures. In Section 5, we will indicate, in particular, first examples of invariant Killing $f$-structures.

It should be mentioned that invariant $f$-structures on flag manifolds were recently investigated in [19]. More precisely, invariant $f$-structures on the classical maximal flag manifolds $\mathbb{F}(n) = U(n)/T$ ($n \geq 2$, $T$ is a maximal torus in the unitary group $U(n)$) were considered. Using graph-theoretic approach, invariant $f$-structures admitting $(1,2)$-symplectic metrics on $\mathbb{F}(n)$ were characterized in the following way:

**Theorem 4.6** *[19] Let $\mathcal{F}$ be an invariant $f$-structure on $\mathbb{F}(n)$, $n \geq 2$. $(\mathbb{F}(n), \mathcal{F})$ admits invariant $(1,2)$-symplectic metrics if and only if the associated with $\mathcal{F}$ digraph $\mathcal{G}$ is locally transitive.*

It is noted in [19] that the problem of classifying locally transitive digraphs is still open. We refer to [19], [48] and many preceding works sited here for details in notions, constructions, and results.

## 5. Invariant $f$-structures on the complex flag manifold $\mathbf{M = SU(3)/T_{max}}$

In this Section, we will consider all invariant $f$-structures on the flag manifold $M = SU(3)/T_{max}$. Note that invariant almost complex structures (i.e. $f$-structures of maximal rank 6) on this space were investigated in [22], [2], [3] and many other papers.

The homogeneous manifold $SU(3)/T_{max}$ is known to be an important example in many branches of differential geometry and beyond. In particular,

$M = SU(3)/T_{max}$ is a Riemannian homogeneous 3-symmetric space not homeomorphic with the underlying manifold $M$ of any Riemannian symmetric space (see [42]). Further, $M$ is a homogeneous $k$-symmetric space for any $k \geq 3$. Moreover, $M$ is a naturally reductive Riemannian homogeneous space that is *non-commutative* (see [30]). It means that the algebra of invariant differential operators $\mathcal{D}(SU(3)/T_{max})$ is not commutative (see [28]). It follows that $M = SU(3)/T_{max}$ is not even a *weakly symmetric space* (see, for example, [55]).

Besides, $M$ is the twistor space for the projective space $\mathbb{C}P^2$ (see, for example, [13], Chapter 13). It was a key point for constructing the first examples of 6-dimensional Riemannian manifolds admitting a real Killing spinor (see [15]). More exactly, the flag manifold $M = SU(3)/T_{max}$ with the nearly Kähler structure $(g, J)$ just possesses a real Killing spinor (see [15], [26]). Moreover, using the duality procedure for this space $SU(3)/T_{max}$, one can effectively construct pseudo-Riemannian homogeneous manifolds with the real Killing spinors (see [31]).

Let $\Phi = I(s)$ be an inner automorphism of the Lie group $SU(3)$ defined by the element $s = diag\,(\varepsilon, \overline{\varepsilon}, 1)$, where $\varepsilon$ is a primitive third root of unity. Then the subgroup $H = G^\Phi$ of all fixed points of $\Phi$ is of the form:

$$G^\Phi = \{diag(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}) | \beta_1 + \beta_2 + \beta_3 = 0,\ \beta_j \in \mathbb{R}\}.$$

Obviously, $G^\Phi$ is isomorphic to $T^2 = T_{max}$ diagonally imbedded into $SU(3)$. It means that the flag manifold $M = SU(3)/T_{max}$ is a homogeneous 3-symmetric space defined by the automorphism $\Phi$.

Consider the canonical reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$ for the homogeneous $\Phi$-space $M$. Using the notations in [46], we obtain:

$$\mathfrak{g} = \mathfrak{su}(3) = \left\{ \begin{pmatrix} \alpha_1 & a & \overline{c} \\ -\overline{a} & \alpha_2 & b \\ -c & -\overline{b} & \alpha_3 \end{pmatrix} \middle| \begin{array}{l} \alpha_1, \alpha_2, \alpha_3 \in Im\,\mathbb{C}, \\ a, b, c \in \mathbb{C}, \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \end{array} \right\}$$

$$= E(\alpha_1, \alpha_2, \alpha_3) \oplus D(a, b, c) = \mathfrak{h} \oplus \mathfrak{m}.$$

If we put $X = D(a, b, c), Y = D(a_1, b_1, c_1), Z = E(\alpha_1, \alpha_2, \alpha_3)$, then the Lie brackets can be briefly indicated (see [47]):

$$[X, Y] = D\,(\overline{bc_1 - b_1c}, \overline{ca_1 - c_1a}, \overline{ab_1 - a_1b})$$
$$-2E\,(Im(a\overline{a_1} + \overline{c}c_1), Im(\overline{a}a_1 + b\overline{b_1}), Im(c\overline{c_1} + \overline{b}b_1)),$$

$$[Z, X] = D\,(\alpha_1 a - a\alpha_2, \alpha_2 b - b\alpha_3, \alpha_3 c - c\alpha_1).$$

Further, we put$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, where

$$\mathfrak{m}_1 = \{X \in \mathfrak{su}(3) | X = D(a, 0, 0), a \in \mathbb{C}\},$$
$$\mathfrak{m}_2 = \{X \in \mathfrak{su}(3) | X = D(0, b, 0), b \in \mathbb{C}\},$$
$$\mathfrak{m}_3 = \{X \in \mathfrak{su}(3) | X = D(0, 0, c), c \in \mathbb{C}\}.$$

Using the Killing form of the Lie algebra $\mathfrak{su}(3)$, we define an invariant inner product on $\mathfrak{m}$:

$$g_o(X, Y) = \langle X, Y \rangle_o = -\tfrac{1}{2} Re\, tr\, XY.$$

Then (see [46]) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ is $\langle \cdot, \cdot \rangle_o$-orthogonal decomposition satisfying the following relations:

$$[\mathfrak{h}, \mathfrak{m}_j] \subset \mathfrak{m}_j,\ [\mathfrak{m}_j, \mathfrak{m}_j] \subset \mathfrak{h},\ [\mathfrak{m}_j, \mathfrak{m}_{j+1}] \subset \mathfrak{m}_{j+2},$$

where $j = 1, 2, 3$ and the index $j$ should be reduced by modulo 3. Besides, the $H$-modules $\mathfrak{m}_j$ are pairwise non-isomorphic.

Now we turn to invariant Riemannian metrics on $M$. Taking into account the well-known one-to-one correspondence between $G$-invariant Riemannian metrics on $G/H$ and $Ad(H)$-invariant inner products on $\mathfrak{m}$ (see [40]), we will make use of the following fact:

**Lemma 5.1** *[46] Any $SU(3)$-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ on the flag manifold $M = SU(3)/T_{max}$ can be written in the form*

$$g = \langle \cdot, \cdot \rangle = \lambda_1 \langle \cdot, \cdot \rangle_{o|\mathfrak{m}_1 \times \mathfrak{m}_1} + \lambda_2 \langle \cdot, \cdot \rangle_{o|\mathfrak{m}_2 \times \mathfrak{m}_2} + \lambda_3 \langle \cdot, \cdot \rangle_{o|\mathfrak{m}_3 \times \mathfrak{m}_3},$$

*where $\lambda_j > 0,\ j = 1, 2, 3$.*

A triple $(\lambda_1, \lambda_2, \lambda_3)$ is called [46] a *characteristic collection* of a Riemannian metric $g$ above mentioned . Considering Riemannian metrics up to homothety, one can assume that $(\lambda_1, \lambda_2, \lambda_3) = (1, t, s),\ t > 0, s > 0$. For convenience we will denote this correspondence in the following way: $g = (\lambda_1, \lambda_2, \lambda_3)$ or $g = (1, t, s)$.

We also recall the following result:

**Theorem 5.1** *[57],[4],[46] There are exactly (up to homothety) the following invariant Einstein metrics on the flag manifold $SU(3)/T_{max}$ :*

$$(1, 1, 1), (1, 2, 1), (1, 1, 2), (2, 1, 1).$$

Let $\alpha$ be the Nomizu function (see [45]) of the Levi-Civita connection $\nabla$ for an invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ on a reductive homogeneous space $G/H$. Then

$$\alpha(X, Y) = \frac{1}{2} [X, Y]_{\mathfrak{m}} + U(X, Y), \quad X, Y \in \mathfrak{m}, \tag{3}$$

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is a symmetric bilinear mapping determined by the formula (see[40]):

$$2\langle U(X,Y), Z\rangle = \langle X, [Z,Y]_{\mathfrak{m}}\rangle + \langle [Z,X]_{\mathfrak{m}}, Y\rangle.$$

For our case in these notations we have

**Lemma 5.2** *[56],[47] For the Levi-Civita connection of a Riemannian metric $g = (\lambda_1, \lambda_2, \lambda_3)$ on the flag manifold $SU(3)/T_{max}$ the following conditions are satisfied*:

$$U(X,Y) = 0, \quad if \quad X, Y \in \mathfrak{m}_j, \; j \in \{1,2,3\};$$

$$U(X,Y) = -(2\lambda_j)^{-1}(\lambda_{j+1} - \lambda_{j+2})[X,Y], \quad if \quad X \in \mathfrak{m}_{j+1}, Y \in \mathfrak{m}_{j+2},$$

*where $j = 1,2,3$ and the numbers $j$ are reduced by modulo 3.*

Let us now turn to invariant $f$-structures on $M = SU(3)/T_{max}$. Keeping the above notations, any invariant $f$-structure on $M$ can be expressed by the mapping

$$f : D(a,b,c) \to D(\zeta_1 ia, \zeta_2 ib, \zeta_3 ic), \tag{4}$$

where $\zeta_j \in \{1, 0, -1\}$, $j = 1,2,3$, $i$ is the imaginary unit. We will call the collection $(\zeta_1, \zeta_2, \zeta_3)$ a *characteristic collection* of the invariant $f$-structure and for convenience denote $f = (\zeta_1, \zeta_2, \zeta_3)$. Obviously, all invariant $f$-structures on $M$ pairwise commute.

If we agree to consider $f$-structures up to sign, then there are the following invariant $f$-structures on $M = SU(3)/T_{max}$:

1) *invariant $f$-structures of rank* 6 *(invariant almost complex structures)*:

$$J_1 = (1,1,1), \quad J_2 = (1,-1,1), \quad J_3 = (1,1,-1), \quad J_4 = (1,-1,-1).$$

2) *invariant $f$-structures of rank* 4:

$$f_1 = (1,1,0), \quad f_2 = (1,0,1), \quad f_3 = (0,1,1),$$
$$f_4 = (1,-1,0), \quad f_5 = (1,0,-1), \quad f_6 = (0,1,-1).$$

3) *invariant $f$-structures of rank* 2:

$$f_7 = (1,0,0), \quad f_8 = (0,1,0), \quad f_9 = (0,0,1).$$

Our description of all invariant $f$-structures and all invariant Riemannian metrics evidently implies that any invariant $f$-structure $f = (\zeta_1, \zeta_2, \zeta_3)$ is a metric $f$-structure with respect to any invariant Riemannian metric $g = (\lambda_1, \lambda_2, \lambda_3)$. In particular, $J_j$, $j = 1,2,3,4$ are invariant almost Hermitian structures with respect to all invariant Riemannian metrics $g = (\lambda_1, \lambda_2, \lambda_3)$.

Now we are able to investigate all invariant $f$-structures in the sense of generalized Hermitian geometry, i.e. the special classes **Kf**, **NKf**, **Kill f**,

**Hf**, $\mathbf{G_1 f}$.

A key point of our consideration belongs to the expression $\nabla_X(f)Y$. Using formula (3), we get:

$$\begin{aligned}\nabla_X(f)Y &= \nabla_X fY - f\nabla_X Y = \alpha(X, fY) - f\alpha(X,Y) \\ &= \frac{1}{2}\left([X, fY]_{\mathfrak{m}} - f[X,Y]_{\mathfrak{m}}\right) + U(X, fY) - fU(X,Y).\end{aligned}$$

As a result, we can obtain:

$$\begin{aligned}\nabla_X(f)Y &= \frac{1}{2}\, D(A,B,C), \quad \text{where} \\ A &= \overline{i((\zeta_1+\zeta_3)(1+s-t)bc_1 + (\zeta_1+\zeta_2)(s-t-1)b_1c)}, \\ B &= \overline{i((\zeta_2+\zeta_1)(1+\frac{1-s}{t})ca_1 + (\zeta_2+\zeta_3)(\frac{1-s}{t}-1)c_1a)}, \\ C &= \overline{i((\zeta_3+\zeta_2)(\frac{t-1}{s}+1)ab_1 + (\zeta_3+\zeta_1)(\frac{t-1}{s}-1)a_1b)}. \qquad (5)\end{aligned}$$

## 5.1. *Kähler f-structures*

Kähler $f$-structures are defined by the condition $\nabla_X(f)Y = 0$ (see Section 4). Using formula (5), this condition is equivalent to the following system of equations:

$$\begin{cases}(\zeta_1+\zeta_3)(s-t+1) = 0\\ (\zeta_1+\zeta_2)(s-t-1) = 0\\ (\zeta_2+\zeta_3)(s+t-1) = 0\end{cases} \qquad (6)$$

Solving (6) for all invariant $f$-structures, we obtain the following result:

**Proposition 5.1** *The flag manifold $M = SU(3)/T_{max}$ admits the following invariant Kähler $f$-structures with respect to the corresponding invariant Riemannian metrics only:*

$$\begin{aligned}J_2 &= (1,-1,1), \qquad & g_t &= (1,t,t-1),\ t>1;\\ J_3 &= (1,1,-1), \qquad & g_t &= (1,t,t+1),\ t>0;\\ J_4 &= (1,-1,-1), \qquad & g_t &= (1,t,1-t),\ 0<t<1.\end{aligned}$$

*Hence there are no invariant Kähler $f$-structures of rank 2 and 4 on $M$.*

We note that the result is known for invariant almost complex structures (see [22],[3]). We can also observe that for each of Kähler $f$-structures $J_2, J_3, J_4$ the corresponding 1-parameter set $g_t$ of invariant Riemannian metrics contains exactly one Einstein metric excluding the naturally reductive metric $g = (1, 1, 1)$ (see Theorem 5.1). Taking into account Theorem 3.2, the latter fact implies that the structures $J_2, J_3, J_4$ cannot be realized as the canonical almost complex structures $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ for some homogeneous $\Phi$-spaces of order 3.

In addition, Lie brackets relations for the subspaces $\mathfrak{m}_j,\ j = 1, 2, 3$ imply that all invariant $f$-structures of rank 2 and 4 are non-integrable. It immediately follows that these $f$-structures cannot be Kähler $f$-structures.

### 5.2. *Killing f-structures*

The defining condition for Killing $f$-structures can be written in the form $\nabla_X(f)X = 0$ (see Section 4). From (5), it follows

$$\nabla_X(f)X = \frac{1}{2}\, D(A_0, B_0, C_0), \quad \text{where}$$
$$A_0 = \overline{ibc((\zeta_1 + \zeta_3)(1 + s - t) + (\zeta_1 + \zeta_2)(s - t - 1))},$$
$$B_0 = \overline{ica((\zeta_2 + \zeta_1)(1 + \frac{1-s}{t}) + (\zeta_2 + \zeta_3)(\frac{1-s}{t} - 1))},$$
$$C_0 = \overline{iab((\zeta_3 + \zeta_2)(\frac{t-1}{s} + 1) + (\zeta_3 + \zeta_1)(\frac{t-1}{s} - 1))}.$$

It easy to show that the condition $\nabla_X(f)X = 0$ is equivalent to the following system of equations:

$$\begin{cases} (\zeta_1 + \zeta_3)(s - t + 1) + (\zeta_1 + \zeta_2)(s - t - 1) = 0 \\ (\zeta_1 + \zeta_2)(s - t - 1) + (\zeta_2 + \zeta_3)(s + t - 1) = 0 \end{cases}$$

Analyzing this system for all invariant $f$-structures, we obtain the following result:

**Proposition 5.2** *All invariant strictly Killing (i.e. non-Kähler) $f$-structures on the flag manifold $M = SU(3)/T_{max}$ and the corresponding invariant Riemannian metrics (up to homothety) are indicated below:*

$$\begin{aligned} J_1 &= (1, 1, 1), & g &= (1, 1, 1);\\ f_1 &= (1, 1, 0), & g &= (3, 3, 4);\\ f_2 &= (1, 0, 1), & g &= (3, 4, 3);\\ f_3 &= (0, 1, 1), & g &= (4, 3, 3). \end{aligned}$$

*In particular, there are no invariant Killing $f$-structures of rank 2 on $M$.*

Note the structure $J_1$ is a well-known non-integrable nearly Kähler structure on a naturally reductive space $M$ (see [21], [22], [34], [3] and others). The structures $f_1, f_2, f_3$ present first invariant non-trivial Killing $f$-structures [12]. The important feature of these structures is that the corresponding invariant Riemannian metrics are not Einstein (see Theorem 5.1). It illustrates a substantial difference between non-trivial strictly Killing $f$-structures and strictly $NK$-structures at least in the 6-dimensional case (see Theorem 3.3).

**Remark 5.1** It is interesting to note that all strictly Killing $f$-structures above indicated are canonical $f$-structures for suitable homogeneous $\Phi$-spaces of the Lie group $SU(3)$. We already mentioned that $M = SU(3)/T_{max}$ is a homogeneous $k$-symmetric space for any $k \geq 3$. It means $M$ as an underlying manifold could be generated by various automorphisms $\Phi$ of the Lie group $SU(3)$. In particular, $J_1$ is the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ for the homogeneous $\Phi$-space of order 3, where $\Phi = I(s),\ s = diag\ (\varepsilon, \bar{\varepsilon}, 1),\ \varepsilon = \sqrt[3]{1}$ (see the beginning of this Section). Further, if we consider the automorphism $\Phi_1 = I(s_1),\ s_1 = diag\ (i, -i, 1)$, where $i = \sqrt[4]{1}$ is the imaginary unit, then $M$ is a homogeneous $\Phi_1$-space of order 4. The corresponding canonical $f$-structure $f = \frac{1}{2}(\theta_1 - \theta_1^3)$ for this $\Phi_1$-space just coincides (up to sign) with the $f$-structure $f_3 = (0, 1, 1)$. The structures $f_1$ and $f_2$ can be obtained in the similar way. Moreover, all the structures $f_1, f_2, f_3$ and $f_7, f_8, f_9$ can be realized as canonical $f$-structures for suitable homogeneous $\Phi$-spaces of order 5.

We also note that all $f$-structures $f_1, f_2, f_3$ are just the restrictions of the structure $J_1$ onto the corresponding distributions $\mathfrak{m}_p \oplus \mathfrak{m}_q,\ p, q \in \{1, 2, 3\}$.

### 5.3. *Nearly Kähler f-structures*

Using (5), we can easily obtain:

$$\begin{aligned}
\nabla_{fX}(f)fX = \frac{1}{2}D(\hat{A}, \hat{B}, \hat{C}), \quad \text{where} \\
\hat{A} &= \overline{-i\zeta_2\zeta_3 bc((\zeta_1 + \zeta_3)(1 + s - t) + (\zeta_1 + \zeta_2)(s - t - 1))}, \\
\hat{B} &= \overline{-i\zeta_1\zeta_3 ca((\zeta_2 + \zeta_1)(1 + \frac{1 - s}{t}) + (\zeta_2 + \zeta_3)(\frac{1 - s}{t} - 1))}, \\
\hat{C} &= \overline{-i\zeta_1\zeta_2 ab((\zeta_3 + \zeta_2)(\frac{t - 1}{s} + 1) + (\zeta_3 + \zeta_1)(\frac{t - 1}{s} - 1))}.
\end{aligned}$$

It follows that the condition $\nabla_{fX}(f)fX = 0$ is reduced to the following system of equations:

$$\begin{cases} \zeta_2\zeta_3((\zeta_1+\zeta_3)(s-t+1)+(\zeta_1+\zeta_2)(s-t-1))=0 \\ \zeta_1\zeta_3((\zeta_2+\zeta_1)(1+t-s)+(\zeta_2+\zeta_3)(1-s-t))=0 \\ \zeta_1\zeta_2((\zeta_3+\zeta_2)(t+s-1)+(\zeta_3+\zeta_1)(t-s-1))=0 \end{cases}$$

Consideration of this system implies

**Proposition 5.3** *The only invariant strictly nearly Kähler $f$-structure of rank 6 on the flag manifold $SU(3)/T_{max}$ is the nearly Kähler structure $J_1 = (1,1,1)$ with respect to the naturally reductive metric $g = (1,1,1)$. Invariant strictly nearly Kähler $f$-structures of rank 4 and the corresponding invariant Riemannian metrics (up to homothety) are:*

$$\begin{aligned} f_1 &= (1,1,0), \qquad g_s = (1,1,s), \quad s > 0; \\ f_2 &= (1,0,1), \qquad g_t = (1,t,1), \quad t > 0; \\ f_3 &= (0,1,1), \qquad g_t = (1,t,t), \quad t > 0. \end{aligned}$$

*The invariant $f$-structures $f_7, f_8, f_9$ of rank 2 are strictly $NKf$-structures with respect to all invariant Riemannian metrics $g = (1,t,s),\ t,s > 0$.*

The structures $f_1, f_2, f_3$ and $f_7, f_8, f_9$ provide invariant examples of $NKf$-structures with non-naturally reductive metrics on the homogeneous space $M = SU(3)/T_{max}$, which belongs to a semi-simple type. Besides, for any invariant strictly $NKf$-structure on $M$ there exists at least one (up to homothety) corresponding Einstein metric. More exactly, for these $NKf$-structures of rank 6, 4, and 2 there are (up to homothety) 1, 2, and 4 Einstein metrics respectively (see Theorem 5.1). This is a certain analogy with the result of Theorem 3.3. This particular fact and some related general results lead to the following conjecture, which seems to be plausible:

**Conjecture**. For any strictly nearly Kähler $f$-structure on a 6-dimensional manifold there exists at least one corresponding Einstein metric.

**Remark 5.2** The invariant $f$-structures $f_4, f_5, f_6$ on the flag manifold $M = SU(3)/T_{max}$ cannot be canonical $f$-structures for all homogeneous $\Phi$-spaces of orders 4 and 5 of the Lie group $SU(3)$. It evidently follows by comparing the results in Theorem 4.2, Theorem 4.3, and Proposition 5.3.

### 5.4. *Hermitian f-structures*

We calculate the tensor $T$ (see formula (2)) for any invariant $f$-structure on $(SU(3)/T_{max}, g = (1,t,s))$. Combining (5) and (4), one can obtain:

$$T(X,Y) = \frac{1}{8}\, D(\check{A}, \check{B}, \check{C}), \quad \text{where} \tag{7}$$

$$\check{A} = -\zeta_1\zeta_2\zeta_3(1+\zeta_2\zeta_3)((\zeta_1+\zeta_3)(1+s-t)\overline{bc_1} + (\zeta_1+\zeta_2)(s-t-1)\overline{b_1c}),$$

$$\check{B} = -\zeta_1\zeta_2\zeta_3(1+\zeta_1\zeta_3)((\zeta_2+\zeta_1)(1+\frac{1-s}{t})\overline{ca_1} + (\zeta_2+\zeta_3)(\frac{1-s}{t}-1)\overline{c_1a}),$$

$$\check{C} = -\zeta_1\zeta_2\zeta_3(1+\zeta_1\zeta_2)((\zeta_3+\zeta_2)(\frac{t-1}{s}+1)\overline{ab_1} + (\zeta_3+\zeta_1)(\frac{t-1}{s}-1)\overline{a_1b}).$$

We recall that the defining property for a Hermitian $f$-structure is the condition $T(X,Y) = 0$. Now from (7), we get the following result:

**Proposition 5.4** *The invariant $f$-structures $J_2, J_3, J_4$ and $f_1, \dots, f_9$ are Hermitian $f$-structures with respect to all invariant Riemannian metrics $g = (1,t,s),\ \ t,s > 0$ on the flag manifold $M = SU(3)/T_{max}$.*

Notice that the almost complex structure $J_1 = (1,1,1)$ is non-integrable. It agrees with the fact that $J_1$ is not a Hermitian $f$-structure for each Riemannian metric. While we stress that all $f$-structures $f_1, \dots, f_9$ of rank 4 and 2 are non-integrable, but they are Hermitian $f$-structures.

### 5.5. *$G_1f$-structures*

Finally, we consider the condition $T(X,X) = 0$, which is the defining property for $G_1f$-structures. Using (7) and taking into account Propositions 5.3 and 5.4, we get

**Proposition 5.5** *The flag manifold $M = SU(3)/T_{max}$ does not admit invariant strictly $G_1f$-structures (i.e. neither $NKf$-structures nor $Hf$-structures). In particular, there are no invariant strictly $G_1$-structures $J$ (i.e. neither nearly Kähler nor Hermitian) on $M$.*

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